Operational probabilistic theories and cellular automata: how I learned to stop worrying and love C* algebras

School on Advanced Topics in Quantum Information and Foundations
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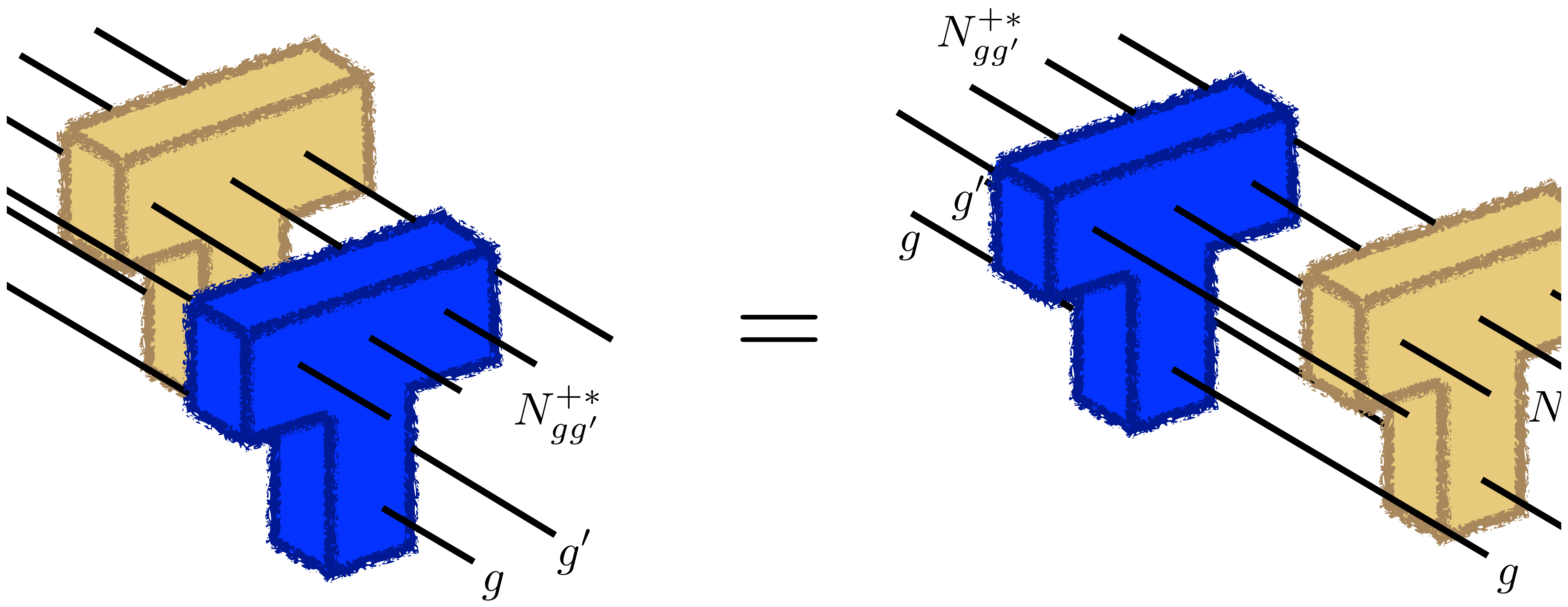
QUit quantum information theory group

Paolo Perinotti - February 8-12 2021

Lecture 3
Update rules in OPTs

Summary

- Infinite CAs: quasi-local effects and algebra
  - Topological closure: sup-norm and op-norm
- States
- Global update rules
- Cellular automata
- Neighbourhood



Causal theories

- In causal theories every conditional test is allowed
    - If $[\![\mathrm{I} \to \mathrm{I}]\!] \neq \{0,1\}$ then
        - $[\![\mathrm{I} \to \mathrm{I}]\!] = [0,1]$
        - all the sets are convex
    - Every system A has a unique deterministic effect e_{A}

Infinite composite systems

Starting from the quantum case

- In the quantum case: quasi-local algebra
  - Bonus 1: definition of QCA through local action on effects

$$(\rho| \overset{\mathrm{A}}{\text{—}} \boxed{\mathscr{U}} \overset{\mathrm{A}'}{\text{—}} |Q) = (\rho| \overset{\mathrm{A}}{\text{—}} |\mathscr{U}^{\dagger}(Q)) = (\hat{\mathscr{U}}(\rho)| \overset{\mathrm{A}'}{\text{—}} |Q)$$

