On the violation of ***positivity bounds*** due to ***non-locality***

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Refs. : JT arXiv: 1902:XXXX

2019 2.11-15 AP Winter School @YITP

plan

- 1. Introduction & Motivation (pages.)

  What are positivity bounds?

- 2. Our work (4 pages.)

  ・Information of **unitarity**, **analyticity**, and **Lorentz invariance** of UV completion is encoded **secretly** in low energy EFT.

- 3. Summary (1 page.)

Quest for Fundamental theory…

- Phenomenology: Low-energy effective field theories (EFT)

Energy scale

$S_{\mathrm{UV}}[\phi, \chi, \dots]$ Fundamental theory (Superstring?)

Heavy field ($E_{\mathrm{IR}} \ll M_\chi$) →integrated out

$$e^{iS_{\mathrm{EFT}}[\phi]} = \int \mathfrak{D}\mathrm{Heavy}\ e^{iS_{UV}[\phi,\,\mathrm{Heavy}]}$$

E_{IR}

$S_{\mathrm{EFT}}[\phi]$ e.g.) : Modified Gravity models

$$\mathcal{L}_{\mathrm{EFT}} = -\frac{1}{2}(\partial\phi)^2 + \frac{c_1}{\Lambda^4}(\partial\phi)^4 + \cdots$$

Constraints on EFT parameters → seeking for signal of new physics

Positivity bounds

- Assuming UV completion is

A. Adams *et al* ('06)

① Unitary $\hat{S}\hat{S}^{+} = 1$

② Lorentz invariant

③ Analytic (↔ Causal)

④ Local $[\hat{\phi}(x), \hat{\phi}(y)] = 0$ for spacelike $(x - y)$.

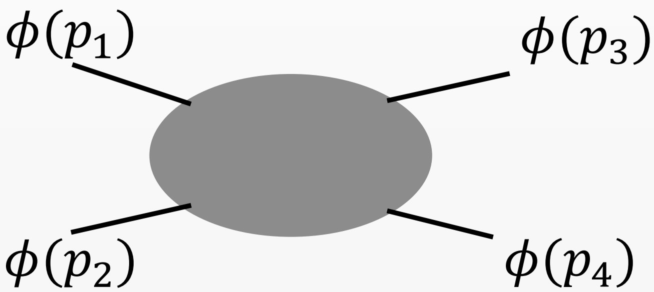
Low-energy EFT scattering amplitudes must satisfy an infinite number of inequalities.

"positivity bounds"

Essential properties of UV completion ①-④ is secretly encoded in EFT !

Positivity bounds

- Low-energy 2 to 2 scat. amplitude $F(s,t)$ is constrained.



$= F(s,t)$

wl. Mandelstam variables (s,t,u)

$s \equiv -(p_1+p_2)^2$: center of mass energy

$t \equiv -(p_1-p_3)^2$: momentum transfer

$u \equiv -(p_1-p_4)^2$

$s+t+u=4m^2$

- Positivity bounds are obtained:

