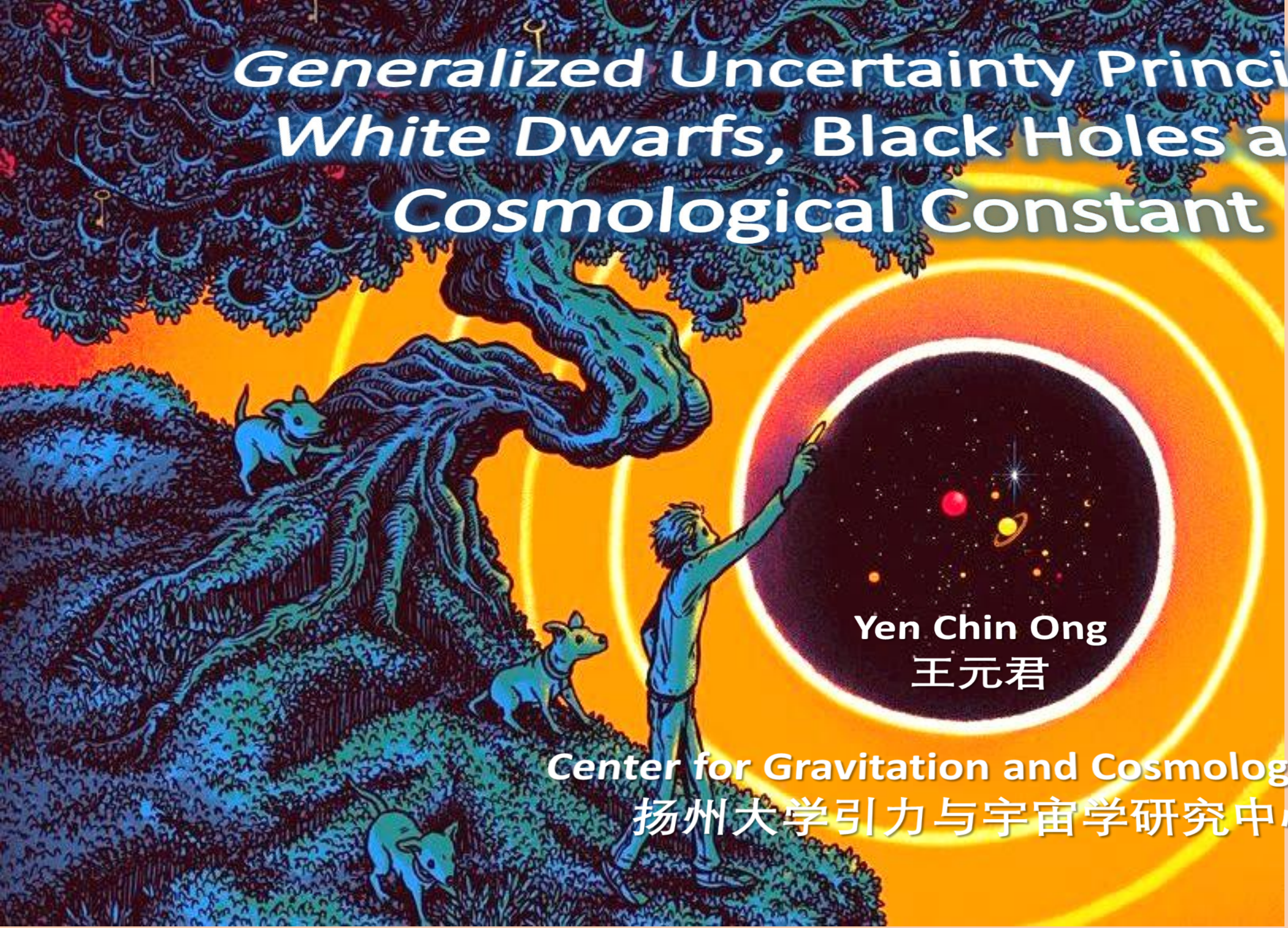
Generalized Uncertainty Principle: White Dwarfs, Black Holes and Cosmological Constant



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Generalized Uncertainty Principle

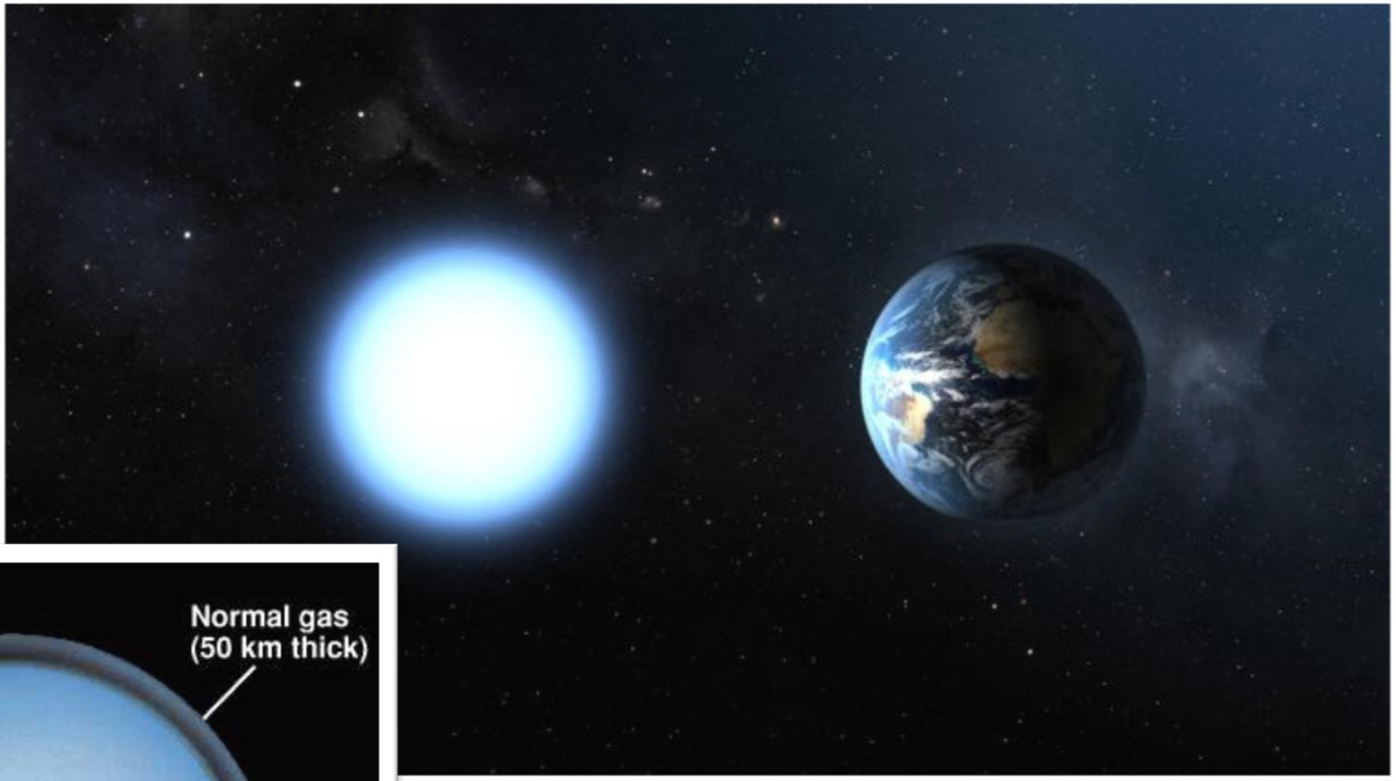
$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \frac{\alpha L_p^2 \Delta p^2}{\hbar}\right]$$

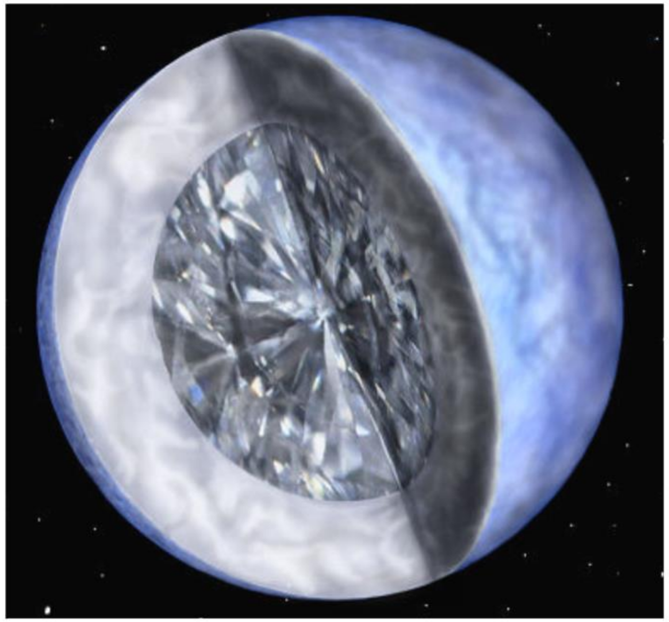
$$L_p^2 = \hbar G / c^3$$

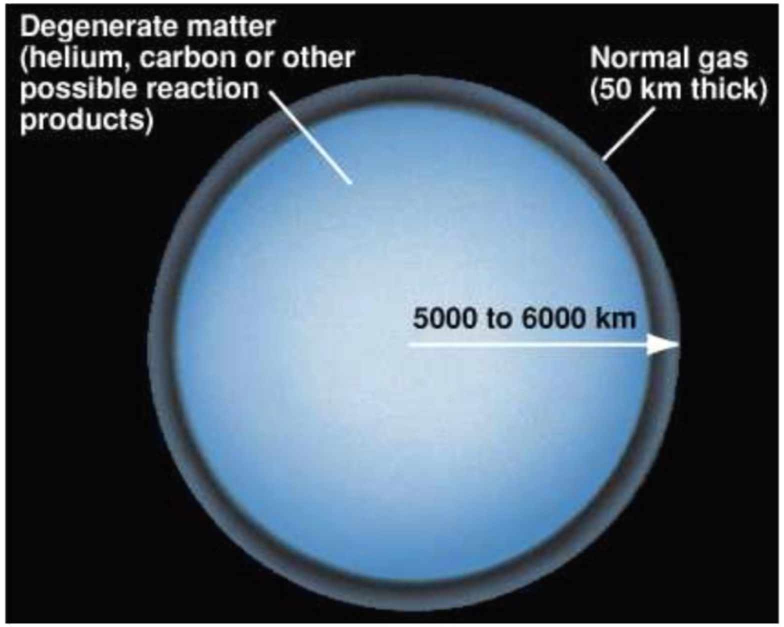
Let us start with white dwarfs

End stage of solar-mass stars

Supported by electron degenerative pressure







Textbook material:

Total kinetic energy of a non-relativistic white dwarf

Number of electron → N

$$E_k = \frac{N\Delta p^2}{2m_e} \sim \frac{N\hbar^2}{(\Delta x)^2 2m_e}$$

$$\Delta x \Delta p \sim \hbar$$

Number density,

$$n = \frac{N}{V} = \frac{M}{V m_e}$$

$$\Delta x \sim n^{-\frac{1}{3}} = \left(\frac{V}{N}\right)^{1/3}$$

$$E_k = \frac{N\hbar^2 n^{\frac{2}{3}}}{2m_e} = \frac{N\hbar^2}{2m_e}\left(M^{\frac{2}{3}} m_e^{-\frac{2}{3}} R^{-2}\right) = \frac{M^{\frac{5}{3}}\hbar^2}{2m_e^{\frac{8}{3}} R^2}.$$

Heuristic Derivation of White Dwarf Properties

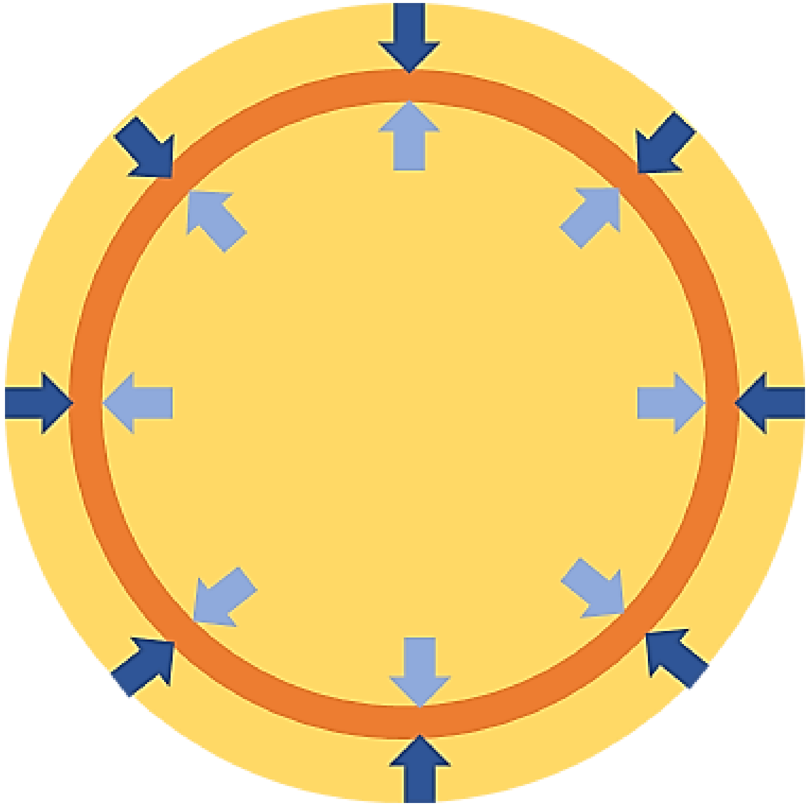
To withstand gravitational collapse, we must balance the kinetic energy with gravitational binding energy