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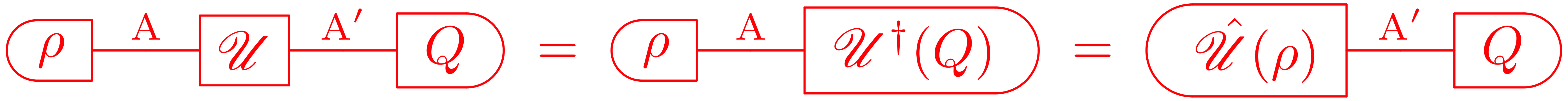
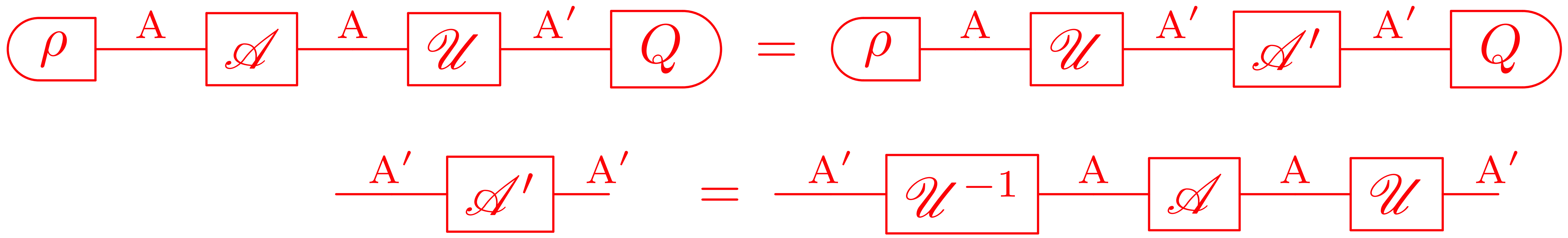


  - Bonus 2: evolution of transformations

$\forall \rho, a$



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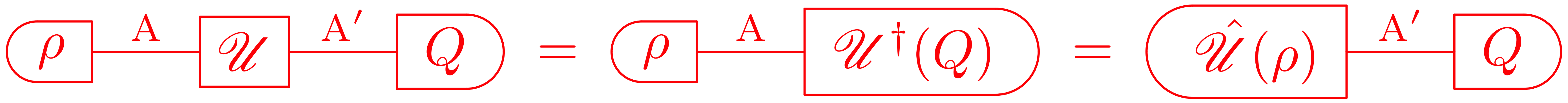

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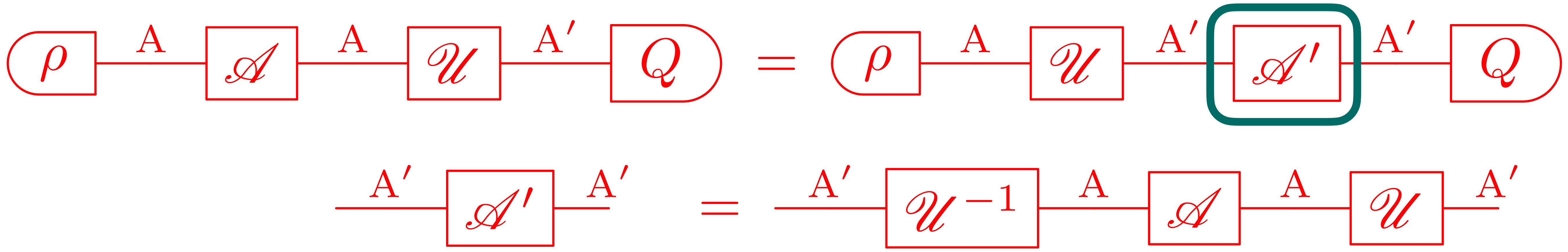

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$$\left(\rho\right) \xrightarrow{\mathrm{A}} \boxed{\mathscr{U}} \xrightarrow{\mathrm{A}'} \left(Q\right) = \left(\rho\right) \xrightarrow{\mathrm{A}} \left(\mathscr{U}^{\dagger}(Q)\right) = \left(\hat{\mathscr{U}}(\rho)\right) \xrightarrow{\mathrm{A}'} \left(Q\right)$$

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$$\xrightarrow{\mathrm{A}'} \boxed{\mathscr{A}'} \xrightarrow{\mathrm{A}'} = \xrightarrow{\mathrm{A}'} \boxed{\mathscr{U}^{-1}} \xrightarrow{\mathrm{A}} \boxed{\mathscr{A}} \xrightarrow{\mathrm{A}} \boxed{\mathscr{U}} \xrightarrow{\mathrm{A}'}$$

$$\hat{\mathscr{U}}(\rho) = U\rho U^{\dagger}, \quad \mathscr{A}(\rho) = \sum_i K_i \rho K_i^{\dagger} \qquad \Rightarrow \qquad \mathscr{A}'(\rho) = \sum_i H_i \rho H_i^{\dagger}, \quad H_i = U K_i U^{\dagger} = (\mathscr{U}^{-1})^{\dagger}(K_i)$$

