# Universal structure of covariant holographic two-point functions in massless higher-order gravities

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# Abstract and main result

We manily consider massless high-order gravities in general $D = d + 1$ dimensions, which are Einstein gravity extended with higher-order curvature invariants in such a way that the linearized spectrum around the AdS vacca involes only the massless graviton. We derive the covariant holographic two-point function and find that they have a universal structure.

- the theory-dependent overall coefficient factor $C_T$ can be universally expressed by $(d-1)C_T = \ell(\partial a/\partial\ell)$, where $a$ is the holographic a-charge and $\ell$ is the AdS radius.
- We verify this relations in Gauss-Bonnet, Lovelock and Einstein cubic gravities.
- In $d = 4$, we also find an intriguing relation between the holographic $c$ and $a$ charges, namely $c = \frac{1}{3}\ell\frac{\partial a}{\partial\ell}$, which implies $C_T = c$.

# Introduction

According to the basic idea of the AdS/CFT Correspondence providing a more manageable approach to compute n-point functions is the field-operator duality. The boundary value $\phi_0$ of $\phi$ is identified as the source coupled to the operator, and furthermore the partition function of the boundary $\mathrm{CFT}_d$ is identified with the on-shell action of $\mathrm{AdS}_{d+1}$ gravities

$$S_{\mathrm{gr}}|_{\phi_0} =< \exp(-\int d^d x \phi_0 O_\phi) > . \tag{1}$$

With the identification, the n-point function of the operator $O_\phi$ of the $\mathrm{CFT}_d$ can be computed by evaluating on-shell action of the gravity theories:

$$< O_\phi(x_1) \cdots O_\phi(x_n) >= \frac{\delta^n S}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)}. \tag{2}$$

In pure gravities, the only available field is the metric, and the corresponding operator is the energy-momentum tensor of the boundary CFT. The corresponding two-point functions in Einstein gravity in general dimensions were previously obtained.[hep-th/9804083]

# Introduction

In this paper, we compute covariant holographic two-point functions in massless higher-order gravities. We mainly follow the method of [Arxiv:1205.5804][Arxiv:1411.3158]developed for four dimensions. We generalized to general $D = d + 1$ dimensions and obtain that

$$< T_{ij}(x)T_{kl}(0) >= \frac{N_2 C_T \mathcal{I}_{ijkl}(x)}{x^{2d}}. \tag{3}$$

where $N_2$ is a numerical constant depending only on $d$. The boundary spacetime tensor $\mathcal{I}_{ijkl}(x)$ is defined by

$$\mathcal{I}_{ijkl} = \frac{1}{2}(I_{ik}I_{jl} + I_{il}I_{jk}) - \frac{1}{d}\eta_{ij}\eta_{kl} \tag{4}$$

$$I_{ij} = \eta_{ij} - \frac{2x_i x_j}{x^2}. \tag{5}$$

where $x_i$ are the cartesian coodinates of the boundary Minkowski spacetime $\eta_{ij}$. The structure matches the result of CFT[hep-th/930710][hep-th/96050009][Arxiv:1203.1339], and also matches the result of Einstein gravity and Gauss-Bonnet Gravity [Arxiv:0911.4257]

# Introduction

The coefficient $C_T$ depend on the detail of the theory, however, for massless higher-order gravities, we find that there is a universal expression relating $C_T$ to $a$-charge:

$$\boxed{C_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell}} \tag{6}$$

where $a$ is the coefficient of the Euler density in the holographic conformal anomaly.(See,e,g.[hep-th/9806087][hep-th/9812032][hep-th/9910267])
The parameter $\ell$ is the radius of the AdS vacuum. It is important to emphasize that the a-charge must be expressed in terms of the $\ell$ and the bare coupling constants of the higher-curvature invariants as independent parameters, with the bare cosmological constant $\Lambda_0$ solved in terms of thesed quantities by the E.O.M.