$$|E_g| \sim \frac{GM^2}{R} = E_k$$

$$E_k = \frac{N\hbar^2 n^{\frac{2}{3}}}{2m_e} = \frac{N\hbar^2}{2m_e}\left(M^{\frac{2}{3}} m_e^{-\frac{2}{3}} R^{-2}\right) = \frac{M^{\frac{5}{3}}\hbar^2}{2m_e^{\frac{8}{3}} R^2}.$$

$$R \sim \frac{\hbar^2}{2m_e^{\frac{8}{3}} G M^{\frac{1}{3}}}.$$

Degenerative matter



(Ultra)Relativistic Case

$$E_k = N(\gamma - 1)m_e c^2, \quad p = \gamma m_e v.$$

$$v \sim c$$

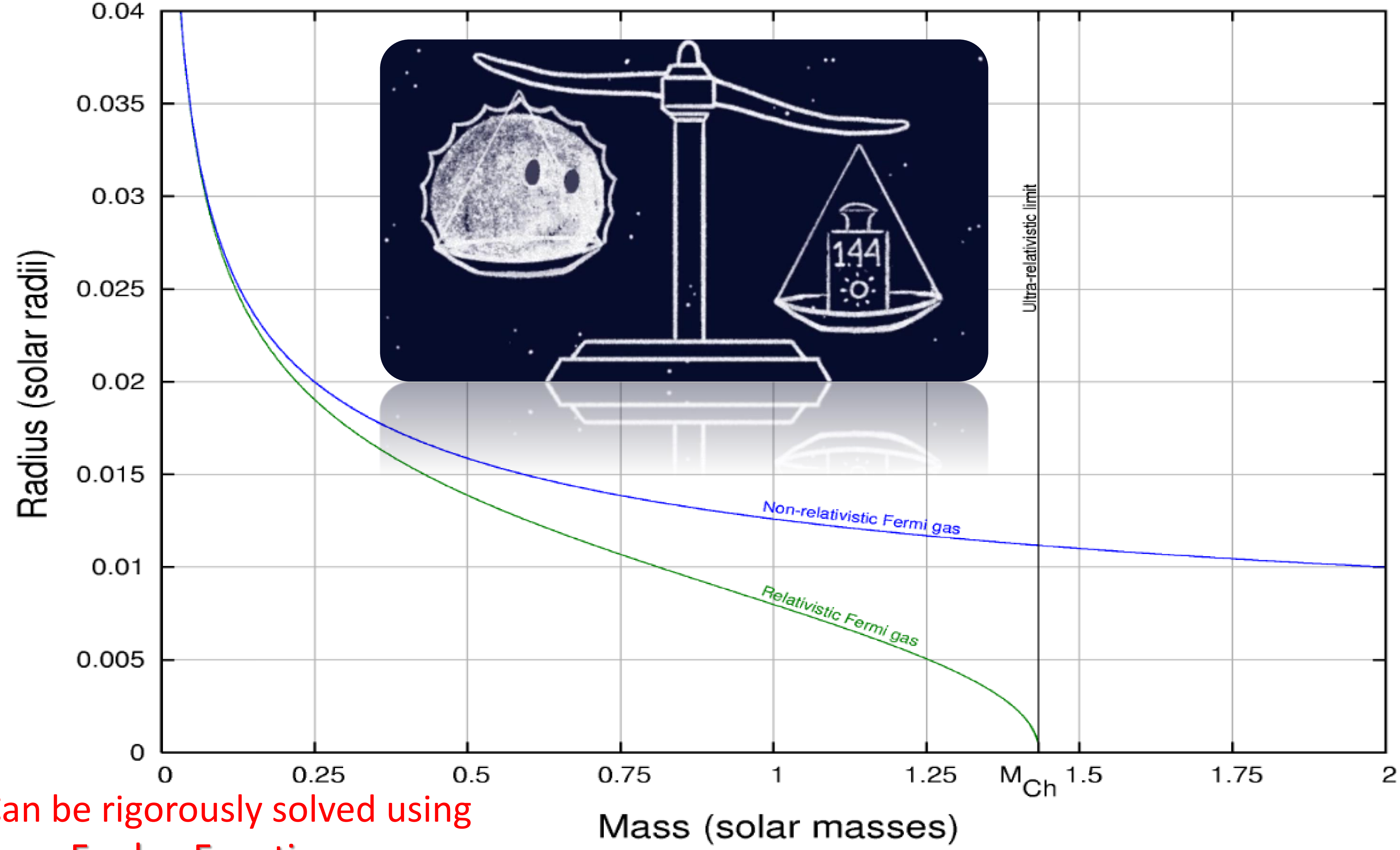
$$\Delta p c - m_e c^2 \sim \frac{\hbar c}{\Delta x} - m_e c^2 \sim \left(\frac{M}{m_e V}\right)^{\frac{1}{3}} \hbar c - m_e c^2.$$

$$\frac{GM^2}{R} \sim N\left[\left(\frac{M}{m_e V}\right)^{\frac{1}{3}} \hbar c\right] \sim \frac{M^{\frac{1}{3}}\hbar c}{m_e^{\frac{4}{3}} R}.$$

$$M_{\rm Ch} \sim \frac{1}{m_e^2}\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}.$$

Chandrasekhar limit
/Chandrasekhar mass
(1930)
1983 Nobel Prize in Physics

Subrahmanyan
Chandrasekhar
(1910-1995)



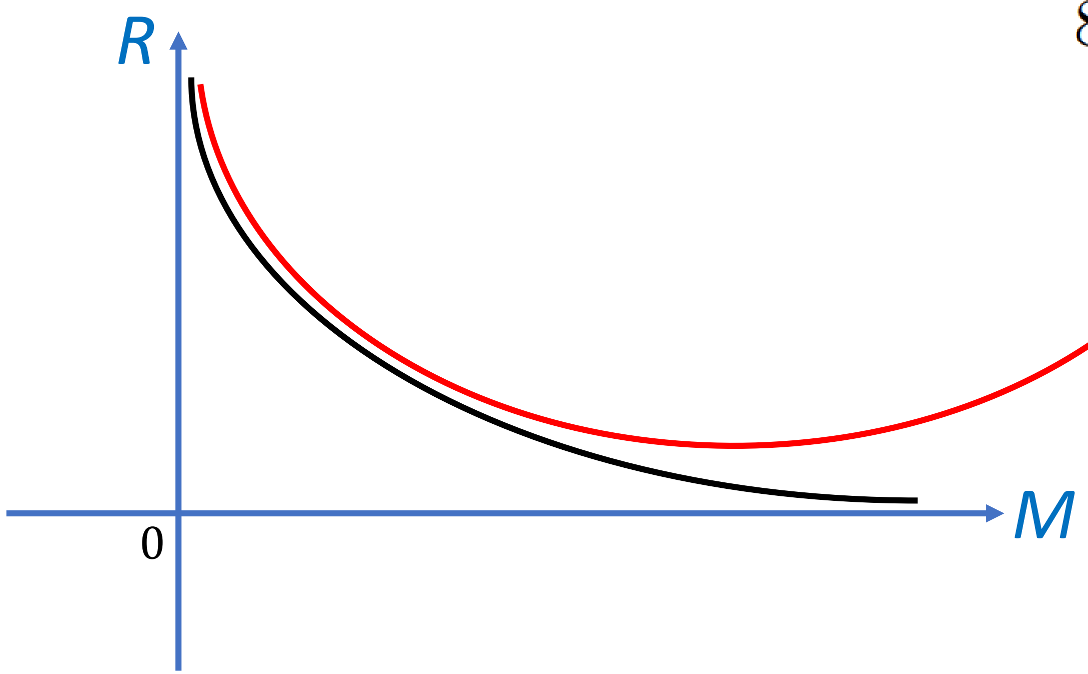
Can be rigorously solved using
Lane-Emden Equation

White Dwarfs with GUP

$$\Delta x \Delta p \sim \hbar + \alpha L_p^2 \Delta p^2/\hbar \quad \Rightarrow \quad \Delta p \sim \frac{\hbar \Delta x}{2\alpha L_p^2}\left[1 \pm \sqrt{1 - \frac{4\alpha L_p^2}{\Delta x^2}}\right]$$

Non-relativistic case:

$$M^{\frac{5}{3}} \sim \frac{\hbar^2}{8Gm_e^{\frac{4}{3}}\alpha^2 L_p^4} R^3 \left(1 \pm \sqrt{1 - \frac{4\alpha L_p^2 M^{\frac{2}{3}}}{m_e^{\frac{2}{3}} R^2}}\right)^2$$



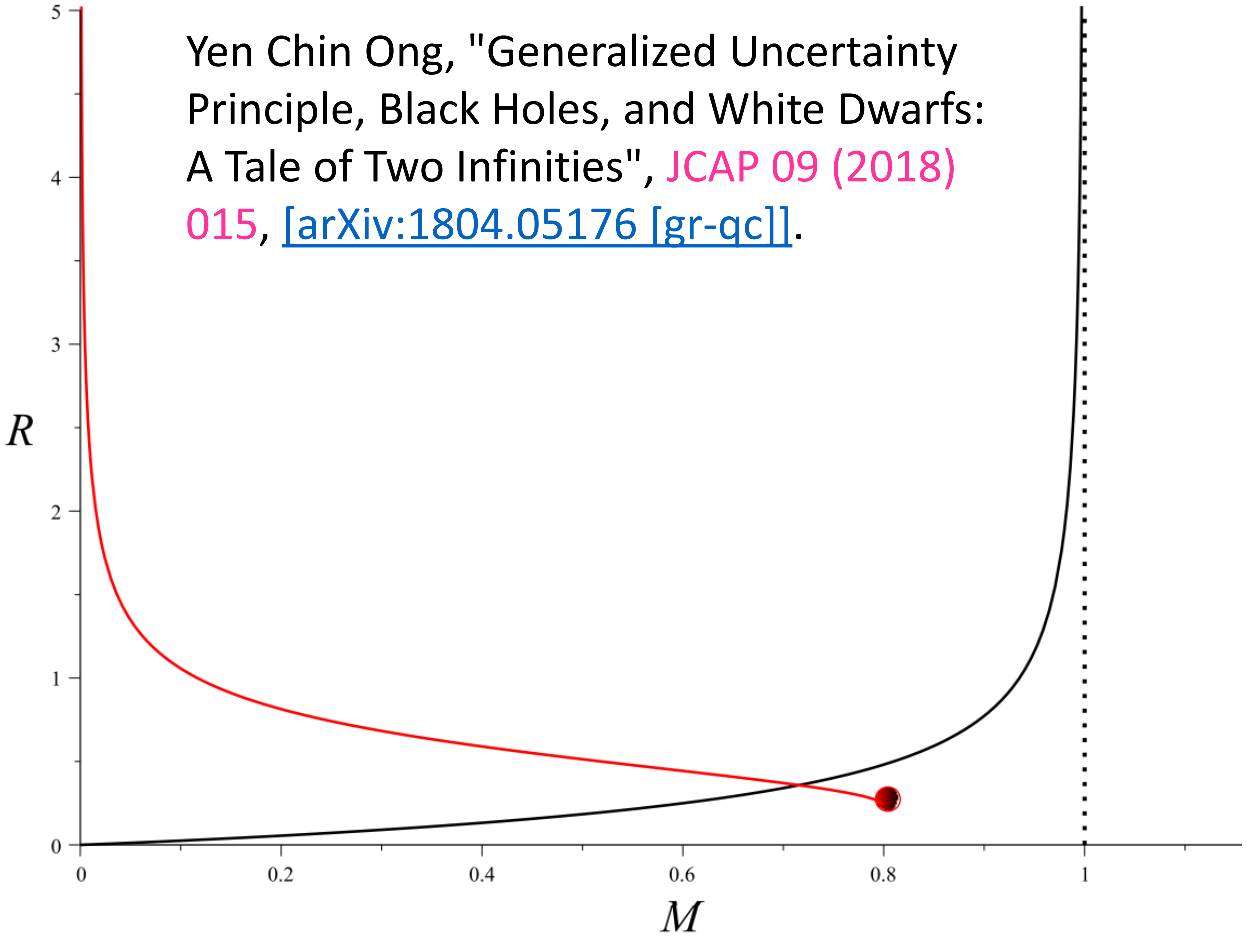
Mohamed Moussa, "Effect of Generalized Uncertainty Principle on Main-Sequence Stars and White Dwarfs", Adv. High Energy Phys. **2015** (2015) 343284, arXiv:1512.04337 [physics.gen-ph].