Infinite composite systems

In general OPTs

- Difference: in OPTs effects are not an algebra

$$a \in [\![\bar{\mathrm{A}}]\!], \quad b \in [\![\bar{\mathrm{A}}]\!], \quad ab = ?$$

- Problem: locality on effects does not grant locality on transformations

$$\overset{\mathrm{A}'}{\longrightarrow}\boxed{\mathscr{A}'}\overset{\mathrm{A}'}{\longrightarrow} \;=\; \overset{\mathrm{A}'}{\longrightarrow}\boxed{\mathscr{U}^{-1}}\overset{\mathrm{A}}{\longrightarrow}\boxed{\mathscr{A}}\overset{\mathrm{A}}{\longrightarrow}\boxed{\mathscr{U}}\overset{\mathrm{A}'}{\longrightarrow}$$

- In OPTs transformations of a given system are an algebra
- In view of these considerations we will define quasi-local transformations, and adapt the definition of CA

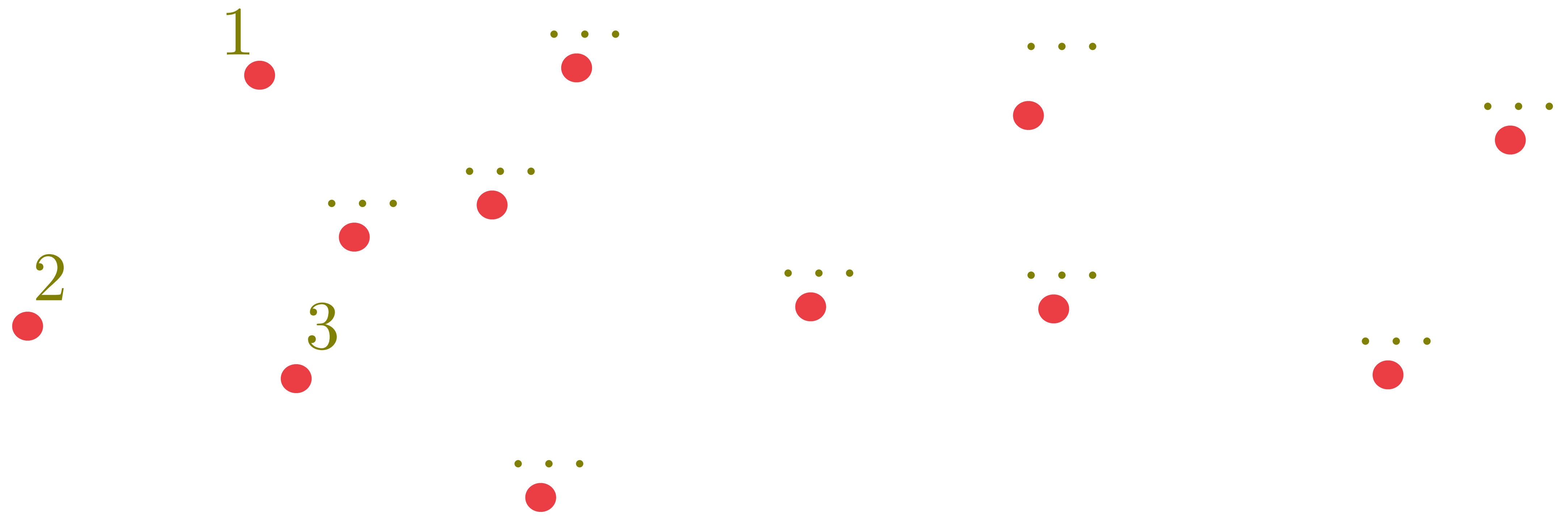
Important difference

- In the definition of QCA: assumed structure of the array of cells $\mathbb{Z}^d$
- We will avoid this, and reconstruct the structure from the CA itself
  - Every cell is a system of the OPT at hand
  - We want to make sense of infinite arrays: infinite composite systems