A. Adams *et al* ('06)

$\partial_s^2 B(s,0) > 0$

$\partial_s^4 B(s,0) > 0$

$\partial_s^6 B(s,0) > 0$

$\vdots$

wl. $B(s,0) \equiv F(s,0) - (\text{light poles})$

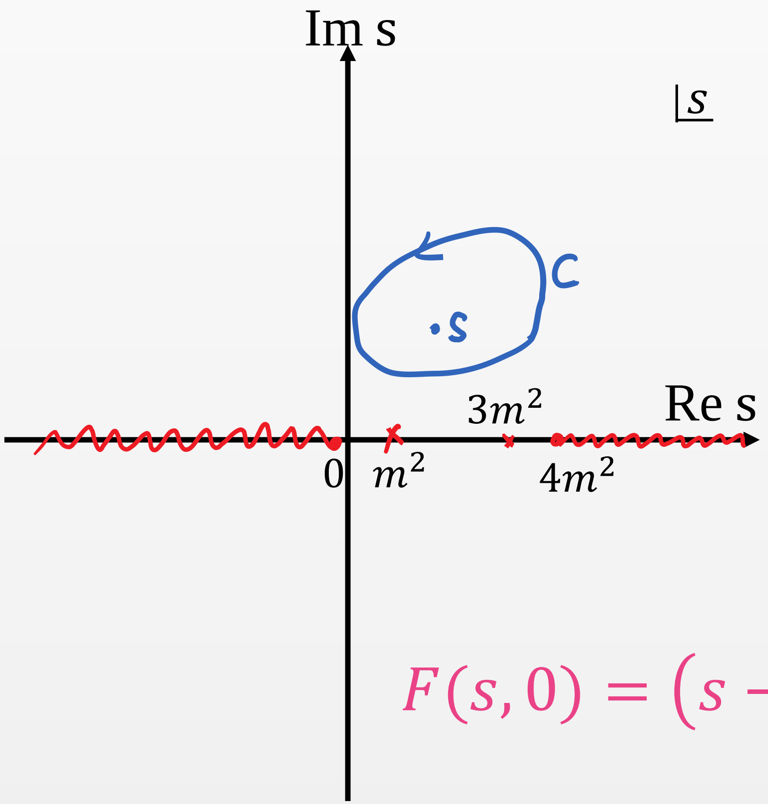
e.g.) $\mathcal{L}_{\text{EFT}} = -\frac{1}{2}(\partial\phi)^2 + \frac{c_1}{\Lambda^4}(\partial\phi)^4 + \cdots$

$c_1 > 0$

- Extended to $t > 0$ case (beyond forward limit) C. de Rham *et al.* ('17,…)

Derivation of Positivity bounds

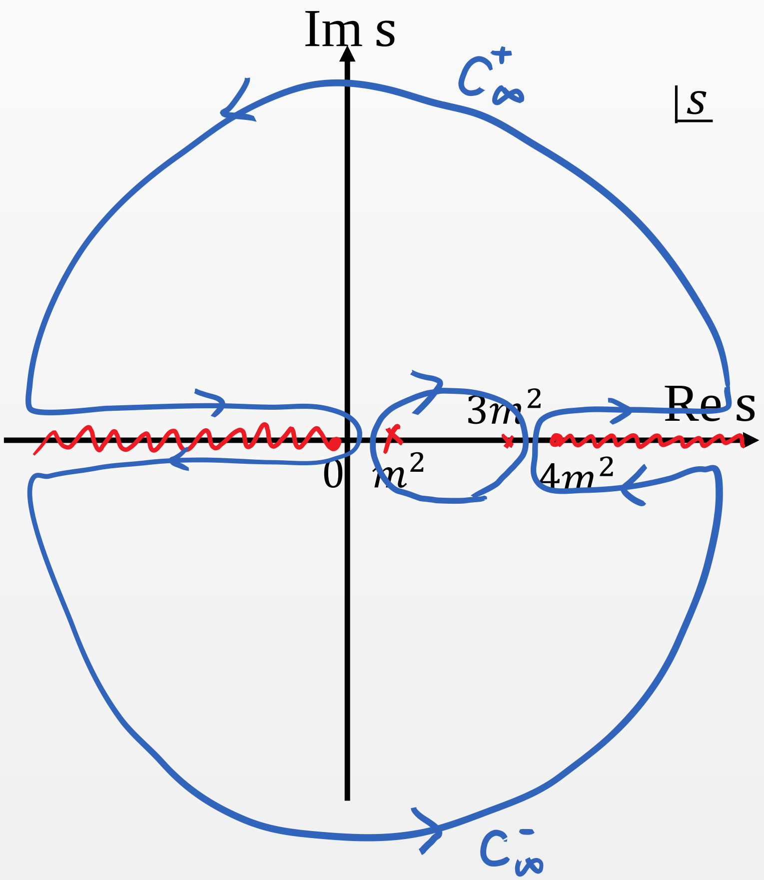
- Analytic structure in complex *s*-plane



$$F(s,0) = \left(s - 2m^2\right)^{2N} \oint_c \frac{\mathrm{d}s'}{2\pi i} \frac{F(s',0)}{(s'-s)(s'-2m^2)^{2N}}$$