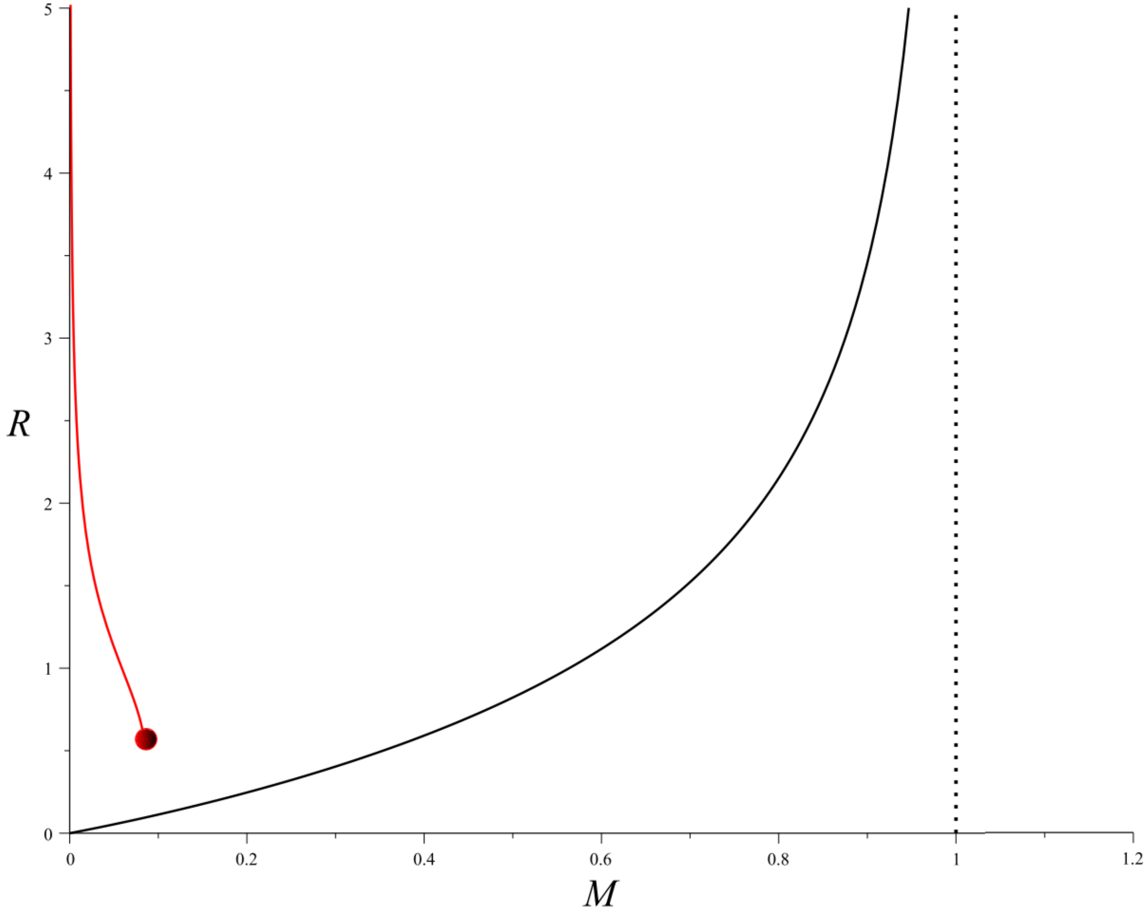
Reza Rashidi, "Generalized Uncertainty Principle and the Maximum Mass of Ideal White Dwarfs", Annals Phys. 374 (2016) 434, [arXiv:1512.06356 [gr-qc]].

A simple fix! Simply Choose $\alpha < 0$

$$\Delta x \Delta p \sim \hbar + \alpha L_p^2 \Delta p^2 / \hbar$$

Yen Chin Ong, "Generalized Uncertainty Principle, Black Holes, and White Dwarfs: A Tale of Two Infinities", JCAP 09 (2018) 015, [arXiv:1804.05176 [gr-qc]].





Negative GUP parameter

Previously suggested in

Petr Jizba, Hagen Kleinert, Fabio Scardigli, "Uncertainty Relation on World Crystal and its Applications to Micro Black Holes", Phys. Rev. D **81** (2010) 084030, [arXiv:0912.2253 [hep-th]].

Also, if one takes the generalized Hawking temperature,

$$T = \frac{Mc^2}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha\hbar c}{GM^2}}\right)$$

and make the reasonable assumption that one should be able to obtain it from Wick-rotating a deformed static Schwarzschild metric with metric coefficient

$$g_{tt} = -\left(1 - \frac{2M}{r} + \varepsilon\frac{M^2}{r^2}\right)$$

then for $|\varepsilon| \ll 1$, we have

$$\alpha = -4\pi^2\varepsilon^2\left(\frac{M}{2M_p}\right)^2 < 0.$$

Fabio Scardigli, Roberto Casadio, "Gravitational tests of the Generalized Uncertainty Principle", Eur. Phys. J. C **75** (2015) 425, arXiv:1407.0113 [hep-th].

How to Understand Negative Alpha Correction?

$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \frac{\alpha L_p^2 \Delta p^2}{\hbar}\right]$$

At large enough energy, RHS becomes smaller: vanishes at Planck scale!

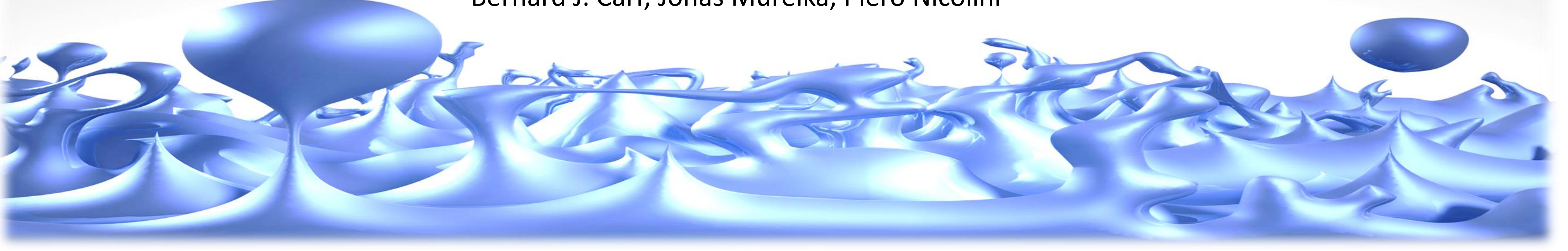
$$\frac{\alpha L_p^2 \Delta p^2}{\hbar} = \frac{\alpha\left(\frac{\hbar G}{c^3}\right)\Delta p^2}{\hbar}$$

$$\sim \frac{\alpha G}{c^3}\left(\frac{E_p}{c}\right)^2 =$$

$$\frac{\alpha G}{c^3}\left(\frac{1}{c^2}\right)\frac{\hbar c^5}{G} = \alpha\hbar$$

Planck Scale Physics Becomes Classical!

Petr Jizba, Hagen Kleinert, Fabio Scardigli;
Bernard J. Carr, Jonas Mureika, Piero Nicolini



Previously in Literature:

- $\hbar$ as a dynamical field that goes to zero in the Planckian limit (Hossenfelder)
- Asymptotic Safe Gravity: If Planck mass is fixed, equivalent to zero G limit since $G = \hbar c / M_p^2$.
- Singularity of dilaton charged black hole (naïve but suggestive):

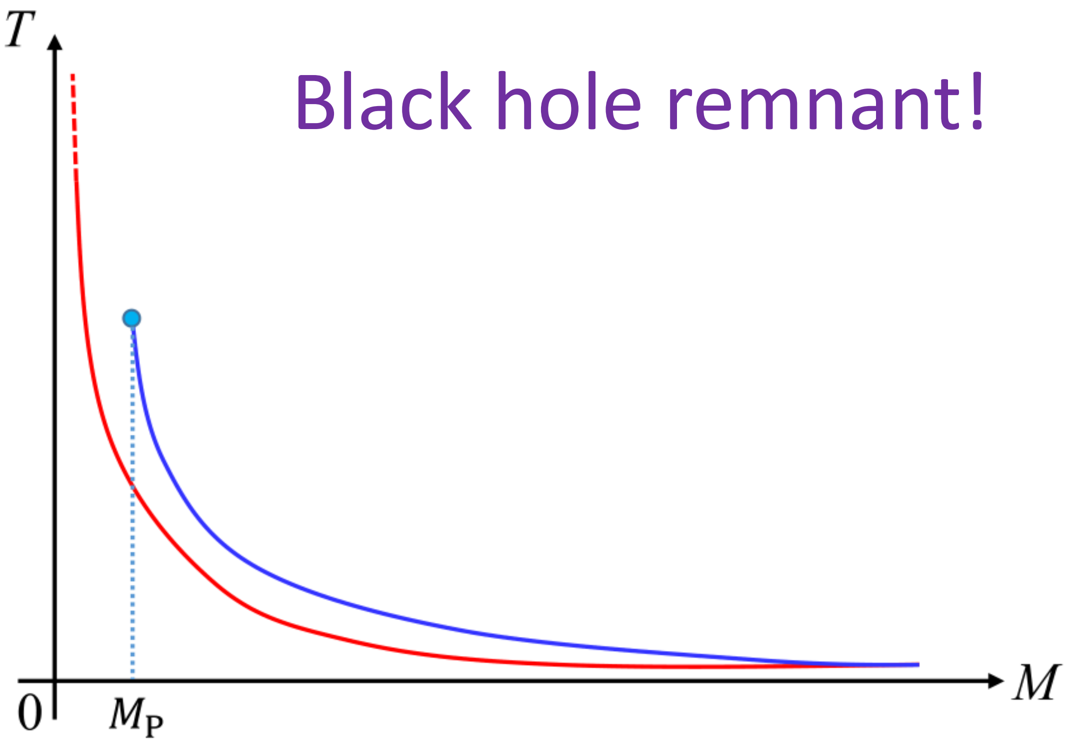
"[...] the string coupling is becoming very weak near the singularity. As we have discussed, we have no right to trust this solution near the singularity, but its difficult to resist speculating about what it might mean if the exact classical solution had a similar behavior. *It would suggest that, contrary to the usual picture of large quantum fluctuations and spacetime foam near the singularity, quantum effects might actually be suppressed.* The singularity would behave classically."

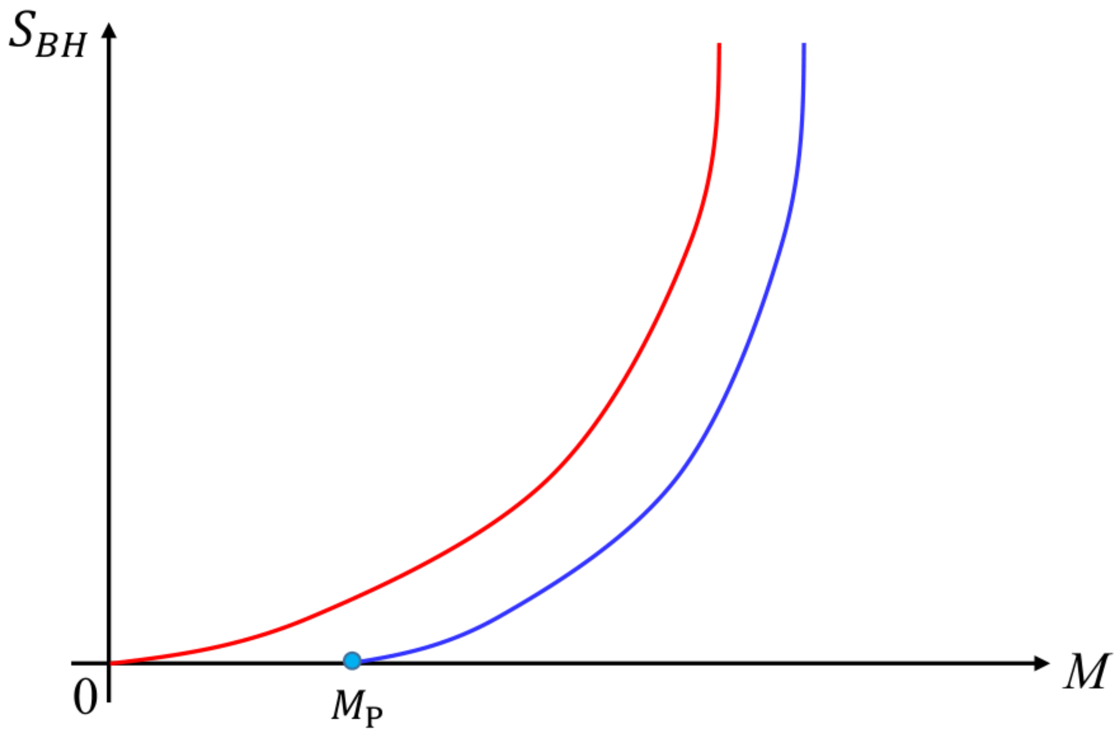
Gary T. Horowitz, "The Dark Side of String Theory: Black Holes and Black Strings", [arXiv:hep-th/9210119].

Hawking Evaporation with GUP

$$T = \frac{Mc^2}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha\hbar c}{GM^2}}\right)$$

Ronald J. Adler, Pisin Chen, David I. Santiago, "The Generalized Uncertainty Principle and Black Hole Remnants", Gen. Rel. Grav. 33 (2001) 2101, arXiv:gr-qc/0106080





Does Negative α Affect Black Hole Physics?

Yes, but OK!