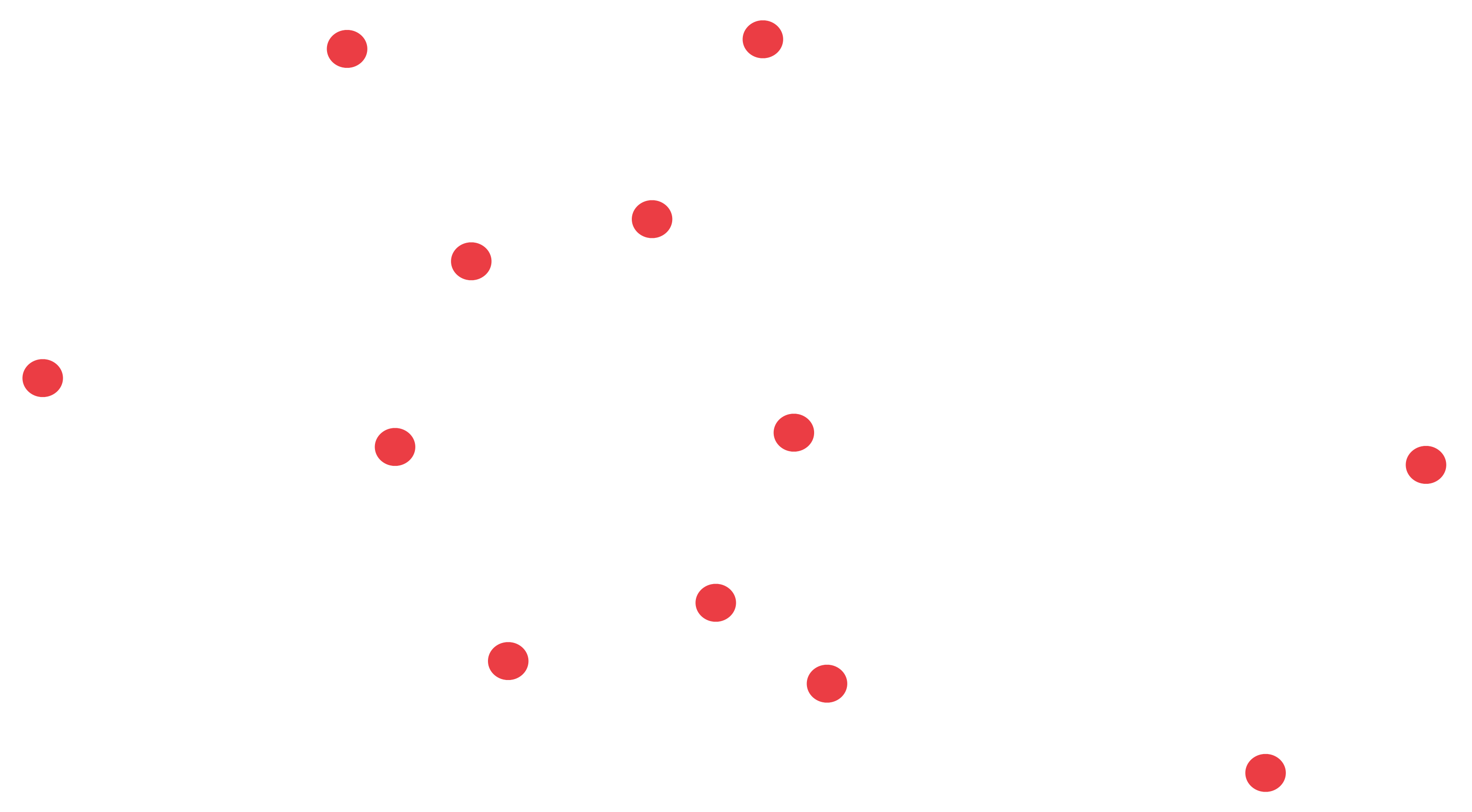
Infinite composite systems

In causal OPTs

- Overarching assumption: denumerable set G of systems
  - Infinite case Address of a cell: $n \in \mathbb{N}$ no immediate geometric meaning



