Regge-Teitelboim analysis of the symmetries of electromagnetic and gravitational fields on asymptotically null spacelike surfaces

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Introduction

One of the main objectives of this work is the study of electromagnetic and gravitational radiation using the canonical approach **[Dirac (1958)]**, **[Regge, Teitelboim (1974)]**.

Gravitational radiation and asymptotic symmetries:

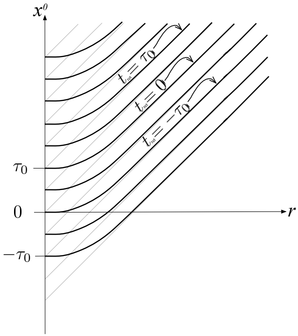
- Asymptotic symmetries at null infinity $\longrightarrow$ BMS algebra (Extension of the Poincaré algebra). **[Bondi, Metzner, van der Burg (1962)]** , **[Sachs (1962)]**.

  Poincaré+ Supertranslations

$$\left[\mathcal{P}\left(\theta,\phi\right),\mathcal{P}\left(\theta',\phi'\right)\right]=0$$

- The Bondi-Sachs formalism was developed using null coordinates. As a consequence, we cannot directly apply the Hamiltonian formalism to construct the symmetry generators (constant u (or v) hypersurfaces are not spacelike).

- We use a foliation of Minkowski spacetime which consists in a sequence of hyperboloids with fixed radius τ_0 and varying center.



- The possibility of using hyperboloids was considered by Dirac in 1940, and he called it the "point form of field dynamics."

We can introduce the following set of (retarded) coordinates (t, χ, θ, ϕ)

$$
\begin{aligned}
x^0 &= t + \tau_0 \cosh \chi \\
r &= \tau_0 \sinh \chi
\end{aligned}
$$

in these coordinates the Minkowski metric reads

$$
ds^2 = -dt^2 - 2\tau_0 \sinh \chi d\chi dt + \tau_0^2 \left[d\chi^2 + \sinh^2 \chi \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]
$$

- Using this foliation of Minkowski spacetime in terms of hyperboloids, we can describe electromagnetic and gravitational waves at null infinity within the Hamiltonian formalism.

Review of the Regge-Teitelboim formalism

In the formulation of field dynamics in which the state is defined on a general spacelike surface developed by Dirac, and completed by Regge and Teitelboim, the generator of the "most general permissible motion" has the form

$$H[\xi;\lambda] = H_0[\xi;\lambda] + Q[\xi;\lambda]$$

where $H_0[\xi;\lambda]$ is an integral over the spacelike surface on which the state is defined

$$H_0[\xi;\lambda] = \int d^3x\,(\xi^\mu \mathcal{H}_\mu + \lambda \mathcal{G})$$

and $Q[\xi,\lambda]$ is a surface integral over the asymptotic boundary of that spacelike surface.

The surface integral $Q[\xi;\lambda]$ is included to make well defined the functional derivatives of $H[\xi;\lambda]$, so that one has,

$$\delta H[\xi;\lambda] = \int d^3x \left(\frac{\delta H}{\delta \phi}\delta\phi + \frac{\delta H}{\delta \pi}\delta\pi \right)$$

without any surface terms.

For improper (large) gauge transformations, the corresponding generators do not vanish weakly

$$H[\xi;\lambda] \approx Q[\xi;\lambda] \neq 0$$

defining the respective conserved charge.

The Hamiltonian action then takes the form

$$I_{\text{can}} = \int d^4x(\pi^{ij}\dot{g}_{ij} + \pi^i \dot{A}_i - H[\xi;\lambda]),$$

where the appropriate boundary term is already included.