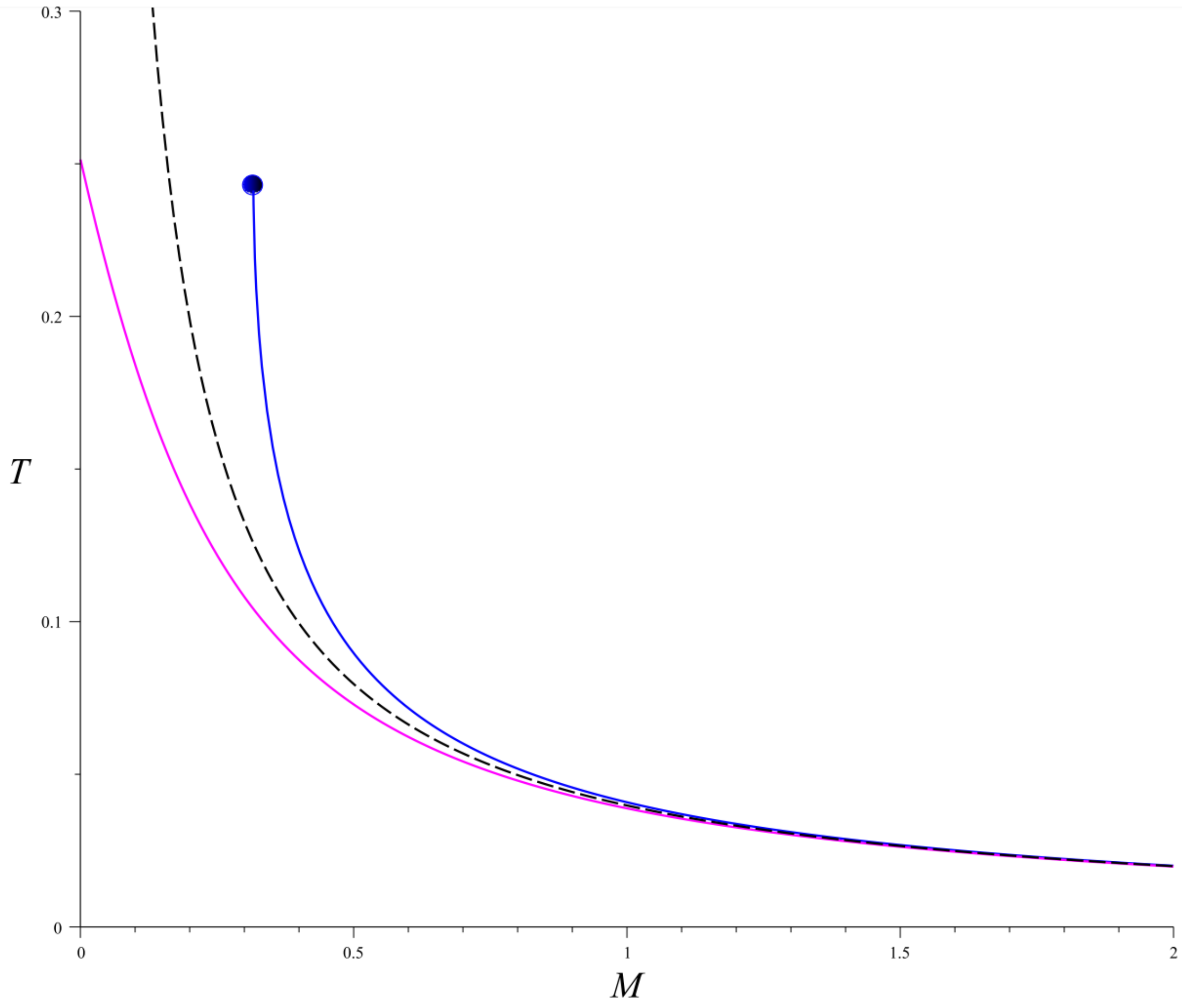
$$T[\alpha > 0] = \frac{M}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha}{M^2}}\right)$$

$$T[\alpha = 0] = \frac{1}{8\pi M}$$

$$T[\alpha < 0] = -\frac{M}{4|\alpha|\pi}\left(1 - \sqrt{1 + \frac{|\alpha|}{M^2}}\right)$$

Small mass limit:

$$T[\alpha < 0] \sim \frac{M}{4|\alpha|\pi}\frac{\sqrt{|\alpha|}}{M} = \frac{1}{4\pi\sqrt{|\alpha|}} < \infty$$



Does this Affect Black Hole Physics?

Yes, but OK!

$$T[\alpha > 0] = \frac{M}{4\alpha\pi}\left(1 - \sqrt{1 - \frac{\alpha}{M^2}}\right)$$

$$T[\alpha = 0] = \frac{1}{8\pi M}$$

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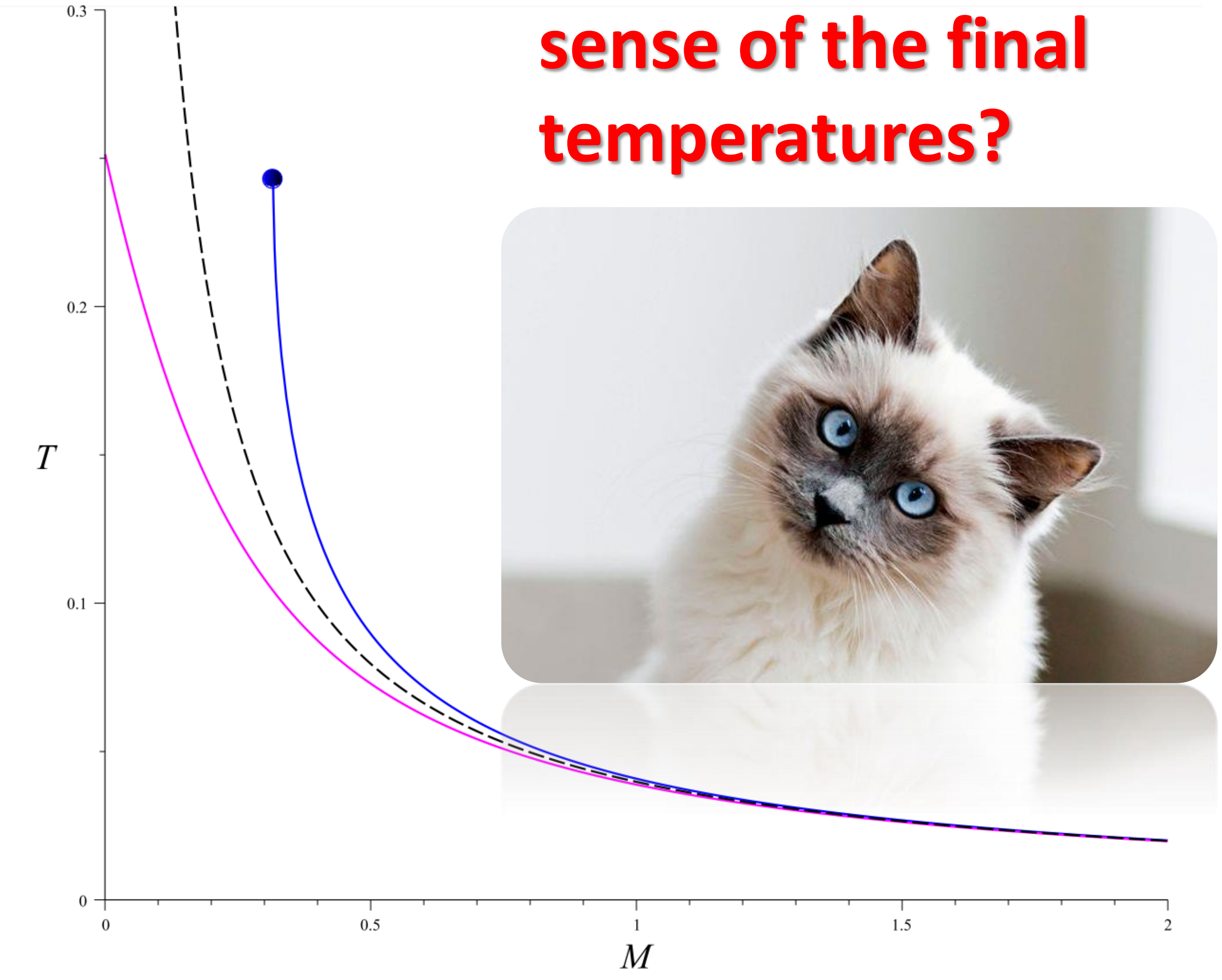
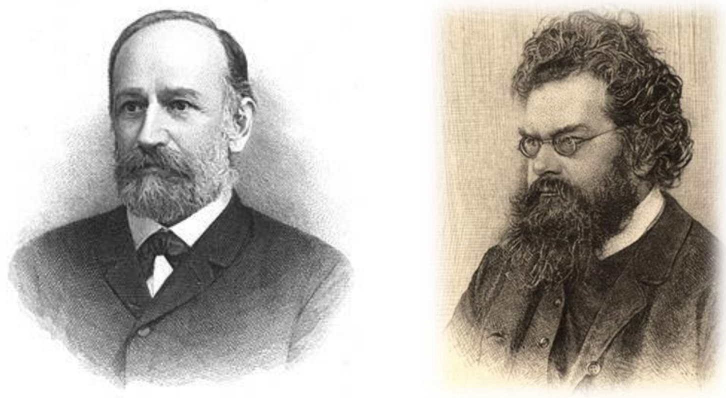


But how to make sense of the final temperatures?

Lifetime of a Black Hole

Hawking temperature:

Stefan-Boltzmann Equation:

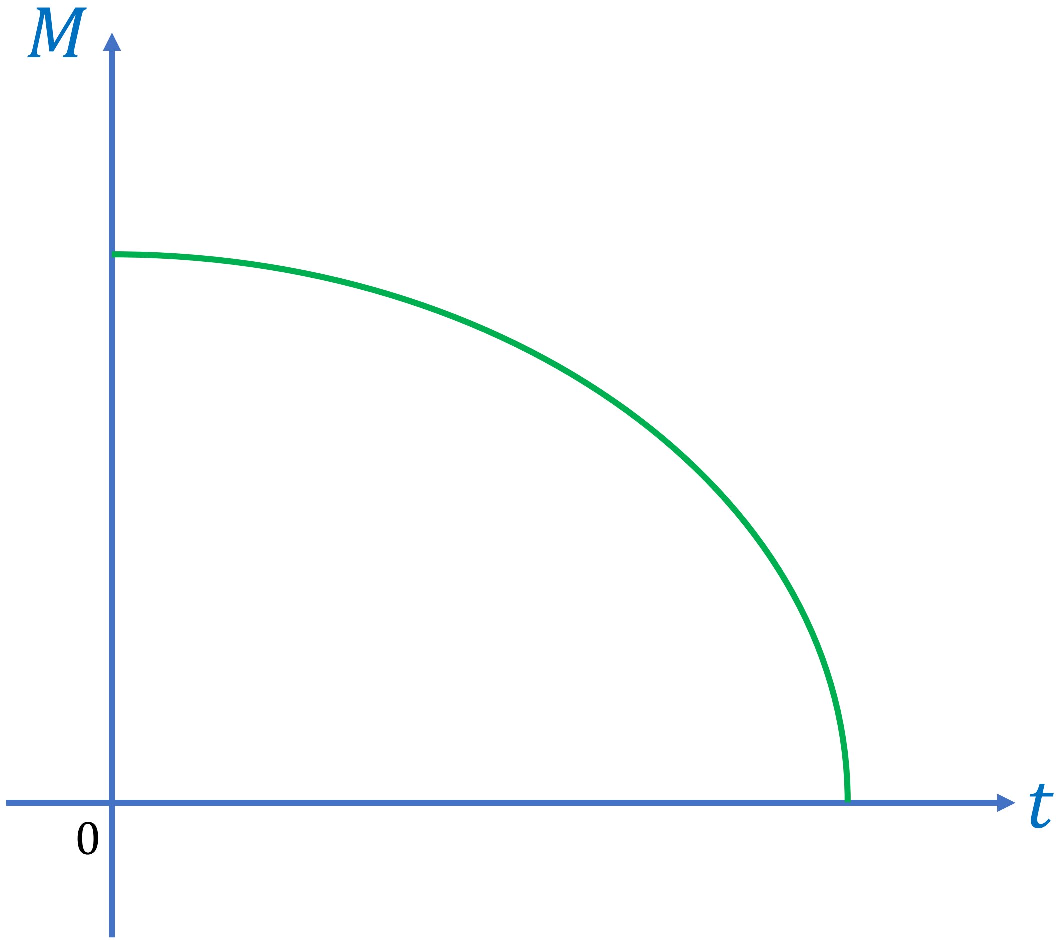


Thermal Mass Loss (up to greybody factor):

$$\frac{dM}{dt} \sim -AT^4 \sim -\frac{1}{M^2}$$

Thus the lifetime of a black hole is of order M^3.

Lifetime for solar mass black hole = O(10^{67}) years



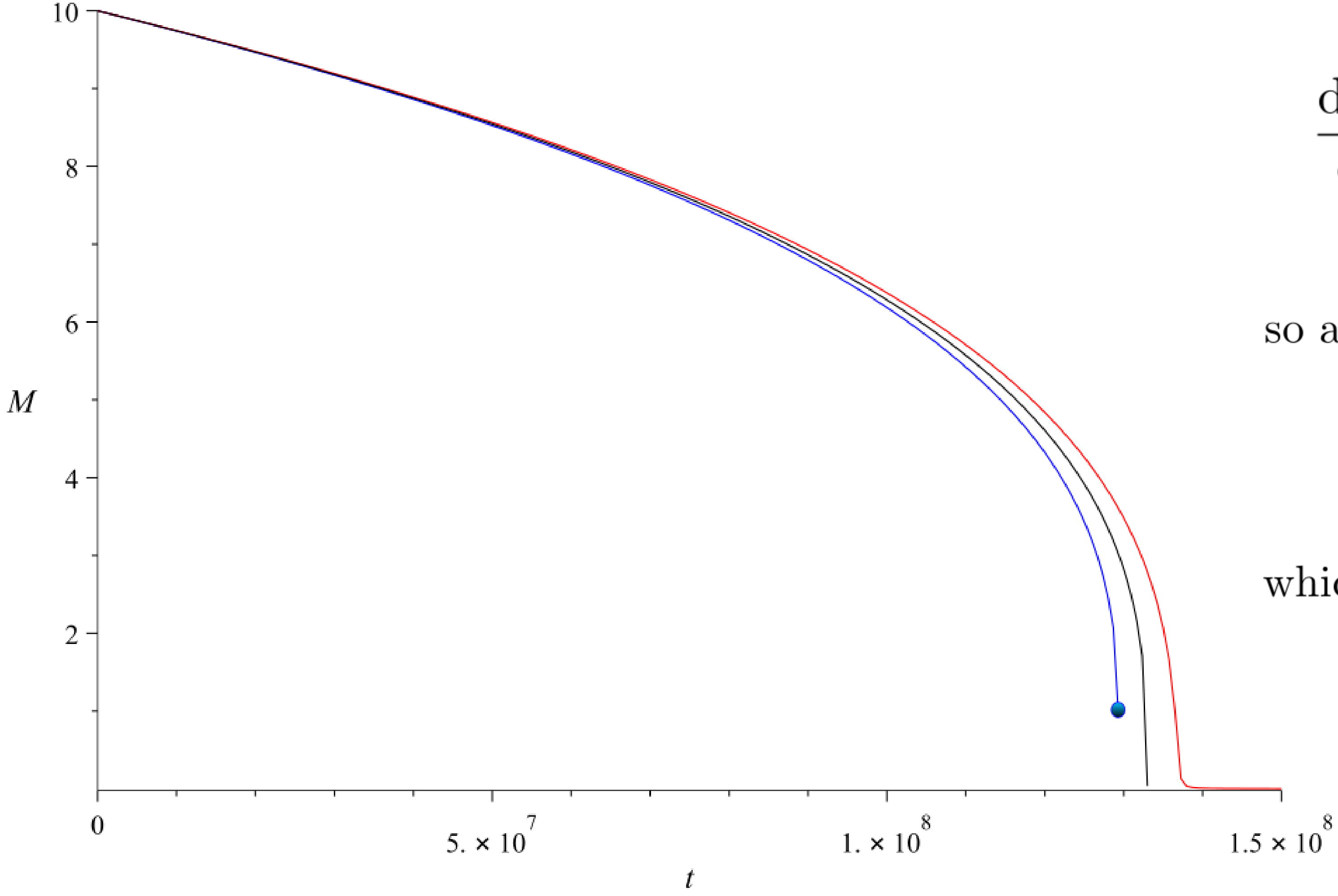


FIG. 2: Mass evolution of Schwarzschild black holes with no GUP correction (black, middle curve), positive GUP correction (blue, left curve), and negative GUP correction (red, right curve). The positive GUP correction leads to a remnant in finite time, while negative GUP correction yields infinite lifetime. These contrast with the usual case without GUP correction, in which the black hole completely evaporates in finite time.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{M^6}{(4|\alpha|\pi)^4}\left(1-\sqrt{1+\frac{|\alpha|}{M^2}}\right)^4.$$

so as M becomes sufficiently small, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \sim -\frac{M^2}{(4\pi)^4\alpha^2},$$