Derivation of Positivity bounds

- Analytic structure in complex s-plane

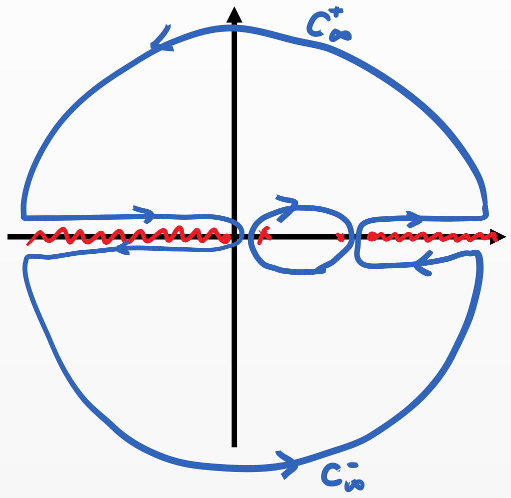


$$F(s,0) = \text{(light poles)}$$

$$+ \int_{\text{branch cuts}}$$

$$+ \int_{C_\infty^\pm}$$

Derivation of Positivity bounds



- Locality & Unitarity→2-subtraction

$$F(s,0) = (\text{light poles}) + \int_{\text{cuts}} \quad + \int_{C_\infty^\pm}$$

$$= \left[-\frac{\mathrm{Res}_{s=m^2}F(s,0)}{m^2-s} + \frac{\mathrm{Res}_{u=m^2}F(s,0)}{m^2-u}\right] + \sum_{k=0}^{2N-1} a_k s^k$$

$$+ \frac{2(s-2m^2)^{2N}}{\pi}\int_{4m^2}^{\infty} \mathrm{d}\mu \left(\frac{\mathrm{Im}\,F(\mu+i\epsilon,0)}{(\mu-2m^2)^{2N-1}[(\mu-2m^2)^2-(s-2m^2)^2]}\right)$$

$$+(s-2m^2)^{2N}\int_{C_\infty^\pm}\frac{\mathrm{d}s'}{2\pi i}\frac{F(s',0)}{(s'-s)(s'-2m^2)^{2N}}$$

Locality & Unitarity $\rightarrow \lim_{|s|\to\infty}\left|\frac{F(s,0)}{s^2}\right| = 0$ "Froissart (-like) bound"

M. Froissart (1961)

$\int_{C_\infty^\pm} \quad = 0$ for $2N \geq 2$: 2-subtraction is sufficient.

Derivation of Positivity bounds

- Locality & Unitarity→2-subtraction

$$\mathrm{B}(s,0) \equiv F(s,0) - (\text{light poles})$$

$$\partial_s^{2N} B(s,0)\Big|_{s=2m^2} = \frac{2(2N)!}{\pi}\int_{4m^2}^{\infty} \mathrm{d}\mu \left(\frac{\mathrm{Im}\, F(\mu+i\epsilon,0)}{(\mu-2m^2)^{2N+1}}\right) > 0$$

for $2N \geq 2$.

Optical theorem (**Unitarity**)

Locality & Unitarity : $\lim_{|s|\to\infty}\left|\frac{F(s,0)}{s^2}\right| = 0$

Derivation of Positivity bounds

- Positivity bounds are obtained:

① Unitarity
② Lorentz invariance
③ Analyticity
④ Locality

→

$$\partial_s^2 B(s,0) > 0$$
$$\partial_s^4 B(s,0) > 0$$
$$\partial_s^6 B(s,0) > 0$$
$$\vdots$$

- So far it's impossible to falsify unitary, analytic, and Lorentz invariant UV completion via the violation of positivity bounds.

Is it impossible to derive these bounds only from ①-③?

① Unitarity
② Lorentz invariance
③ Analyticity

→ ?

Jaffe's non-localizable theories

- Definition

A. M. Jaffe (1967)…

Growth rate of Lehmann-Källén spectral density $\rho(-k^2)$

$$\rho(-k^2) \sim (-k^2)^N \exp\left[\sigma(-k^2)^\alpha\right]$$

$0 \leq \alpha < \frac{1}{2}$: strictly localizable

$\alpha > \frac{1}{2}$: non-localizable

$\alpha = \frac{1}{2}$: quasi-local

e.g.) Little string theories : $\alpha = \frac{1}{2}$ A. Kapustin ('01)

Galileon theories : $\alpha > \frac{1}{2}$ A.J. Tolley *et al.* ('15)

- Spectral representation:

$$W(x,y) = \int \mathrm{d}\mu \;\rho(\mu)\, W_{\mathrm{free}}(x,y{:}\,\mu)$$: ill-defined for $\alpha \geq \frac{1}{2}$

↑ Wightman function of a free field with $m^2 = \mu$

- Unitary S-matrix with standard properties (such as crossing symmetry, LSZ construction, etc) can be constructed.