# The covariant structure of two-point functions

We begin with a brief review of how to compute the two-point function based on the holographic dictionary. The metrics of the asymptotic AdS in $D = d + 1$ dimensions take the form

$$ds^2 = \frac{\ell^2}{r^2} dr^2 + r^2 g_{ij} dx^i dx^j . \tag{7}$$

At the asymptotic region, the FG expansion of $g_{ij}$ is

$$g_{ij} = g_{ij}^{(0)} + \frac{g_{ij}^{(d)}}{r^d} + \cdots . \tag{8}$$

The leading $g_{ij}^0$ is interpreted as the source of the boundary CFT in the context of the holographic dictionary, and the two-point function is given as

$$< T_{ij} T_{kl} >= \frac{\delta}{\delta g^{(0)kl}} < T_{ij} > . \tag{9}$$

The holographic dictionary then given us

$$< T_{ij} >= T_{ij}(h) r^{d-2}|_{r\to\infty} \sim g_{ij}^{(d)} . \tag{10}$$

Thus the computation of the two-point function now involes the evaluation of the quantity $\frac{\delta g_{ij}^{(d)}}{\delta g^{(0)kl}}$.

# The covariant structure of two-point functions

In other words, the main task is to determine how to response mode $g_{ij}^{(d)}$ depends on the source $g_{ij}^{(0)}$ around the AdS vacuum. We turn on the perturbation $h_{ij} = r^2\eta_{ij}$, and $\hat{h}_{ij} = r^2 f_{ij}$.For simplicity, we restrict ourselves to the transverse-traceless gauge:

$$\nabla^j \hat{h}_{ij} = 0, \quad \hat{h} = 0. \tag{11}$$

The linearized equation is then given by

$$\kappa_{\rm eff}(\tilde{\Box} + \frac{2}{\ell^2})\hat{h}_{ij} = 0. \tag{12}$$

where $\kappa_{\rm eff} = \frac{1}{16\pi} + \cdots$ denote the effective Newton constant.The above equation can be solved by separation of variables

$$\hat{h}_{ij} = e^{-ip\dot{x}} f_{ij}(r). \tag{13}$$

The solutions are

$$f_{ij} = e^{-iEt} r^{-\frac{d}{2}}\Big(c_{ij}^a J_{\frac{d}{2}}(\frac{E\ell}{r}) + c_{ij}^b Y_{\frac{d}{2}}(\frac{E\ell}{r}))\Big). \tag{14}$$

where $J$ and $Y$ are the first and second Bessel functions respectively and $(c_{ij}^a, c_{ij}^b)$ are integration constants. Thus we see that the perturbation functions $f_{ij}$ can be expanded as

$$f_{ij} = f_{ij}^{(0)} + \cdots + \frac{f_{ij}^{(d)}}{r^d} + \cdots. \tag{15}$$

# The covariant structure of two-point functions

After PBH transformation,we can solve $f_{ij}$ in Momentum space:

$$f_{ij}^{(d)} = \frac{1}{2}N(d,p)\sum_{I=d}^{\frac{d^2+d-4}{2}} \bar{E}^I_{kl}\bar{E}^I_{ij}f^{(0)kl}. \tag{16}$$

It is instructive to introduce the boundary spacetime tensors

$$\Theta_{ij}(p) = \eta_{ij}p^2 - p_i p_j; \tag{17}$$
$$\Delta^d_{ijkl} = \frac{1}{2}(\Theta_{ik}\Theta_{jl} + \Theta_{il}\Theta_{jk}) - \frac{1}{d-2}\Theta_{ij}\Theta_{kl}.$$

We can verify that the metric basis is satisfied that

$$\sum_{I=d}^{\frac{d^2+d-4}{2}} \bar{E}^I_{kl}\bar{E}^J_{ij} = \frac{2}{p^4}\Delta^d_{ijkl}(p). \tag{18}$$