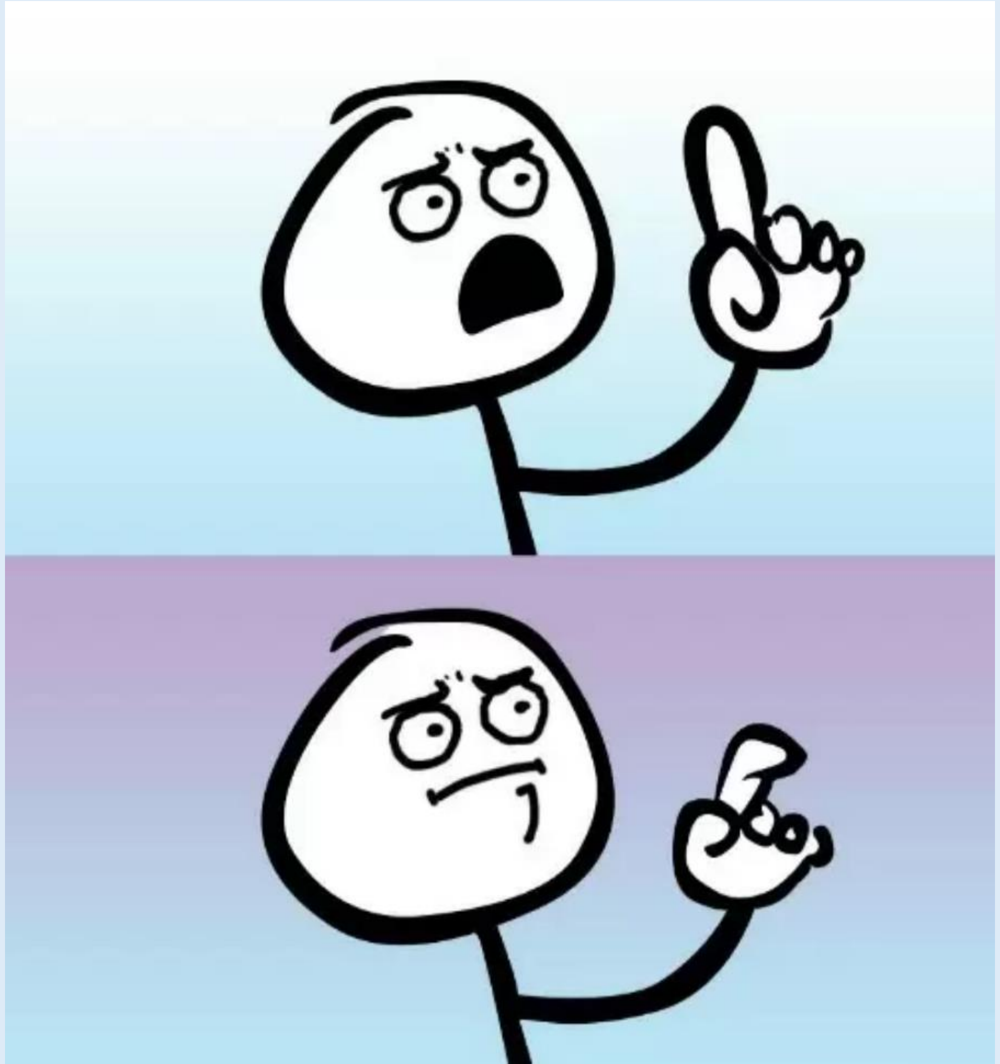
which leads to

$$M = M_0\left(\frac{256\pi^4\alpha^2}{256\pi^4\alpha^2 + M_0 t}\right),$$

Yen Chin Ong, "An Effective Black Hole Remnant via Infinite Evaporation Time Due to Generalized Uncertainty Principle", JHEP 10 (2018) 195, [arXiv:1806.03691 [gr-qc]].

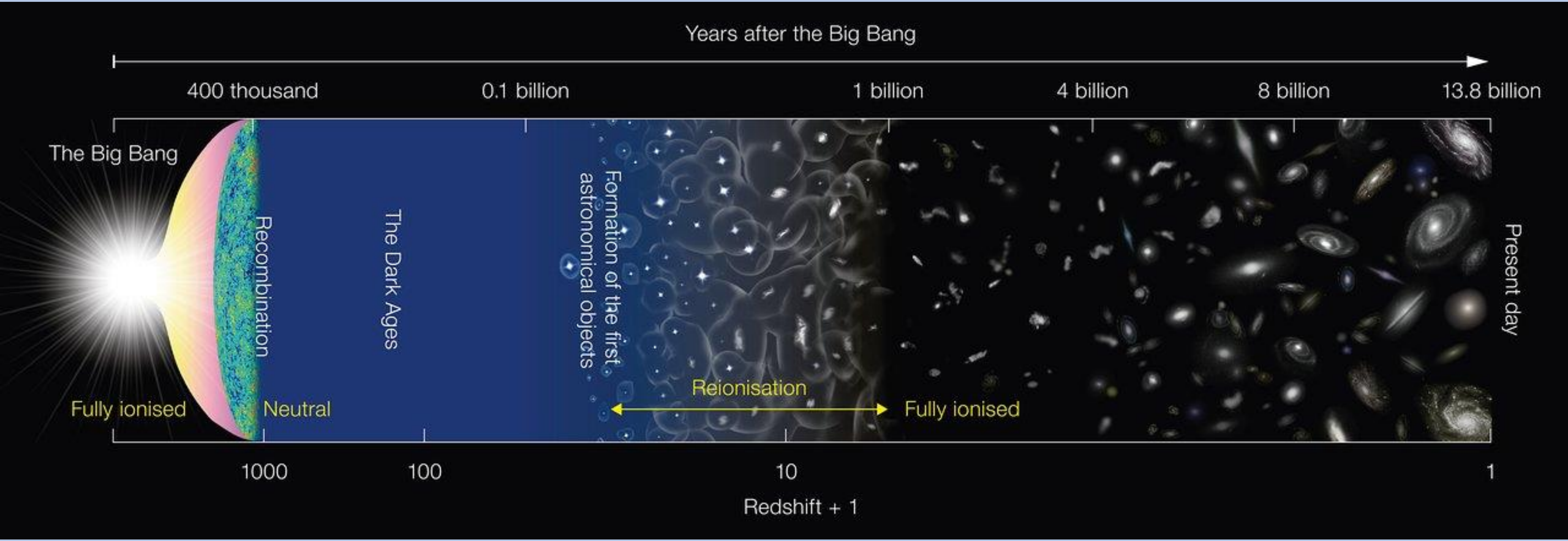
Still … Not so satisfying?

Despite its virtue in preventing arbitrarily large white dwarf, and being consistent with some models of quantum gravity, such a GUP lacks theoretical derivation.

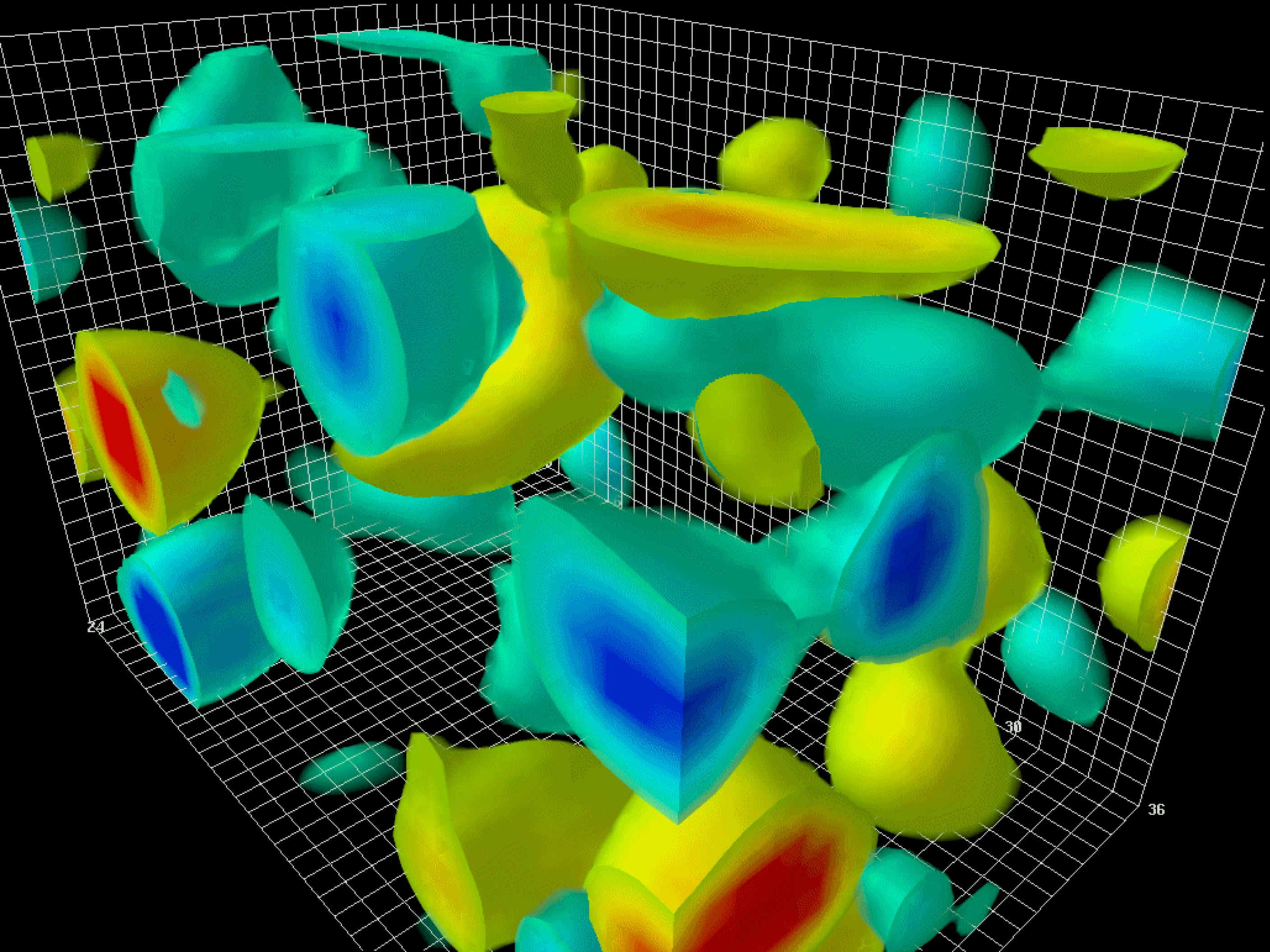


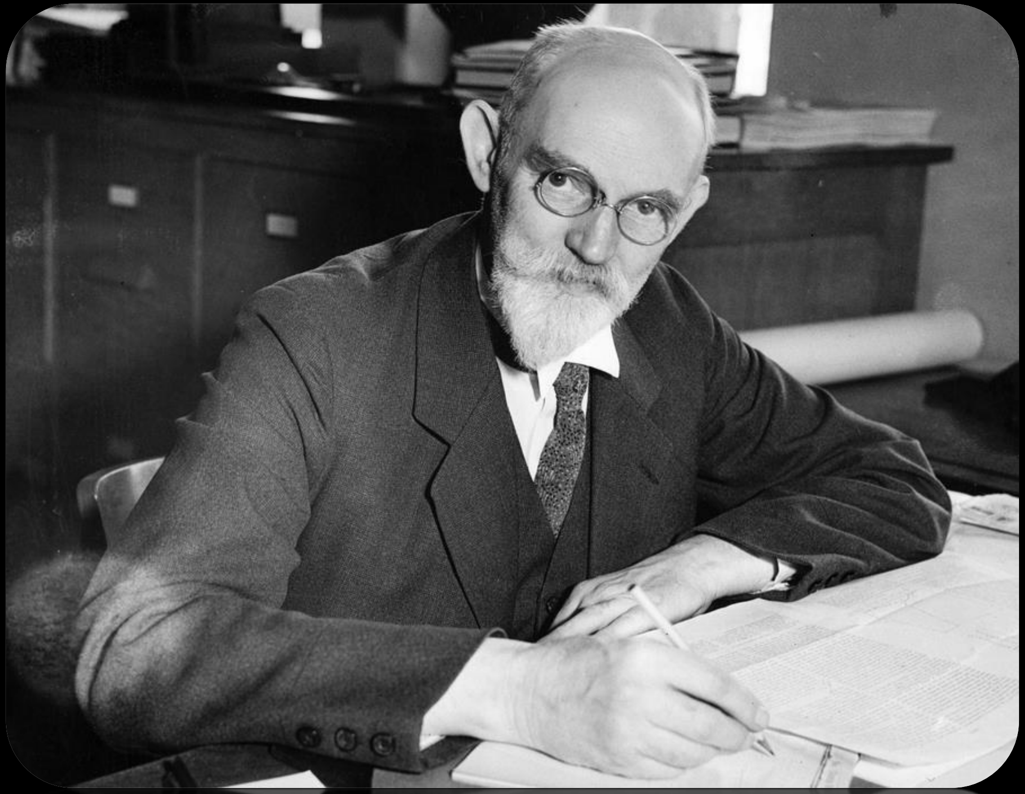
Can we resolve the white dwarf problem with another approach?