Electromagnetic field in Minkowski space

We propose the following set of *asymptotic conditions* ($\chi \to \infty$) for the electromagnetic field:

$$\begin{aligned} A_a &= a_a^{(0)} + a_a^{(1)} e^{-\chi} + \mathcal{O}(e^{-2\chi}) \,, \\ A_\chi &= a_\chi^{(1)} e^{-\chi} + \mathcal{O}(e^{-2\chi}) \;, \\ \pi^a &= \pi^a_{(1)} e^{-\chi} + \mathcal{O}(e^{-2\chi}) \,, \\ \pi^\chi &= \pi^\chi_{(0)} + \mathcal{O}(e^{-\chi}) \,. \end{aligned}$$

The asymptotic conditions are preserved under the Poincaré group, and under gauge transformations of the form

$$\lambda = \lambda_{(0)}(\theta, \phi) + \mathcal{O}(e^{-\chi})$$

The variation of the Hamiltonian is given by

$$\delta H_0 = -\delta \left[\alpha^\mu P_\mu + \vec{\omega} \cdot \vec{J} + \vec{\beta} \cdot \vec{K} + \oint d\theta d\phi \lambda_{(0)} \,(\theta, \phi)\, \pi^\chi_{(0)} \,(\theta, \phi) \right] - \oint d\theta d\phi \,(\alpha^\mu k_\mu)\, h^a \delta a_a^{(0)}$$

where $k_\mu = (-1, \hat{r})$, and $h^a = -\frac{1}{2\tau_0} \left(\pi^{a(1)} + \gamma^{\frac{1}{2}} \gamma^{ab} (a_b^{(1)} + a_{\chi,b}^{(1)}) \right)$.

- The vector h^a corresponds to a wave emitted by a confined source, and is the electromagnetic analogue of the "Bondi news" that will be encountered in gravitation.

When the non-integrable term does not vanish, the equations of motion *do not* follow from demanding that the improved action should be stationary.

- The symmetries remain symmetries of the equations of motion but since there is no action principle they do not yield conserved charges.
- One expects that when $h^a \neq 0$ the response of the charges under spacetime translations should depart from that dictated by the symmetry algebra.

This lack of conservation indeed occurs. One finds for example,

$$\frac{\partial Q(\theta,\phi)}{\partial t} = -\partial_a h^a$$

where $Q(\theta,\phi) = \pi^{\chi}_{(0)}(\theta,\phi)$ are the improper gauge charges, one per each point of the two sphere at infinity, and

$$\frac{dP_0}{dt} = -\oint \frac{d\theta d\phi}{\sqrt{\gamma}} \gamma_{ab} h^a h^b$$

One should *not* expect to be able to "take the delta outside" when there are news.

If one succeeded in doing so, one would have obtained a conserved quantity, energy for example. But this cannot happen because, on an asymptotically null surface, the energy carried away by the news reaches null infinity.

This is precisely why the formulation enables one the study of the details of the radiation process.

- So the "failure" is really a success, and a useful one.

Gravitational field

Correspondence with electromagnetism

$$\underbrace{\left(Q\left(\theta,\phi\right),P_\mu\right)}_{\text{electromagnetism}} \longleftrightarrow \underbrace{\mathcal{P}\left(\theta,\phi\right)}_{\text{gravitation}}$$

$$\underbrace{J_{\mu\nu}}_{\text{electromagnetism}} \longleftrightarrow \underbrace{J_{\mu\nu}}_{\text{gravitation}}$$

$$\underbrace{h_a}_{\text{electromagnetism}} \longleftrightarrow \underbrace{h_{ab}}_{\text{gravitation}}$$

Asymptotic conditions

We propose the following set of *asymptotic conditions* ($\rho = \sinh\chi$)

$$
\begin{aligned}
g_{\rho\rho} &= \tau_0^2 \left(\frac{1}{\rho^2} + \frac{f_{\rho\rho}^{(-4)}}{\rho^4} + \frac{f_{\rho\rho}^{(-5)}}{\rho^5} + O\left(\rho^{-6}\right) \right) , \\
g_{\rho a} &= \tau_0^2 \left(\frac{f_{\rho a}^{(-2)}}{\rho^2} + \frac{f_{\rho a}^{(-3)}}{\rho^3} + O\left(\rho^{-4}\right) \right) , \\
g_{ab} &= \tau_0^2 \left(\gamma_{ab}\rho^2 + f_{ab}^{(1)}\rho + f_{ab}^{(0)} + \frac{f_{ab}^{(-1)}}{\rho} + O\left(\rho^{-2}\right) \right) , \\
\pi^{\rho\rho} &= -\sqrt{\gamma}\rho^3 + p_{(1)}^{\rho\rho}\rho + p_{(0)}^{\rho\rho} + O\left(\rho^{-1}\right) , \\
\pi^{\rho a} &= \frac{p_{(-1)}^{\rho a}}{\rho} + \frac{p_{(-2)}^{\rho a}}{\rho^2} + O\left(\rho^{-3}\right) , \\
\pi^{ab} &= -\sqrt{\gamma}\gamma^{ab}\frac{1}{\rho} + \frac{3\sqrt{\gamma}f^{ab(1)}}{4\tau_0}\frac{1}{\rho^2} + \frac{p_{(-3)}^{ab}}{\rho^3} + \frac{p_{(-4)}^{ab}}{\rho^4} + O\left(\rho^{-5}\right) .
\end{aligned}
$$

