On the mass quadrupole at the 4th post-Newtonian approximation

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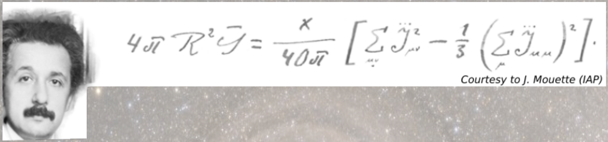
Motivation: The mass quadrupole and the GW



In a modern language it reads

$$\mathcal{F}_{GW} = \frac{G}{5c^5}\left[\dddot{\mathfrak{I}}_{ab}\dddot{\mathfrak{I}}_{ab} + \mathcal{O}\left(\frac{1}{c^2}\right)\right],$$

with the symmetric trace-free mass quadrupole, at linear order:

$$\mathfrak{I}_{ab} = \int \mathrm{d}^3\vec{x}\,\rho(\vec{x})\left[x_{(a}x_{b)} - \frac{1}{3}r^2\delta_{ab}\right] + \mathcal{O}\left(\frac{1}{c^2}\right).$$

The phase evolution of the GW is computed by integrating

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}_{GW}.$$

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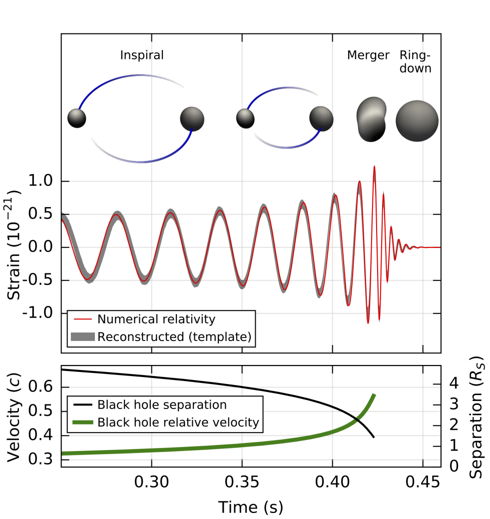
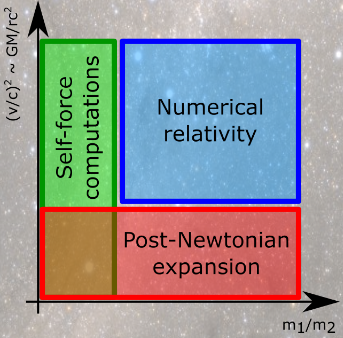


Figure from *LSC & Virgo*, PRL **116**(2016)061102.

For bound systems,



NB: nPN $\equiv \mathcal{O}\left(c^{-2n}\right)$.

Introduction: Current state of the art

To derive the phase, one needs to compute E and $\mathcal{F}_{GW}$.

- Three approaches are used today to compute E *via* the EOM:
  - ✓ *ADM*-Hamiltonian: full 4PN order, up to one *ambiguity* parameter,[1]
  - ✓ Fokker Lagrangian: full 4PN order,[2]
  - ⇝ EFT: on their way toward the 4PN order.[3]

- To compute $\mathcal{F}_{GW}$, one needs:
  - ⇝ the source multipole moments (*this work, in progress*),
  - ✓ the non-linear "tails" interactions, computed up to 4.5PN.[4]

- And in alternative theories of gravitation ?
  - ✓ L. Bernard computed the full 3PN dynamics of scalar-tensor theory.[5]

[1] T. Damour, P. Jaranowski & G. Schäfer, Phys.Rev.D **93**, 084014 (2016).
[2] T. Marchand, L. Bernard, L. Blanchet & G. Faye, Phys.Rev.D **97**, 044023 (2018).
[3] S. Foffa, P. Mastrolia, R. Sturani & C. Sturm, Phys.Rev.D **95**, 104009 (2017).
[4] T. Marchand, L. Blanchet & G. Faye, CQG **33**(2016)244.
[5] L. Bernard, Phys.Rev.D **98**, 044004 (2018).