After transforming the two-point funtion in momentum space into configuration space. we can finally obtain

$$< T_{ij}T_{kl} >= N_2 C_T \mathcal{I}_{ijkl}x^{-2d}. \tag{19}$$

# The covariant structure of two-point functions

where

$$N_2 = \frac{\Gamma(d+2)}{16(-\pi)^{(}d+2+1)(d-1)\Gamma(\frac{d}{2})}. \tag{20}$$

and the constant $C_T$ depends on the details of a specific theory. The simplest example is Einstein gravity, we have

$$C_T = \ell^{d-1}. \tag{21}$$

For general massless higher-order gravities with linear equation, we verified that

$$\boxed{C_T = 16\pi\kappa_{\text{eff}}\ell^{d-1}} \tag{22}$$

At first sight, this theory-dependent expression should not be called universal enough, However we found a more universal expression:

$$C_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell} \tag{23}$$

# Example: Einstein-Guass-Bonnet gravity

First we consider Einstein gravity extended with the Guass-Bonnet term in $D = d + 1 > 4$ dimensions. With bare cosmological constant $\Lambda_0 = \frac{d(d-1)}{2\ell_0^2}$ and the bare coupling constant $\alpha$. For the flat AdS boundary condition, we obtain the total action:

$$
\begin{aligned}
S_{\text{tot}} &= S_{\text{bulk}} + S_{\text{GH}} + S_{\text{ct}} \\
S_{\text{bulk}} &= \frac{1}{16\pi}\int_M d^{d+1}x\sqrt{-g}\Big(R + \frac{d(d-1)}{\ell_0^2} + \alpha(R^2 - R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\Big) \\
S_{\text{GH}} &= \frac{1}{8\pi}\int_{\partial M} d^d x\sqrt{-h}\Big(K - \frac{2\alpha}{3}(K^3 - 3KK^{(2)} + 2K^{(3)})\Big) \\
S_{\text{ct}} &= \frac{1}{8\pi}\int_{\partial M} d^d x\sqrt{-h}(d-1)(\ell^2 - \frac{2}{3}(d-3)(d-2)\alpha)\ell^{-3}.
\end{aligned}
\tag{24}
$$

where $K^{(2)} = K^i_j K^j_i$ and $K^{(3)} = K^i_j K^k_j K^k_i$. For the theory admitting the AdS vacuum, $(\ell_0, \alpha, \ell)$ are related via the E.O.M as

$$\frac{1}{\ell_0^2} - \frac{1}{\ell^2} + \frac{(d-2)(d-3)\alpha}{\ell^4} = 0. \tag{25}$$

If the massless gravity case is under consideration, we can read the effective Newtonian constant as

$$\kappa_{\text{eff}} = \frac{1}{16\pi}(1 - \frac{2\alpha(d-2)(d-3)}{\ell^2}). \tag{26}$$

# Example: Einstein-Guass-Bonnet gravity

In this paper we still treat $\ell$ and $\alpha$ as independent parameters, while solving $\ell_0$ associated with the bare cosmological constant $\Lambda_0$ in terms of these independent parameters $(\ell, \alpha)$ according to the E.O.M.
The Brown-York energy-momentum tensor is given as

$$
\begin{aligned}
T_{ij} &= \frac{1}{8\pi}(K_{ij} - Kh_{ij} + 2\alpha(3J_{ij} - Jh_{ij}) + \frac{(d-1)(3\ell^2 - 2(d-3)(d-2)\alpha)}{3\ell^2}h_{ij}), \\
J_{ij} &= \frac{1}{3}(2KK_{ik}K_j^k + K^{(2)}K_{ij} - 2K_{ik}K^{kl}K_{lj} - K^2K_{ij}.)
\end{aligned}
\tag{27}
$$