Our Universe is undergoing an accelerated expansion



Cosmological Constant? Λ





Extended Uncertainty Principle

$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \beta \frac{\hbar (\Delta x)^2}{L^2}\right]$$

Remark: To recover the correct black hole temperature via the heuristic method, $\beta = \pm 3$

c.f. for S^1: $\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 - C(\Delta x)^2\right)$

B. Bolen, M. Cavaglia, *(Anti-)de Sitter Black Hole Thermodynamics and the Generalized Uncertainty Principle*, Gen. Relativ. Grav. 37, 1255 (2005), arXiv:gr-qc/0411086v1.

M.I. Park, *The Generalized Uncertainty Principle in (A)dS Space and the Modification of Hawking Temperature from the Minimal Length*, Phys. Lett. B659, 698 (2008),
arXiv:0709.2307v4 [hep-th].

C. Bambi, F. R. Urban, *Natural Extension of the Generalised Uncertainty Principle*, Class.Quant.Grav.25:095006 (2008), arXiv:0709.1965v2 [gr-qc].

S. Mignemi, *Extended Uncertainty Principle and the Gometry of (Anti)-de Sitter Space*, Mod.Phys.Lett. A25 (2010) 1697-1703, arXiv:0909.1202v2 [gr-qc].

Extended Generalized Uncertainty Principle

$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \alpha \frac{L_p^2 (\Delta p)^2}{\hbar} + \beta \frac{\hbar (\Delta x)^2}{L^2}\right]$$

$$M^{\frac{4}{3}} \sim \frac{c^4 R^2}{2\alpha m_e^{\frac{2}{3}} G^2}\left[1 - \sqrt{1 - \frac{4G\hbar\alpha}{c^3}\left(\frac{\beta}{L^2} + \frac{M^{\frac{2}{3}}}{m_e^{\frac{2}{3}} R^2}\right)}\right]$$

$$R_{1,2}(M) := \frac{\sqrt{2}}{2}\left[\frac{L^2 M^{\frac{4}{3}} m_e^{\frac{4}{3}} - L^2 M^{\frac{2}{3}} \pm \sqrt{\mathscr{F}(\alpha, \beta, M, L)}}{\beta m_e^{\frac{2}{3}}}\right]^{\frac{1}{2}},$$

where

$$\mathscr{F}(\alpha, \beta, M, L) := L^4 (M m_e)^{\frac{8}{3}} - 4L^2 (M m_e)^{\frac{8}{3}} \alpha\beta - 2L^4 M^2 m_e^{\frac{4}{3}} + L^4 M^{\frac{4}{3}}.$$

Extended Generalized Uncertainty Principle

$$\Delta x \Delta p \geqslant \frac{1}{2}\left[\hbar + \alpha\frac{L_p^2(\Delta p)^2}{\hbar} + \beta\frac{\hbar(\Delta x)^2}{L^2}\right]$$

QG-correction Classical geometry-correction

$$M^{\frac{4}{3}} \sim \frac{c^4R^2}{2\alpha m_e^{\frac{2}{3}}G^2}\left[1 - \sqrt{1 - \frac{4G\hbar\alpha}{c^3}\left(\frac{\beta}{L^2} + \frac{M^{\frac{2}{3}}}{m_e^{\frac{2}{3}}R^2}\right)}\right]$$

$$R_{1,2}(M) := \frac{\sqrt{2}}{2}\left[\frac{L^2M^{\frac{4}{3}}m_e^{\frac{4}{3}} - L^2M^{\frac{2}{3}} \pm \sqrt{\mathscr{F}(\alpha,\beta,M,L)}}{\beta m_e^{\frac{2}{3}}}\right]^{\frac{1}{2}},$$

where

$$\mathscr{F}(\alpha,\beta,M,L) := L^4(Mm_e)^{\frac{8}{3}} - 4L^2(Mm_e)^{\frac{8}{3}}\alpha\beta - 2L^4M^2m_e^{\frac{4}{3}} + L^4M^{\frac{4}{3}}.$$

Yen Chin Ong, Yuan Yao, "Generalized Uncertainty Principle and White Dwarfs Redux: How Cosmological Constant Protects Chandrasekhar Limit", Phys. Rev. D 98 (2018) 126018, [arXiv:1809.06348 [gr-qc]].

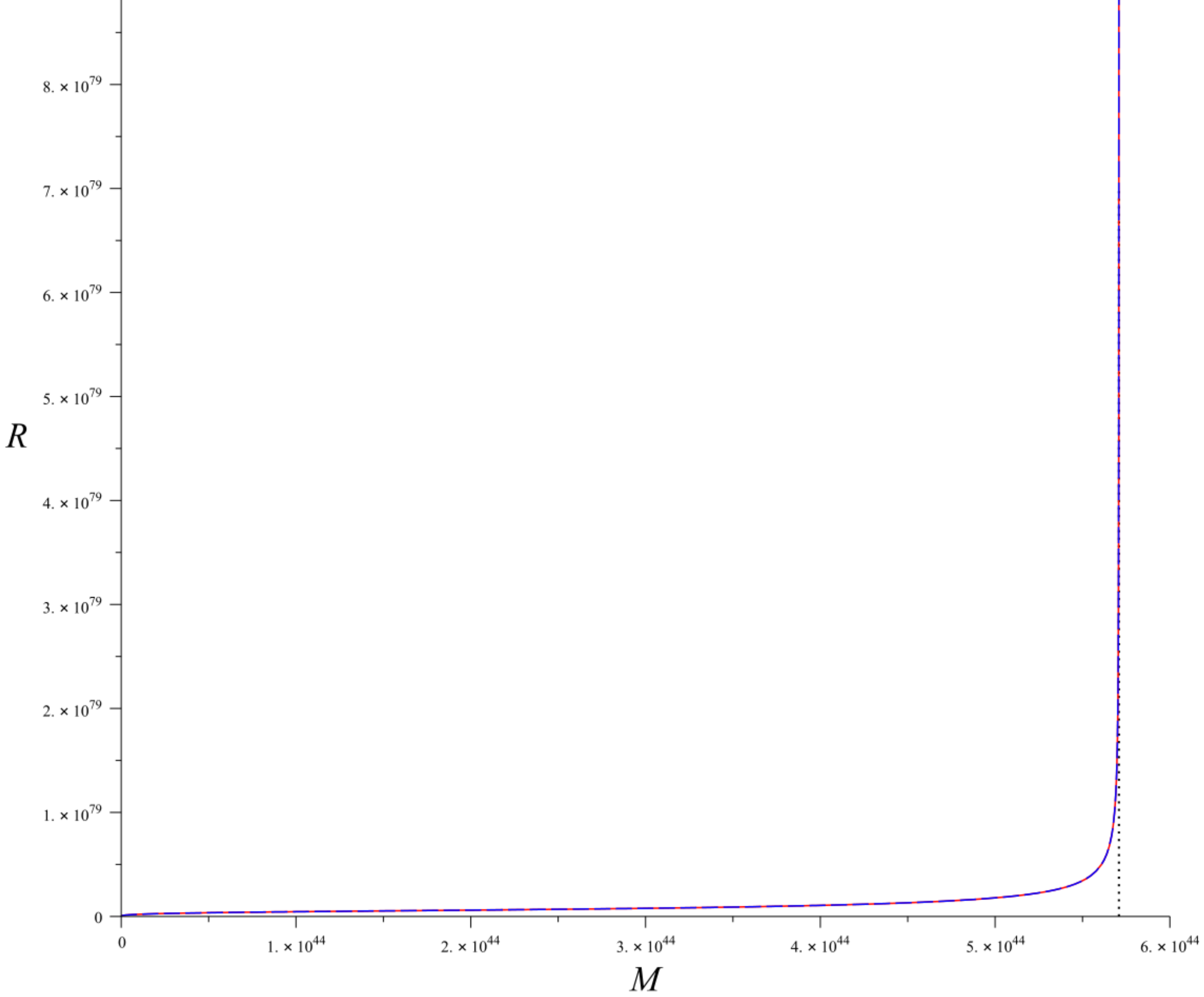


FIG. 2: The mass-radius relationship of an ultra-relativistic white dwarf with EGUP correction, $R_1(M)$. Without EGUP correction, it is simply the vertical Chandrasekhar limit. The effect of EGUP is to cause sufficiently small white dwarfs to deviate away from the Chandrasekhar limit, but note that no star can exist above the limit. Red curve and blue curve correspond to $\alpha = 1$ and $\alpha = -1$ respectively, they pretty much coincide with each other. Varying the magnitude of α up to 10^{11} does not change the result by much.

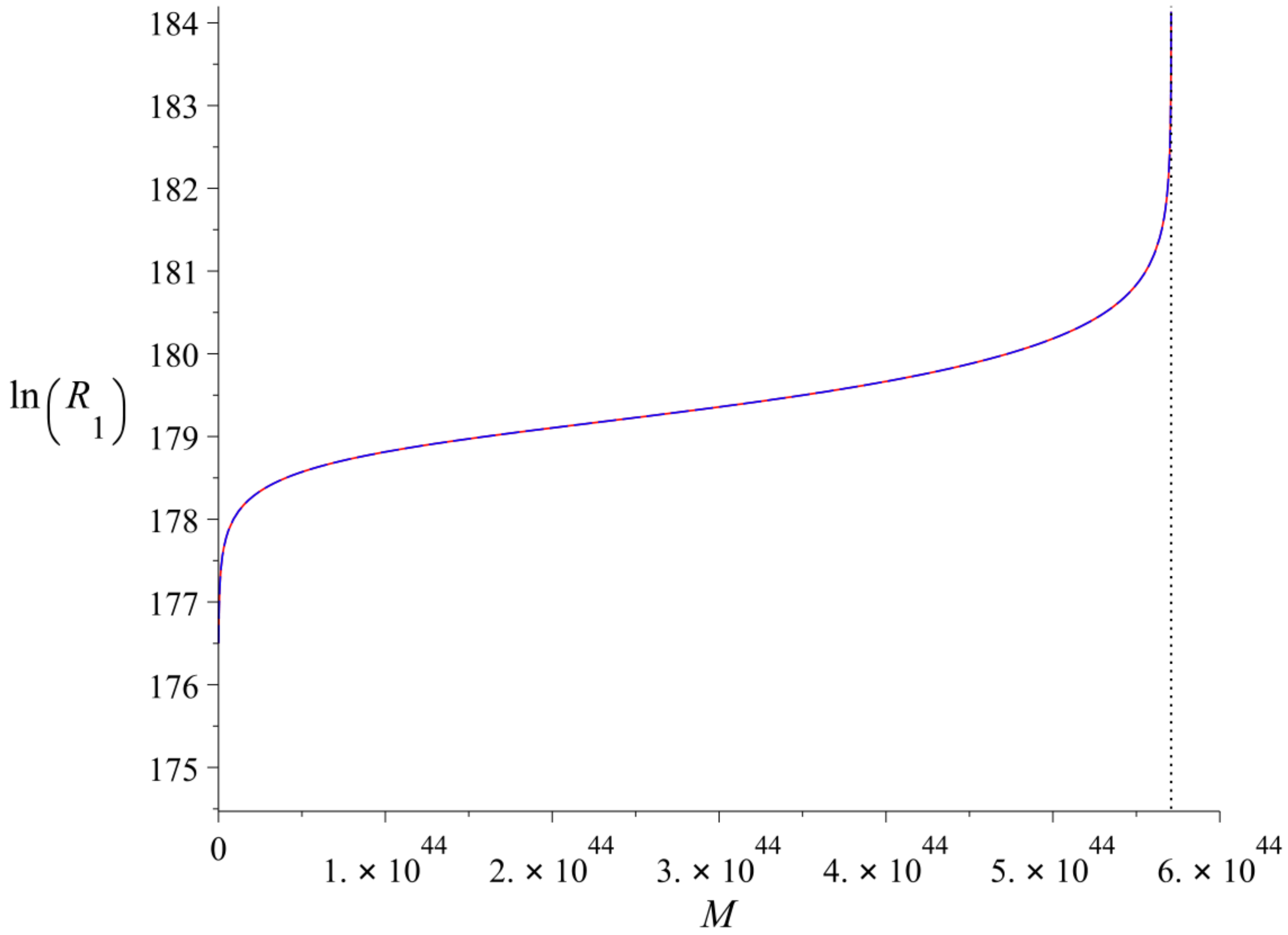


FIG. 4: The mass-radius relationship of an ultra-relativistic white dwarf with EGUP correction, $\log[R_1(M)]$. Red curve and blue curve correspond to $\alpha = 1$ and $\alpha = -1$ respectively, they are still indistinguishable even in log plot. The dashed vertical line corresponds to $M = M_{\rm Ch}$.