Local effects



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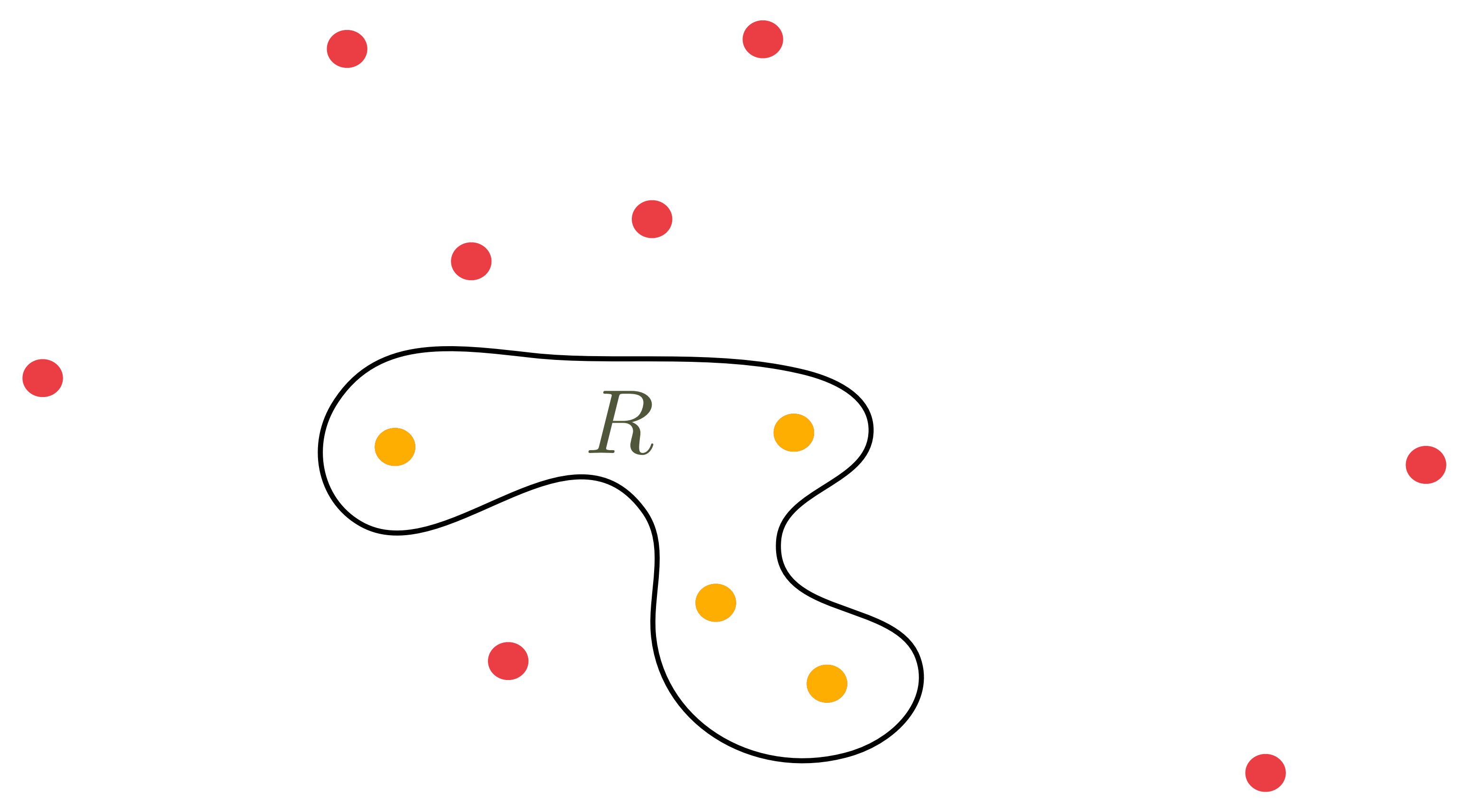

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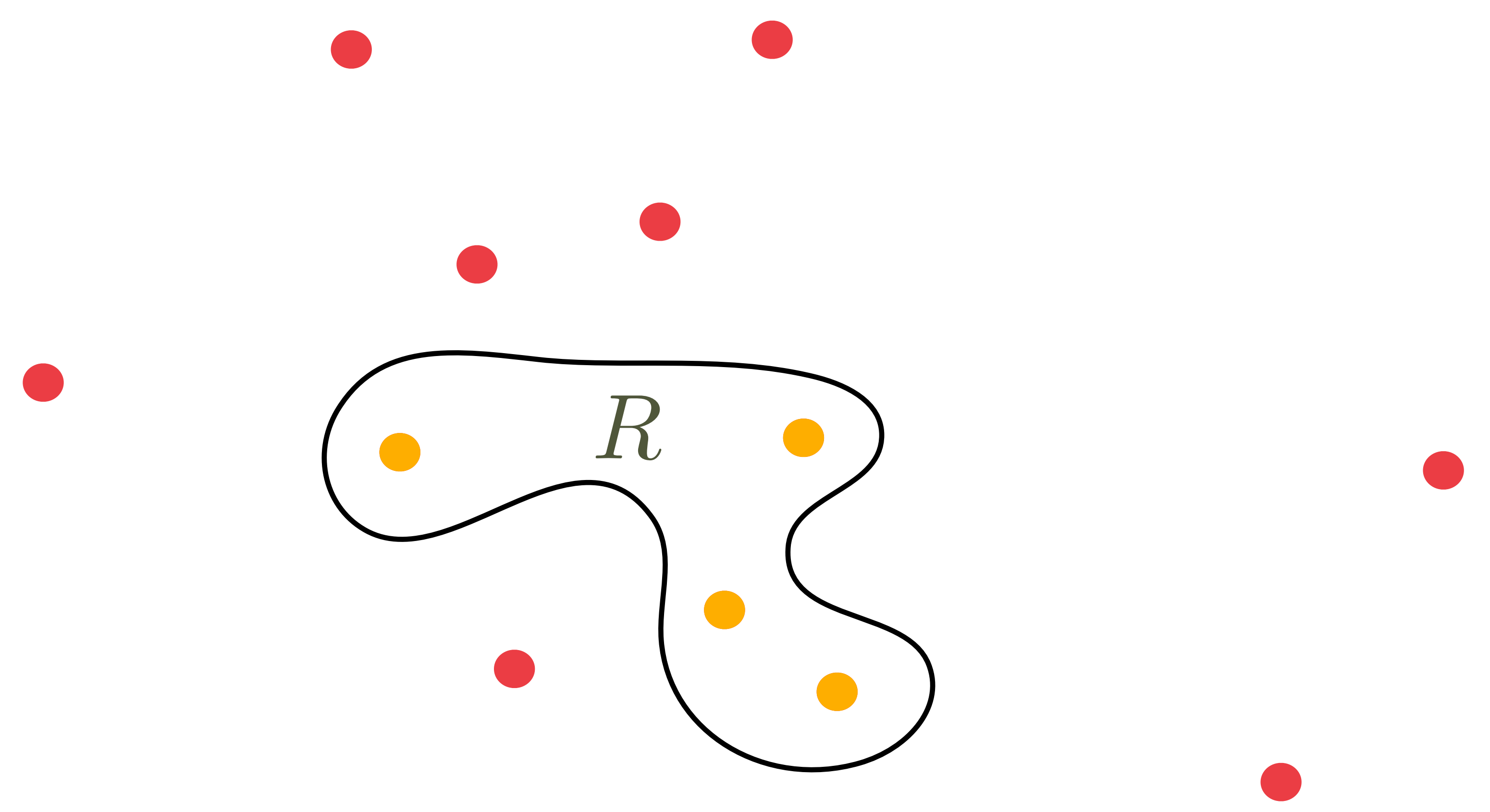