O. Steinmann (1970)

High-energy behavior

- Feynman propagator (spectral representation)

$$G_F(-k^2) = g(-k^2) \int \mathrm{d}\mu \frac{\rho(\mu)}{g(\mu)} \frac{-i}{k^2 + \mu - i\epsilon}$$

an entire function which ensures the convergence of the integral.

$$\Longrightarrow \left|G_F(-k^2)\right| < \left|-k^2\right|^N \exp\left[\sigma\left|-k^2\right|^\alpha\right] \quad \text{as } \left|-k^2\right| \to \infty.$$

- Similarly, 4-point time-ordered correlation functions $G_F^{(4)}$ will be

$$\left|G_F^{(4)}(s, t, \{k_i^2\})\right| < |s|^N \exp[\sigma|s|^\alpha] \quad \text{as } |s| \to \infty \ (t\text{: fixed}).$$

$$\Longrightarrow \quad |F(s,t)| < |s|^N \exp[\sigma|s|^\alpha] \text{ is expected (LSZ reduction).}$$

$\alpha < \frac{1}{2}$ case → confirmed H. Epstein *et al.* (1969)

High-energy behavior

- One can obtain bounds on partial wave amplitude $f_l(s)$

$$F(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l\left(1+\frac{2t}{s-4m^2}\right)$$

$$|f_l(s)| < (const.)\cdot\exp[\sigma s^{\alpha}]\, l^{-\frac{1}{2}}\exp\left[-\frac{ml}{\sqrt{s}}\right] \quad \text{at large } l.$$

$$|f_l(s)| \le 1 : \text{unitarity bound}$$

$$\Longrightarrow |F(s,0)| < \sum_{l=0}^{s^{\alpha-\frac{1}{2}}}(2l+1) \sim s^{1+2\alpha}$$

Consistent with V.F.Fainberg *et al.* (1971)

$$\Longrightarrow |F(s,0)| < |s|^{1+2\alpha}\ (\alpha < 1)$$

Lindelof-Phragmen theorem

Derivation of Positivity bounds

- Non-Locality & Unitarity→$2+\alpha$-subtraction ($\alpha < 1$)

$$\mathrm{B}(s,0) \equiv F(s,0) - (\text{light poles})$$

$$\partial_s^{2N} B(s,0)\Big|_{s=2m^2} = \frac{2(2N)!}{\pi}\int_{4m^2}^{\infty} \mathrm{d}\mu \left(\frac{\mathrm{Im}\, F(\mu+i\epsilon,0)}{(\mu-2m^2)^{2N+1}}\right) > 0$$

for $2N > 1+2\alpha$. ⟵ $|F(s,0)| < |s|^{1+\alpha}$

- Existence of bounds for non-local case ($\frac{1}{2} \leq \alpha < 1$)

$$\partial_s^2 B(s,0) > 0$$

$$\partial_s^4 B(s,0) > 0$$

$$\partial_s^6 B(s,0) > 0$$

$$\vdots$$

⟵ Must be satisfied even if theory is not strictly-localizable! (as long as $\alpha < 1$)

$0 \leq \alpha < \frac{1}{2}$: strictly localizable

$\alpha > \frac{1}{2}$: non-localizable

$\alpha = \frac{1}{2}$: quasi-local

Summary

- So far,…Positivity bounds are obtained assuming locality

① Unitarity
② Lorentz invariance
③ Analyticity
④ Locality

$\Rightarrow$

$$\partial_s^2 B(s,0) > 0$$
$$\partial_s^4 B(s,0) > 0$$
$$\partial_s^6 B(s,0) > 0$$
$$\vdots$$

- We derived bounds without locality.

① Unitarity
② Lorentz invariance
③ Analyticity

$\Rightarrow$

$$\partial_s^4 B(s,0) > 0$$
$$\partial_s^6 B(s,0) > 0$$
$$\vdots$$

(as long as $\alpha < 1$)

- Our results open the new possibility to falsify unitary, analytic, and Lorentz invariant UV completion via the violation of positivity bounds, even if EFT is apparently Lorentz inv.