In the FG expansion frame, we can obtain the asymptotic formation of the extrinsic curvature like

$$
K_{ij} = \frac{r}{2\ell}\partial_r h_{ij} = \frac{r^2}{\ell^2}g^{(0)} - \frac{d-2}{2\ell^2}\frac{g_{ij}^{(d)}}{r^{d-2}} + \cdots \tag{28}
$$

for the flat boundary and note that

$$
\mathrm{Tr}g^{(d)} = 0, \tag{29}
$$

we have

$$
\begin{aligned}
K_{ik}K_j^k &= \frac{r^2}{\ell^2}g^{(0)ij} - \frac{d-1}{\ell^2}\frac{g_{ij}^{(d)}}{r^{d-2}} + \cdots . \\
K_{kl}K^{kl}K_{ij} &= \frac{d}{\ell^3}r^2 g^{(0)ij} - \frac{d(d-2)}{\ell^3}\frac{g_{ij}^{(d)}}{r^{d-2}} + \cdots ,
\end{aligned}
\tag{30}
$$

# Example: Einstein-Guass-Bonnet gravity

Finally, we arrive the one point function

$$< T_{ij} >= -\frac{d}{\ell}\kappa_{\text{eff}}\, g_{ij}^{(d)}. \tag{31}$$

As the our discussion above, we can calculate the two point function of the Einstein-Guass-Bonnet gravity and read the $C_T$ as

$$\boxed{C_T = 16\pi\kappa_{\text{eff}}\ell^{d-1} = \ell^{d-1}(1 - \frac{2\alpha(d-3)(d-2)}{\ell^2}).} \tag{32}$$

On the other hand, the a-charge of Einstein-Gauss-Bonnet gravity in arbitrary dimensions was previously given as [Arxiv:1011.5819]:

$$\boxed{a = \ell^{d-1}\Big(1 - \frac{2\alpha(d-1)(d-2)}{\ell^2}\Big).} \tag{33}$$

We can check that the identity

$$\boxed{C_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell}} \tag{34}$$

is valid.

# Example: Einstein-Guass-Bonnet gravity

In particular, when $D = 5, d = 4$, $C_T = c$ has been shown (Arxiv:0911.4527), This implies that a-charge and c-charge in four-dimension CFT can be simply related by

$$c = \frac{1}{3}\ell\frac{\partial a}{\partial \ell}. \tag{35}$$

# Example: Einstein-Lovelock Gravities

We consider the Einstein-Lovelock Gravities as sample. The bulk action is given by

$$S_{bulk} = \frac{1}{16\pi} \int_M d^{d+1}x \sqrt{-g} \sum_k a_k E_k. \tag{36}$$

where

$$E_k = \frac{(2k)!}{2^k} \delta^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}}_{\nu_1\nu_2\cdots\nu_{2k-1}\nu_{2k}} R^{\nu_1\nu_2}_{\mu_1\mu_2} \cdots R^{\nu_{2k-1}\nu_{2k}}_{\mu_{2k-1}\mu_{2k}}. \tag{37}$$

We further set the bar cosmological constant and Newtonian constant by

$$a_0 = \frac{d(d-1)}{\ell_0^2}, \qquad a_1 = 1. \tag{38}$$

The equation of motion can be read as

$$\sum_{k>0} \frac{d!}{(d-2k)!} \Big(-\frac{1}{\ell^2}\Big)^k a_k = 0. \tag{39}$$

We find that the effective Newton constant of Lovelock gravity is given by

$$\kappa_{\text{eff}} = \frac{1}{16\pi} \sum_{k\geq 1} \frac{(-1)^{k+1} k(d-2) a_k}{\ell^{2(k-1)}(d-2k)!}. \tag{40}$$

The Gibbons-Hawking surface term and the counterterms are

$$S_{\text{GH}} = \frac{1}{16\pi} \int_{\partial M} d^d x \sqrt{-h} \sum_{k\geq 1} \frac{(-1)^{k-1}(2k)! a_k}{2k-1}$$

# Example: Einstein-Lovelock Gravities

According to our discussion of the asymptotic formation of the extrinsic curvature above, we have

$$\delta T_{ij} = \sum_k \frac{(-1)^{k+2} k(d-2) d a_k}{16\pi \ell^{2k-1} (d-2k)!} \delta g_{ij}^{(d)}. \tag{42}$$