- Finite region: finite subset R of G;
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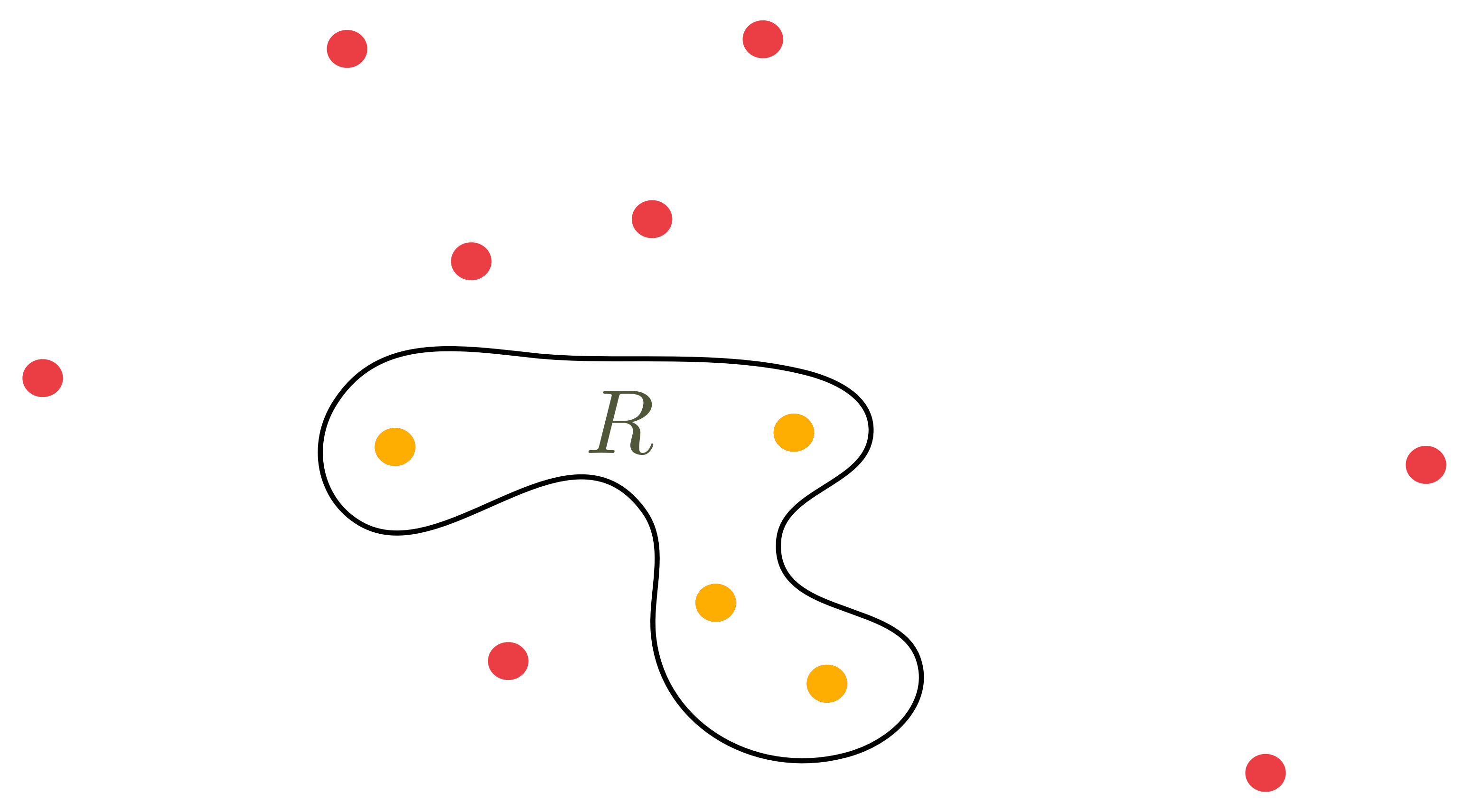