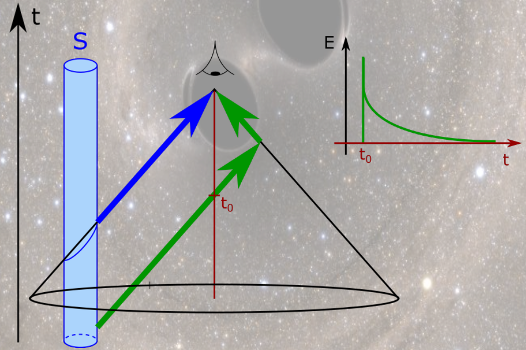
Introduction: The tails of GW

- GR is highly non-linear: scattering of GW,

↪ the source multipoles are not the observed ones,

↪ 1.5PN non-local terms depending on the whole history of the source.

⇒ This effect is controlled and computed up to 4.5PN order[6] !



[6] T. Marchand, L. Blanchet & G. Faye, CQG **33**(2016)244.

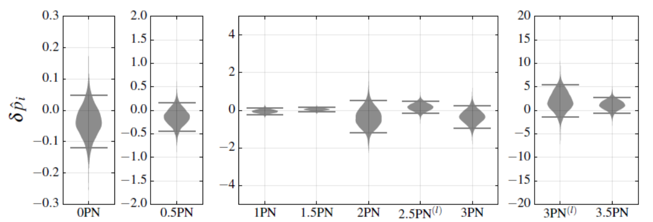
Why shall we compute the mass quadrupole at 4PN ?

- Precision: 1 in 160 000 cycles for a BNS.. extremely small !

↪ but needed for space-borne detectors (*eg.* LISA or DECIGO).

- Better determination of the astrophysical parameters (masses, spins,...).

- Allow for more precise tests of GR *vs.* alternative theories of gravity.



Combined posteriors for GW150914, GW151226 & GW170104.
Figure from *LSC & Virgo*, PRL **118**(2018)221101.

General strategy: The source multipole

In full generality, the source mass multipole reads[7] (with $L = i_1 i_2 \dots i_\ell$)

$$\mathcal{I}_L = \underset{B}{\mathrm{FP}} \int \mathrm{d}^3\vec{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \left\{ \delta_\ell \, \hat{x}_L \, \Sigma - \frac{4(2\ell+1)\,\delta_{\ell+1}}{c^2(\ell+1)(2\ell+3)} \hat{x}_{aL} \, \dot{\Sigma}_a + \frac{2(2\ell+1)\,\delta_{\ell+2}}{c^4(\ell+1)(\ell+2)(2\ell+5)} \hat{x}_{abL} \, \ddot{\Sigma}_{ab} \right\},$$

where $\hat{x}_L = \mathrm{STF}[x_L]$, $\delta_\ell = \frac{(2\ell+1)!!}{2(\ell+1)\ell!}(1-z^2)^\ell$ and

$$\Sigma = \frac{\tau^{00} + \tau^{ii}}{c^2}, \quad \Sigma_i = \frac{\tau^{0i}}{c}, \quad \Sigma_{ij} = \tau^{ij},$$

are evaluated in $(\vec{x}, t - zr/c)$ and $\tau^{\mu\nu}$ is the Landau–Lifshitz pseudo-tensor of matter and gravitation, defined as

$$\Box h^{\mu\nu} = \frac{16\pi\, G}{c^4} \tau^{\mu\nu}.$$

[7] L. Blanchet, Class.Quant.Grav. **15**(1998)1971.

General strategy: The potentials

$$\Sigma = \frac{\tau^{00} + \tau^{ii}}{c^2}, \quad \Sigma_i = \frac{\tau^{0i}}{c}, \quad \Sigma_{ij} = \tau^{ij}, \quad \text{with} \quad \Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}.$$

↪ The Σ's have

- a matter part, sourced by $T^{\mu\nu}$,
- a metric part, sourced by the non-linearities of Einstein's equations.