Thus we see from the two point functions for Einstein-Lovelock gravities are given as

$$\boxed{C_T = 16\pi \kappa_{\text{eff}} \ell^{d-1} = \ell^{d-1} \sum_k \frac{(-1)^{k+1} k(d-2)!}{\ell^{2(k-1)} (d-2k)!} a_k.} \tag{43}$$

Next we focus on calculating the a-charge of Einstein-Lovelock gravity. First we employ the trick of the reduce FG expansion as (Arxiv:1803.08088). Considering a special class of the FG coordinates

$$ds_D^2 = \frac{\ell^2}{4r^2} dr^2 + \frac{f(r)}{r} dx^i dx_i. \tag{44}$$

The bulk action is then given as

$$S_{\text{bulk}} = \frac{1}{16\pi} \int_M d^{d+1} \sum k a_k \left( \frac{d!}{(d-2k)!} F_2^k + \frac{2kd!}{(d-2k+1)!} F_2^{k-1} F_1 \right) \tag{45}$$

where

$$F_1 = -\frac{2r^2 ff'' - r^2 f'^2 + f^2}{\ell^2 f^2}, \quad F_2 = -\frac{-r(\ell^2 + 2f') + r^2 f'^2 + f^2}{\ell^2 f^2}. \tag{46}$$

Next we expand $f$ as

$$f = f_0 + f_2 r + f_4 r^2 + f_6 f^3 + \cdots \tag{47}$$

and have

$$f_2 = -\frac{\ell^2}{2}, \quad f_4 = \frac{\ell^3}{16 f_0}, \quad f_{2i} = 0, \quad , i \geq 3. \tag{48}$$

We can then read off the a-charge:

$$\boxed{a = \ell^{d-1} \sum_{k \geq 1} \frac{(-1)^{k+1} k (d-2)!}{\ell^{2k} (d - 2k + 1)!} a_k.} \tag{49}$$

It is easy to verified that

$$\boxed{C_T = \frac{1}{d-1} \ell \frac{\partial a}{\partial \ell}.} \tag{50}$$

# Example: Einstein-Riemann cubic gravities

The crucial property in our derivation is that the linearized spectrum of the AdS vacuum involes only the massless graviton. In fact these theories are two derivatives in any background. In this section, we may relax this condition and consider Riemann cubic extended gravities.
The bulk action of Einstein gravity extended with generic Riemann cubic invariants is

$$S = \frac{1}{16\pi}\int_M d^{d+1}x\sqrt{-g}L, \qquad L = R + \frac{d(d-1)}{\ell_0^2} + H^{(3)}. \tag{51}$$

where $H^{(3)}$ is given by

$$\begin{aligned} H^{(3)} &= e_1 R^3 + e_2 R R_{\mu\nu} R^{\mu\nu} + e_3 R^{\mu}_{\nu} R^{\nu}_{\rho} R^{\rho}_{\mu} + e_4 R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} \\ &+ e_5 R R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + e_6 R^{\mu\nu} R_{\mu\alpha\beta} R_{\nu}{}^{\alpha\beta\gamma} + e_7 R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} R^{\alpha\beta}{}_{\mu\nu} \\ &+ e_8 R^{\mu\nu\alpha\beta} R_{\nu\rho\beta\gamma} R^{\rho}{}_{\mu}{}^{\gamma}{}_{\alpha} \end{aligned} \tag{52}$$

The bare and effective cosmological constants are related to

$$\begin{aligned} &\frac{1}{\ell_0^2} - \frac{1}{\ell^2} - \frac{(d-5)}{(d-1)\ell^6}\Big(d^2(d+1)^2 e_1 + d^2(d+1)e_2 + \\ &d^2 e_3 + d^2 e_4 + 2d(d+1)e_5 + 2de_6 + 4e_7 + (d-1)e_8\Big) = 0 \end{aligned} \tag{53}$$