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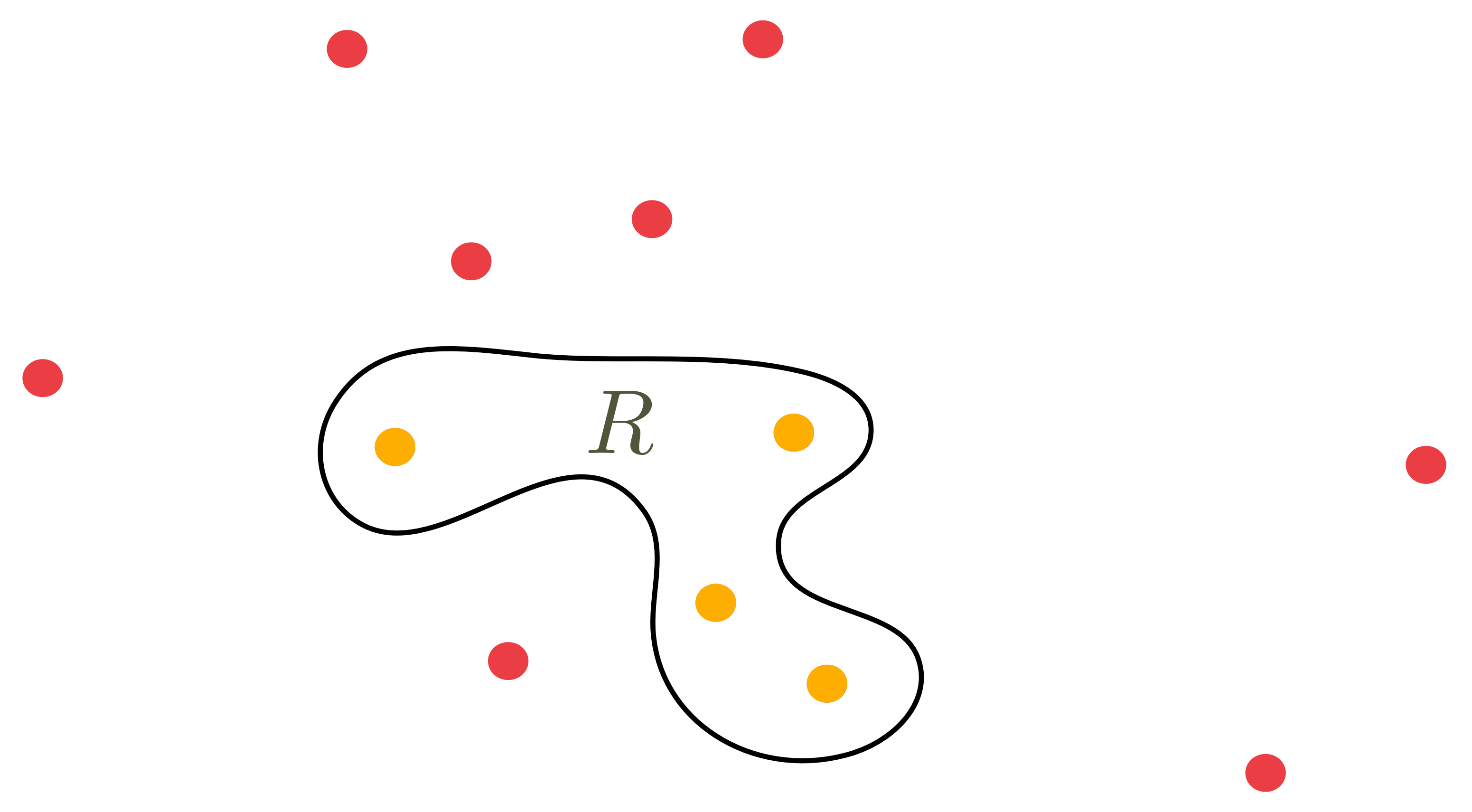