⇒ The matter system is taken as a binary of compact objects,

⇒ The metric can be decomposed in potentials, as for example

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \frac{8}{c^6}\left(\hat{X} + \frac{V^3}{6} + \ldots\right) + \frac{32}{c^8}\left(\hat{T} - \frac{V\hat{X}}{2} + \ldots\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right),$$

with

$$V = \Box^{-1}\left[-4\pi G \frac{T^{00} + T^{ii}}{c^2}\right] = G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) + \mathcal{O}\left(\frac{1}{c^2}\right).$$

General strategy

How to proceed ?

- $\hookrightarrow$ Express the Σ's in terms of potentials, but some of them are not known in the full space,
- $\hookrightarrow$ deal with them by some ingenious integrations by part,
- $\hookrightarrow$ Taylor-expand $\mathfrak{I}_{ij}$ up to 4^{th}PN order,
- $\hookrightarrow$ integrate,
- $\hookrightarrow$ check against known results (self-force, numerical relativity,...),
- $\hookrightarrow$ grab a beer and enjoy !

$\Rightarrow$ Seems rather simple, but many complications appear...

How to deal with UV divergences ?

Point-source approximation for the compact objects

$$\rho(\vec{x},t) = m_1\,\delta^{(3)}\left[\vec{x}-\vec{y}_1(t)\right] + m_2\,\delta^{(3)}\left[\vec{x}-\vec{y}_2(t)\right],$$

$\hookrightarrow$ safe as $\left(\frac{v}{c}\right)^2 \sim \frac{R_s}{r} \ll 1$,

$\hookrightarrow$ but non-physical divergences appear in the integrals !

$\Rightarrow$ Need for a regularisation: Hadamard's one seemed OK up to 2.5PN,

$\hookrightarrow$ but not strong enough for higher orders: it does effectively break the diffeomorphisms...

$\Rightarrow$ Need to use dimensional regularisation...

$\rightarrowtail$ In practice: compute all with Hadamard and add the difference DimReg - Hadamard around the singularities.

Where are we now ?

In terms of potential, one writes

$$h^{ij} = -\frac{4}{c^4}\left(\hat{W}_{ij} - \frac{1}{2}\hat{W}_{kk}\,\delta_{ij}\right) - \frac{16}{c^6}\left(\hat{Z}_{ij} - \frac{1}{2}\hat{Z}_{kk}\,\delta_{ij}\right) + \frac{1}{c^8}\left(\hat{M}_{ij} + \dots\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right),$$

where

$$\Box \hat{W}_{ij} = -4\pi\, G\,(T_{ij} - T_{kk}\,\delta_{ij}) - \partial_i V \partial_j V\,.$$

$\Rightarrow$ The matter part is easily inverted.

$\Rightarrow$ The $\Box^{-1}\left[\partial_i V \partial_j V\right]$ is less trivial, the 2PN part was only computed at infinity.

$$V = \frac{Gm_1}{r_1} + \frac{Gm_1}{c^2}\,\frac{\partial_t^2 r_1}{2} + (1 \to 2) + \mathcal{O}\left(\frac{1}{c^4}\right),$$

$\hookrightarrow$ thus one needs to know the kernels $\Delta^{-n} r_1^{2p-1} r_2^{2q-1}$.

$\hookrightarrow$ $\hat{W}_{ij}^{2PN}$ currently computed, around 200 000 terms expected.

Summary and perspectives

- Computing the mass quadrupole is an important step in deriving the full waveform of GW.
  - ↪ Needed for space-borne detectors (LISA, DECIGO,...).
  - ↪ Currently being computed.

- What's next ?
  - ↪ We need the current quadrupole and the mass octupole at 3PN,
  - ↪ then we can compute the full waveform and energy flux at 4PN,
  - ↪ add spins,
  - ↪ thus derive the evolution of the phase up to 4.5PN,
  - ↪ and so build a bank of templates for the detector.


Thank you for your attention !
Mass octupole @ 3PN
Current quadrupole @ 3PN
Mass quadrupole @ 4PN