The most general surface deformation that preserves these conditions are given by:

$$
\begin{aligned}
N^{\perp} &= \epsilon^{\perp}_{(1)}\rho + \frac{\epsilon^{\perp}_{(-1)}}{\rho} + O\left(\rho^{-2}\right) , \\
N^{\rho} &= -\frac{\epsilon^{\perp}_{(1)}}{\tau_0}\rho^2 - \frac{1}{2}\left(\nabla_a \epsilon^a_{(0)}\right)\rho + \epsilon^{\rho}_{(0)} + O\left(\rho^{-1}\right) , \\
N^{a} &= \epsilon^{a}_{(0)} - \frac{1}{\tau_0}\left(\nabla^a \epsilon^{\perp}_{(1)}\right)\frac{1}{\rho} + \frac{\epsilon^{a}_{(-2)}}{\rho^2} + O\left(\rho^{-3}\right) ,
\end{aligned}
$$

Here

$$
\epsilon^{\perp}_{(1)} = \epsilon^{\perp}_{(1)}(\theta,\phi) \qquad , \qquad \epsilon^{a}_{(0)} = \partial^a(\vec{\beta}\cdot\hat{r}) + \frac{\epsilon^{ab}}{\sqrt{\gamma}}\partial_b(\vec{\omega}\cdot\hat{r}) ,
$$

The variation of the Hamiltonian is

$$
\begin{aligned}
\delta H_0 = &- \delta\left[\oint d\theta d\phi \left(\epsilon^{\perp}_{(1)}\left(\theta,\phi\right)\mathcal{P}\left(\theta,\phi\right)\right) + \vec{\omega}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}\right] \\
&- \oint d\theta d\phi \epsilon^{\perp}_{(1)}\frac{1}{2}h^{ab}\delta\left[\tau_0 f^{(1)}_{ab}\right]
\end{aligned}
$$

where the “news tensor” h^{ab} is given by $h^{ab} = 2\tilde{p}^{ab}_{(-3)} - \sqrt{\gamma}\tilde{f}^{ab(0)}$.

As expected, the charges are not conserved:

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{1}{\sqrt{\gamma}} h^{ab} h_{ab} + \nabla_a \nabla_b h^{ab} \quad , \quad \frac{d\vec{J}}{dt} = -\oint d\theta d\phi \frac{1}{2} h^{ab} \mathcal{L}_{\vec{\xi}_R} \left[\tau_0 f_{ab}^{(1)} \right]$$

There is perfect agreement with Bondi's result. [**Bondi (1960)**].

- In the absence of news the BMS algebra can be canonically realized.
- The Kerr and Taub-NUT solutions can be accommodated within our asymptotic conditions, and the charges are the expected ones.
- A "dictionary" can be provided for translating the variables which appear in the present Hamiltonian treatment with those employed in the original BMS light cone analysis. In the notation used by Sachs

$$f_{zz}^{(1)} = \frac{4c}{\tau_0 (1 + z\bar{z})^2} \quad , \quad f_{\bar{z}\bar{z}}^{(1)} = \frac{4\bar{c}}{\tau_0 (1 + z\bar{z})^2} \, ,$$

$$h_{(1)}^{zz} = \frac{1}{2} \sqrt{\gamma} (1 + z\bar{z})^2 \dot{\bar{c}} \quad , \quad h_{(1)}^{\bar{z}\bar{z}} = \frac{1}{2} \sqrt{\gamma} (1 + z\bar{z})^2 \dot{c} \, .$$

The density associated to supertranslations $\mathcal{P}$ is related to the Bondi mass aspect m_B according to

$$\mathcal{P} = 2\sqrt{\gamma} m_B \, .$$

Thank you