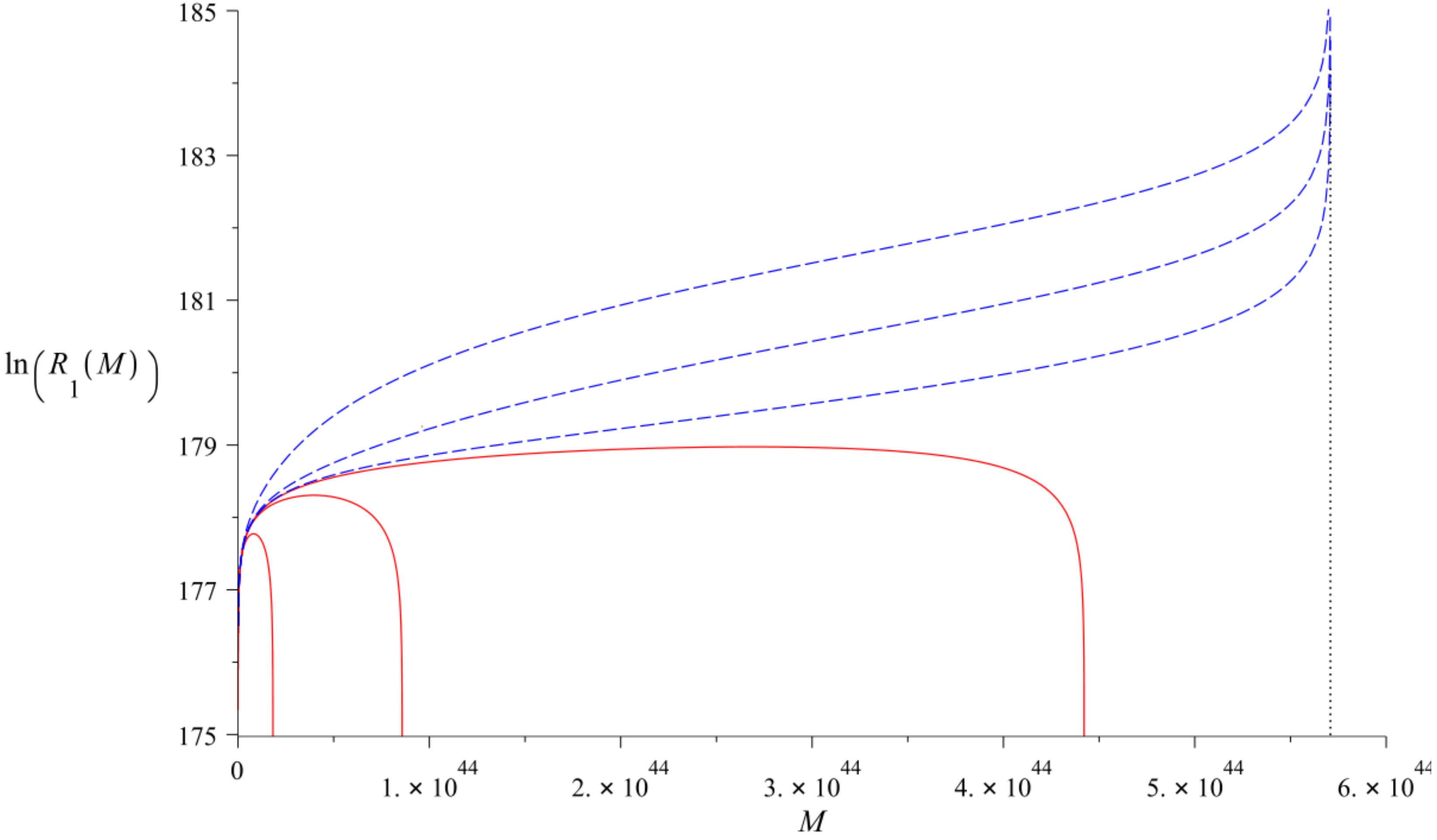


FIG. 5: The mass-radius relationship of an ultra-relativistic white dwarf with EGUP correction, $\log[R_1(M)]$. Solid curves are for $\alpha > 0$ and dashed curves are for $\alpha < 0$. The curves, from top to bottom, correspond respectively to $\alpha = -10^{113}, -10^{112}, -4 \times 10^{110}, 4 \times 10^{110}, 10^{112}, 10^{113}$, respectively. The dashed vertical line corresponds to $M = M_{\rm Ch}$.

A Bound On Cosmological Constant

$$\Lambda < \frac{M^{\frac{2}{3}}}{R^2 m_e^{\frac{2}{3}}} \lesssim \frac{1}{M_{\odot}^{\frac{4}{3}} m_e^{\frac{2}{3}}} = 10^{-36}$$

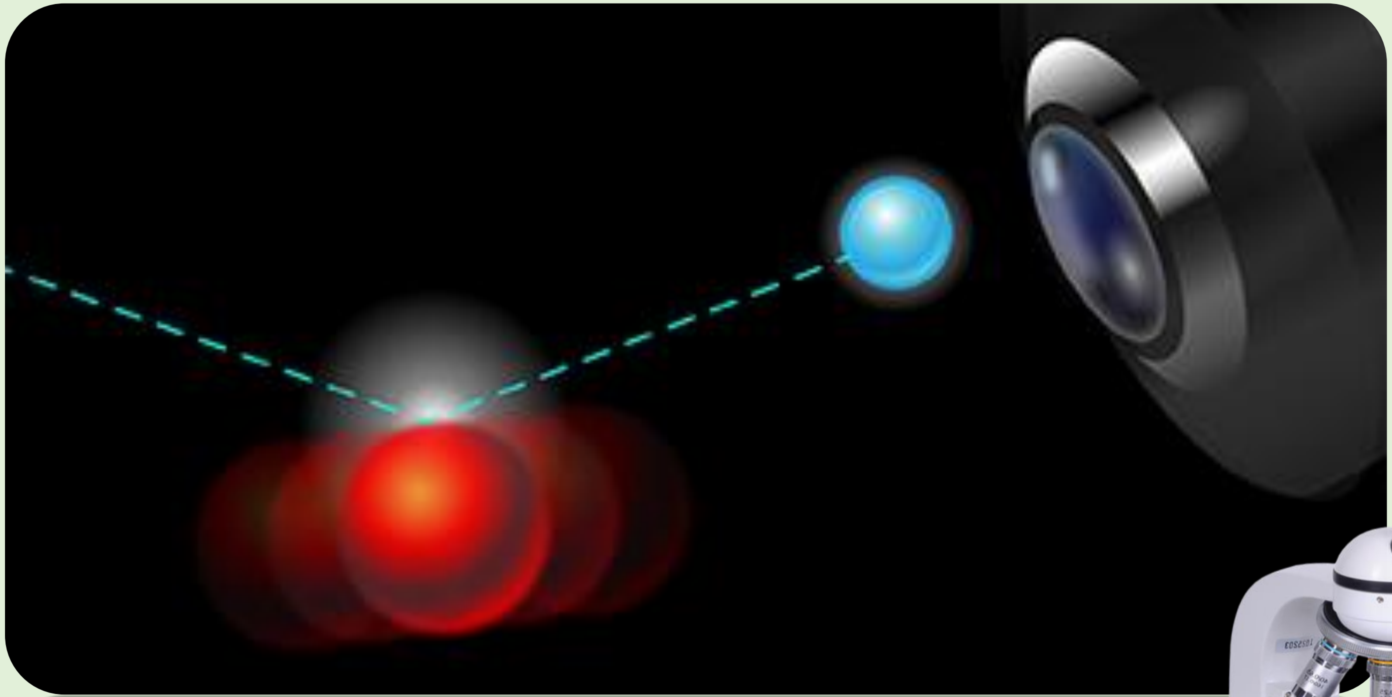
Still large compared to the required $\sim 10^{-122}$, but small compared to the “natural” value O(1).

Talk Based On

- Yen Chin Ong, "Generalized Uncertainty Principle, Black Holes, and White Dwarfs: A Tale of Two Infinities", JCAP 09 (2018) 015, [arXiv:1804.05176 [gr-qc]].
- Yen Chin Ong, "An Effective Black Hole Remnant via Infinite Evaporation Time Due to Generalized Uncertainty Principle", JHEP 10 (2018) 195, [arXiv:1806.03691 [gr-qc]].
- Yen Chin Ong, Yuan Yao, "Generalized Uncertainty Principle and White Dwarfs Redux: How Cosmological Constant Protects Chandrasekhar Limit", Phys. Rev. D 98 (2018) 126018 [arXiv:1809.06348 [gr-qc]].
- Yuan Yao, Meng-Shi Hou, Yen Chin Ong, "A Complementary Third Law for Black Hole Thermodynamics", [arXiv:1812.03136 [gr-qc]].

Extra Slides

Heisenberg’s Microscope



$$\Delta x \Delta p \sim \hbar$$

Heisenberg's Microscope with Gravitational Correction

Ronald J. Adler, "Six Easy Roads to the Planck Scale", Am. J. Phys. 78 (2010) 925, arXiv:1001.1205 [gr-qc].

Photon energy $E = h\nu$, so effective mass $M_{\text{eff}} = h\nu/c^2 = h/c\lambda$ exerts force to accelerate particle: additional fuzziness!

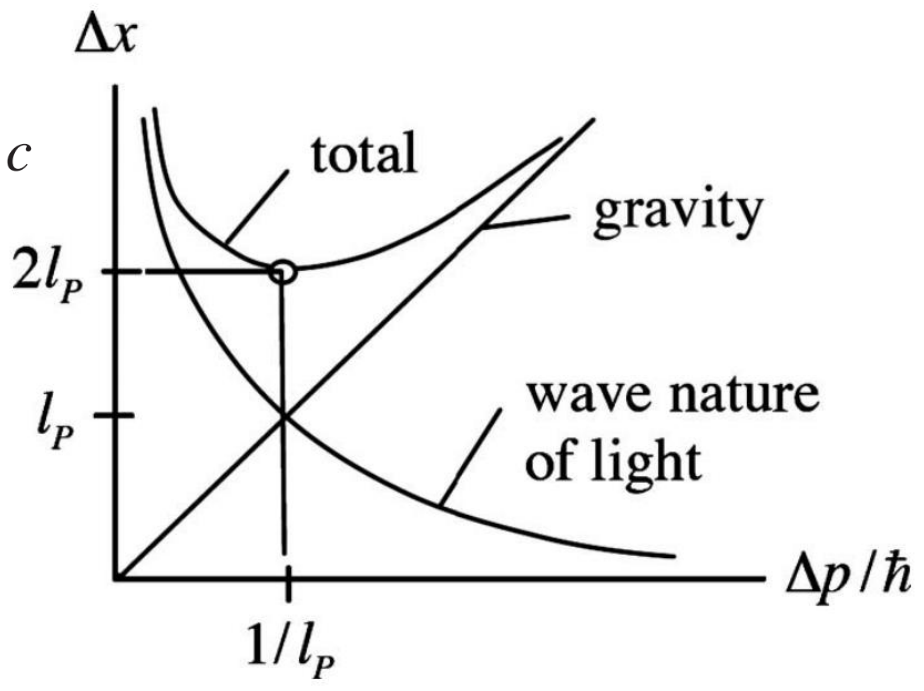
$$\Delta a_g \approx GM_{\text{eff}}/r_{\text{eff}}^2 = G(h/\lambda c)/r_{\text{eff}}^2,$$

$$\Delta x_g \approx \Delta a_g t_{\text{eff}}^2 \approx G(h/\lambda c)(t_{\text{eff}}^2/r_{\text{eff}}^2),$$

$r_{\text{eff}}/t_{\text{eff}} \approx c$

$$\Delta x_g \approx Gh/\lambda c^3 \approx (G\hbar/c^3)/\lambda = \ell_P^2/\lambda.$$

$$\Delta x \approx \left(\frac{\hbar}{\Delta p}\right) + \ell_P^2\left(\frac{\Delta p}{\hbar}\right).$$



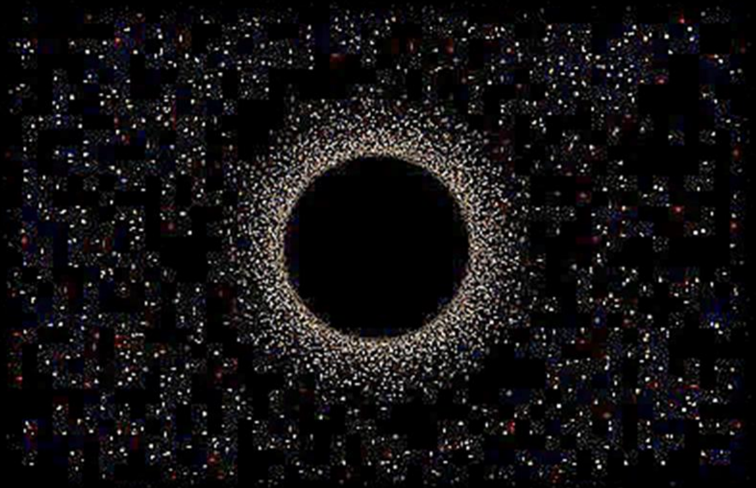
Also, black hole formation?

Heuristic Derivation of Bekenstein-Hawking Temperature of Black Holes via Uncertainty Principle

The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

with the event horizon at $r_+ = 2GM/c^2$.