# Example: Einstein-Riemann cubic gravities

It is essential to introduce the generalized form os the Gibbons-Hawking surface term to make a well-pose variation principle and it is given by [Arxiv:0908.0679]

$$S_{GH} = \frac{1}{4\pi}\int_{\partial M} d^d x\,\sqrt{-h}\Phi^{\mu}_{\nu}K^{\nu}_{\mu}, \quad \Phi^{\mu}_{\nu} = P^{\mu}{}_{\rho\nu\sigma}n^{\rho}n^{\sigma}, \quad P^{\mu\nu\rho\sigma} = \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \tag{54}$$

where

$$n^{\mu} = \frac{r}{\ell}\Big(\frac{\partial}{\partial r}\Big)^{\mu}. \tag{55}$$

It is important to note that in our case $\Phi^{\mu}_{\nu}$ is auxiliary field and does not involve in the variation. We shall need to include the appropriate counter term and make the on-shell action finite. It is given by some algebraic calculation:

$$\begin{aligned} S_{ct} &= -\frac{1}{8\pi}\int d^d x\,\sqrt{-h}\frac{3(d-1)}{\ell^5}\Big(\frac{1}{3}\ell^4 + d^2(d+1)^2 e_1 + d^2(d+1)e_2 + d^2 e_3 + d^2 e_4 \\ &\quad 2d(d+1)e_5 + 2de_6 + 4e_7 + (d-1)e_8\Big). \end{aligned} \tag{56}$$

The decoupling of both the massive scalar and spin-2 modes requires

$$\begin{aligned} &(d+1)de_2 + 3de_3 + (2d-1)e_4 + 4(d+1)de_5 + 4(d+1)e_6 + 24e_7 - 3e_8 = 0 \\ &12(d+1)d^2 e_1 + (d^2+10d+1)de_2 + 3(d+1)de_3 + (2d^2+5d-1)e_4 \\ &+4(d+5)de_5 + 4(2d+1)e_6 + 3(d-1)e_8 + 24e_7 = 0. \end{aligned} \tag{57}$$

# Example: Einstein-Riemann cubic gravities

The remaining six-parameter theory we call massless cubic gravity and the linearized equation of motion around AdS vacuum is given with

$$\kappa_{\text{eff}} = \frac{1}{16\pi}(1 + \frac{1}{\ell^4}(d-5)(d-2)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5)). \tag{58}$$

Next we substitute the flat FG expansion into the total action and then perform the variation associated with $g^{(0)ij}$, we can arrive the one-point function

$$< T_{ij} >= -\frac{d}{\ell}\kappa_{\text{eff}} g_{ij}^{(d)}. \tag{59}$$

Having obtained the one-point function, it follows from our earlier discussions that the two-point functions with the coefficient $C_T$:

$$\boxed{C_T = \ell^{d-1}\Big(1 + \frac{1}{\ell^4}(d-5)(d-2)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5)\Big)} \tag{60}$$

On the other hand, the holographic a-charge for the massless cubic gravity in arbitary dimensions is [Arxiv:1711.03650]

$$\boxed{a = \ell^{d-1}\Big(1 + \frac{1}{\ell^4}(d-2)(d-1)(3(d+1)de_2 + 2de_2 + e_4 + 4e_5)\Big).} \tag{61}$$

# Example: Einstein-Riemann cubic gravities

Thus we see that the relation

$$\boxed{C_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell}} \tag{62}$$

is valid.

It is also important to note that when $D = 5(d = 4)$, the c-charge for the massless cubic gravity is [Arxiv:1711.03650]

$$c = \ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^-1 \tag{63}$$

Thus in d=4, we have $C_T = c$ and the relation

$$\boxed{c = \frac{1}{3}\ell\frac{\partial a}{\partial \ell}} \tag{64}$$

is again established, we have sufficient evidence to conjecture that the relation above between c-charge and a-charge is a general property of CFT in four dimensions.

In this paper, we consider the Einstein gravity extended with general classes of high-order curvature invariants. We derived the covariant holographic two-point functions of these pure gravity theory in AdS vacuum. We presented the results in both momentum and configuration spaces. We found that the c-charge $C_T = 16\pi\kappa_{\rm eff}$ was related to the holographic $a$-charge by a universal expression

$$C_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell},. \tag{65}$$