Heisenberg's Uncertainty Principle then yields, if we identify $\Delta x \sim r_+$, the following approximation

$$\Delta p \approx \frac{\hbar}{2\Delta x} \approx \frac{\hbar}{2r_+} = \frac{\hbar c^2}{4GM}$$

Suppose now we interpret the quantum uncertainty in the kinetic energy of the emitted particles as due to thermal agitation, the uncertainty in the energy of photons emitted during Hawking evaporation is identified with

$$\Delta E = \Delta pc \approx \frac{\hbar c^3}{4GM} \sim k_B T \Rightarrow T = 2\pi\left[\frac{\hbar c^3}{8\pi k_B GM}\right]$$

where

$$T_{BH} = \frac{\hbar c^3}{8\pi k_B GM}$$

is the Bekenstein-Hawking temperature of the Schwarzschild black hole.

Hawking Radiation Does Not Originate from Near Horizon!

de Broglie wavelength:

$$\lambda = \frac{2\pi\hbar}{p}$$

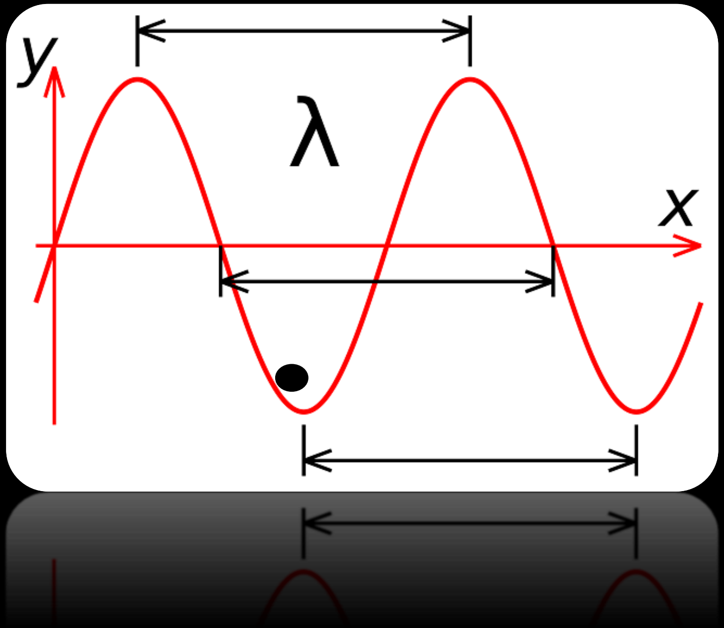
Quantum Mechanics

$$E = pc$$

thermodynamics

$$E = k_B T$$

Wavelength of Hawking Particle:



$$\lambda_T = \frac{2\pi\hbar}{k_B T/c} = \frac{2\pi\hbar c}{k_B} \cdot \frac{8\pi k_B G M}{\hbar c^3}$$

$$= 8\pi^2 \cdot \frac{2GM}{c^2} = 8\pi^2 r_h \approx 79\, r_h$$

S. B. Giddings, "Hawking radiation, the Stefan Boltzmann law, and unitarization," Phys. Lett. B754 (2016) 39, 1511.08221.

A Comparison: Uncertainty Principle on Unit Circle

1963: D. Judge published a single page, ultra-dense paper titled “*On the Uncertainty Relation for L_z and ϕ*” [Physics Letters, Vol. 5, No. 3, 1963]: (Details in 1964)

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On the Uncertainty Relation for Angle Variables.

D. Judge
University College - Dublin

(ricevuto il 10 Luglio 1963)

Summary. — The uncertainty relation between the orbital angular momentum component L_z and the corresponding angle φ is discussed. The uncertainty for an angle variable is defined. The formula $\Delta L_z \cdot \Delta\varphi \geqslant \frac{1}{2}\hbar$, which is sometimes quoted, is shown to be incorrect, and an alternative relation, in full accord with the Heisenberg Uncertainty Principle, is derived.

$$\frac{\Delta\phi \cdot \Delta L_z}{1 - \frac{3}{\pi^2}(\Delta\phi)^2} \geq \frac{\hbar}{2}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 - C(\Delta x)^2\right)$$

Heisenberg's Uncertainty Principle: Modern Statement

Theorem: Assume that the self-adjoint operators A and B admits variances, then

H.P. Robertson, Phys. Rev. **34** (1929) 163.

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\langle\psi|\{\hat{A},\hat{B}\}|\psi\rangle^2 + \frac{1}{4}\langle\psi|\Gamma|\psi\rangle^2,$$

where $\hat{A} := A - \langle A\rangle; \hat{B} := B - \langle B\rangle; [\hat{A},\hat{B}] = i\Gamma$, and $\{\cdot,\cdot\}$ is the anti-commutator.

Corollary (Heisenberg's Uncertainty Principle): For the position operator X and momentum operator $P = -i\hbar\,\partial/\partial x$ in position space basis, we have $[X,P] = i\hbar$, and so

"Canonical" Commutation Relation

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle\{X,P\}\rangle^2 + \frac{\hbar^2}{4} \geq \frac{\hbar^2}{4}.$$

That is,

$$\Delta X \cdot \Delta P \geq \frac{\hbar}{2},$$

with equality attained if and only if

(1) $\hat{X}|\psi\rangle = c\hat{P}|\psi\rangle,\ c \in \mathbb{C}$, and

(2) $\langle\psi|\{\hat{X},\hat{P}\}|\psi\rangle = 0$.

See also E.H. Kennard, Z. Phys. **44** (1927) 326

$\delta p \cdot \delta x \sim \hbar$

Uncertainty Principle on Unit Circle

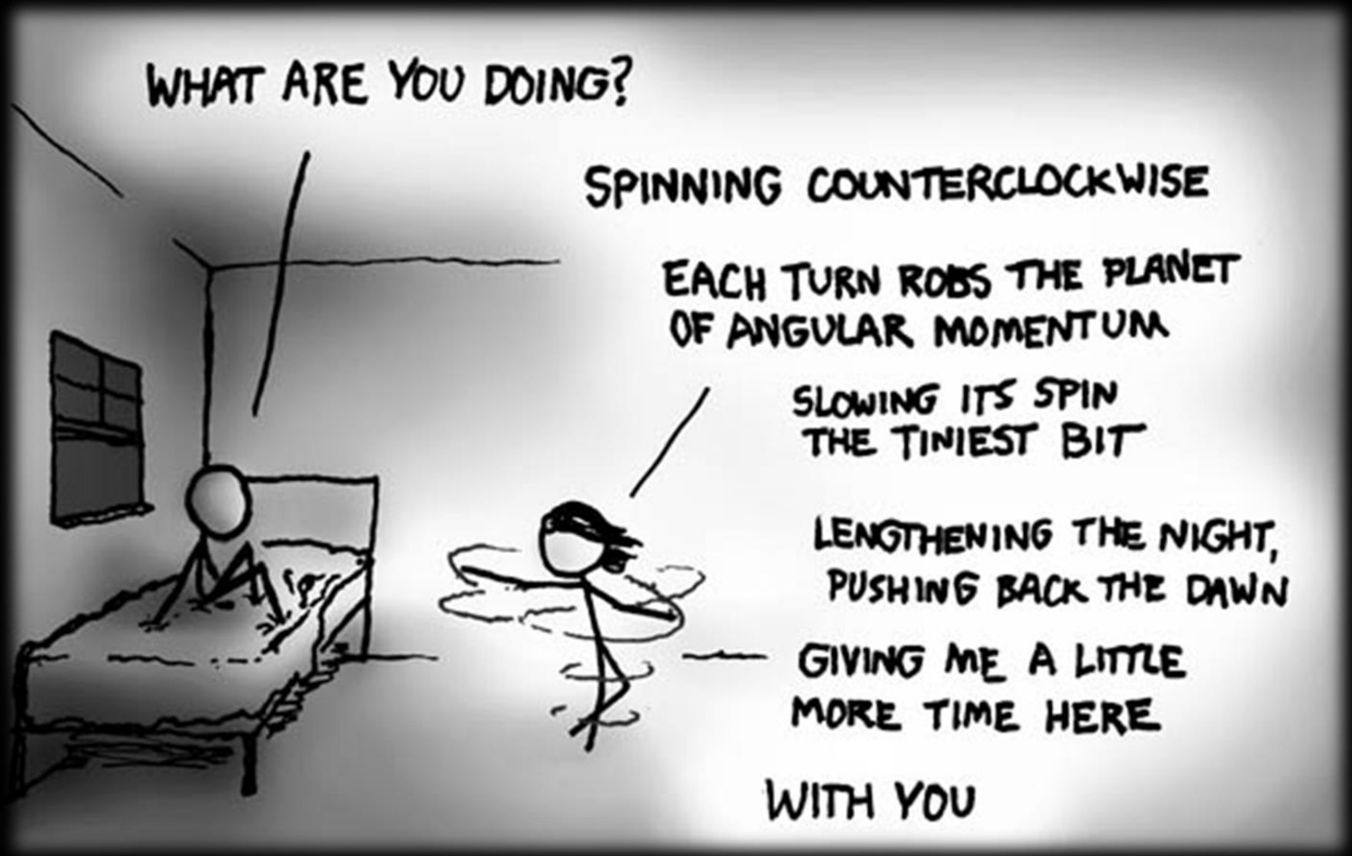
It can be shown that angular momentum and angular coordinate satisfies the canonical commutation relation:

$$[\phi, L_z] = i\hbar \, , \, L_z = -i\hbar \frac{\partial}{\partial \phi}$$

Yet can find states such that ΔL_z is sufficiently small, so that if

$$\Delta L_z \cdot \Delta\phi \geq \frac{\hbar}{2},$$

then $\Delta\phi \geq 2\pi.$ **Something is wrong!**



xkcd

Note: Commutation Relation does *not* imply [the form of] Uncertainty Principle

Recall that the uncertainty relation for any two operators A and B is usually written in the form:

$$(\Delta A)^2(\Delta B)^2 \geq \langle\psi|i[A,B]|\psi\rangle^2,$$

A very important notion that is not usually mentioned in quantum mechanics textbooks is the *domain of definition* of an operator. Like functions, an operator has domain.

Note: Commutation Relation does *not* imply [the form of] Uncertainty Principle

Definition: An *operator* A on the Hilbert space $\mathcal{H}$ is a linear map

$$A: \mathcal{D}(A) \longrightarrow \mathcal{H},$$
$$|\psi\rangle \longmapsto A|\psi\rangle,$$

where $\mathcal{D}(A)$ is a dense subspace of $\mathcal{H}$, called the *domain of definition*, or simply, domain of A.

$$(\Delta A)^2(\Delta B)^2 \geq \langle\psi|i[A,B]|\psi\rangle^2,$$

LHS is defined on $|\psi\rangle \in \mathcal{D}(A) \cap \mathcal{D}(B)$, the subspace of $\mathcal{H}$ containing all states for which the uncertainties $\Delta A, \Delta B$ are well-defined [i.e. have physical meaning].

RHS is defined *only* for states on the subspace $\mathcal{D}([A,B]) = \mathcal{D}(AB) \cap \mathcal{D}(BA) \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$.

Thus commutation relation does not always allow us to derive the correct uncertainty principle!

F. Gieres, "Mathematical Surprises and Dirac's Formalism in Quantum Mechanics", Rep.Prog.Phys. 63 (2000) 1893, arXiv:quant-ph/9907069.

Note: Commutation Relation does *not* imply [the form of] Uncertainty Principle

Take $|\psi\rangle \in \mathcal{D}(A) \cap \mathcal{D}(B)$, define $\hat{A} = A - \langle A\rangle, \hat{B} = B - \langle B\rangle$,
for simplicity, denote $A|\psi\rangle = A\psi$. Then

$$|i\langle A\psi, B\psi\rangle - i\langle B\psi, A\psi\rangle| = \left|i\langle \hat{A}\psi, \hat{B}\psi\rangle - i\langle \hat{B}\psi, \hat{A}\psi\rangle\right|$$

Triangle Inequality →

$$\leq \left|\langle \hat{A}\psi, \hat{B}\psi\rangle\right| + \left|\langle \hat{B}\psi, \hat{A}\psi\rangle\right| = 2\left|\langle \hat{A}\psi, \hat{B}\psi\rangle\right|$$

Cauchy-Schwarz Inequality →

$$\leq 2\left|\hat{A}\psi\right| \cdot \left|\hat{B}\psi\right| = 2\Delta A \cdot \Delta B$$

$$\Delta A \cdot \Delta B \geq \frac{1}{2}|i\langle A\psi, B\psi\rangle - i\langle B\psi, A\psi\rangle|$$

Both sides are now defined on the *same domain*

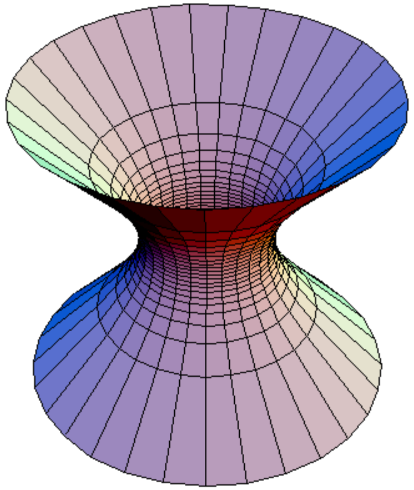
$$|\psi\rangle \in \mathcal{D}(A) \cap \mathcal{D}(B)$$

Thus the uncertainty principle is determined not by commutation relation, but by *Hermitian sesquilinear form*.

In Short:
Uncertainty Principle Depends on Geometry

The uncertainty principle concerns Fourier transforms of functions, which is nontrivial on curved manifolds.

Alexey Golovnev, Lev Vasil'evich Prokhorov, "Uncertainty Relations in Curved Spaces", J. Phys. A **37** (2004) 2765, [arXiv:quant-ph/0306080].
Thomas Schürmann, "Uncertainty Principle on 3-Dimensional Manifolds of Constant Curvature", Found. Phys. **48** (2018) 716, [arXiv:1804.02551 [quant-ph]].



What happens in de Sitter space?