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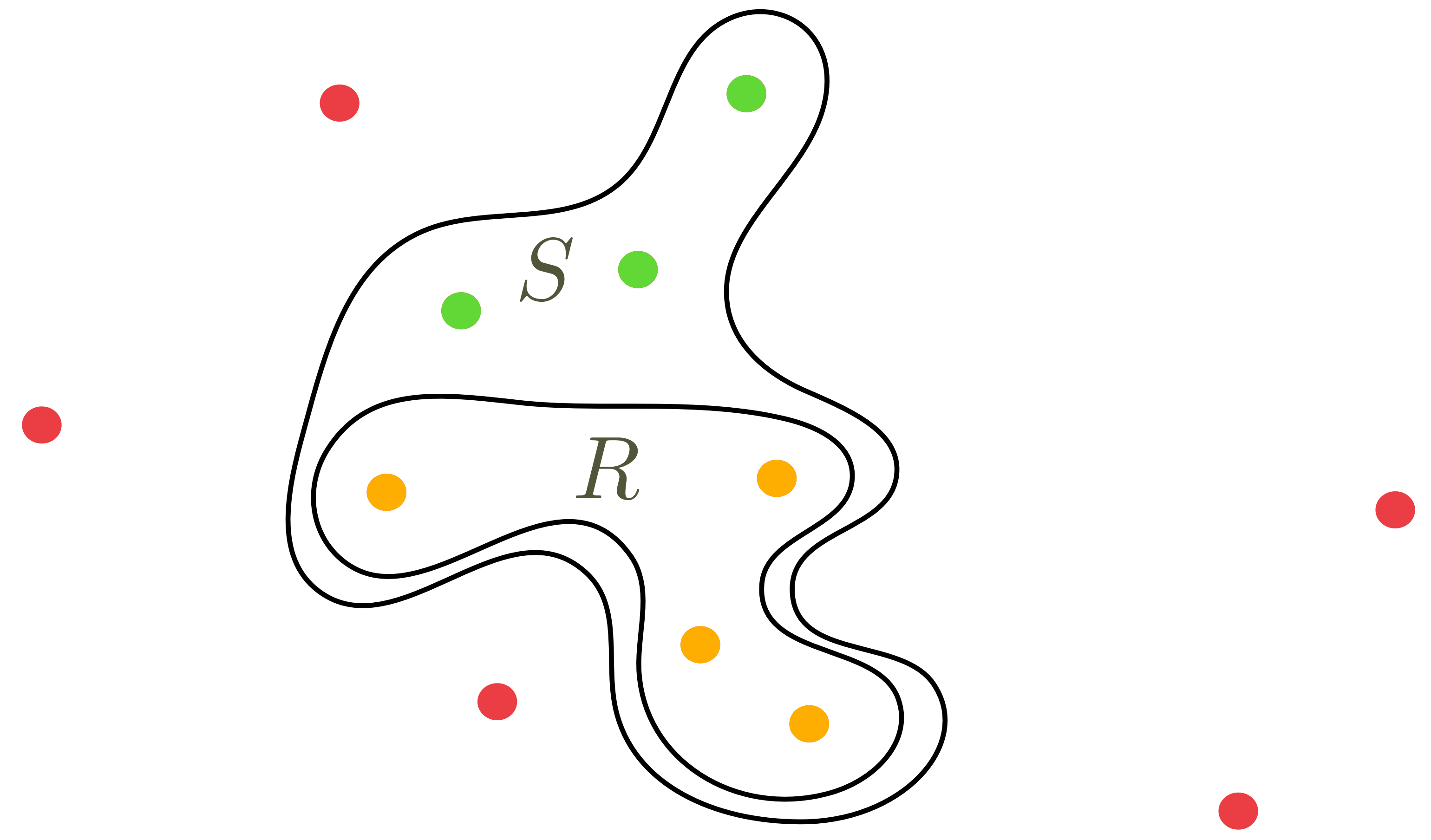
- We collect $\bigsqcup_{R \subseteq G} [\![\bar{\mathrm{A}}_R]\!]_{\mathbb{R}}$
- We define injection functions:
  let $R \subseteq S$, then $f_{R,S} : [\![\bar{\mathrm{A}}_R]\!]_{\mathbb{R}} \to [\![\bar{\mathrm{A}}_S]\!]$
  maps $f_{R,S}(a_R) = (a \otimes e_{S \setminus R})_S$

Local effects



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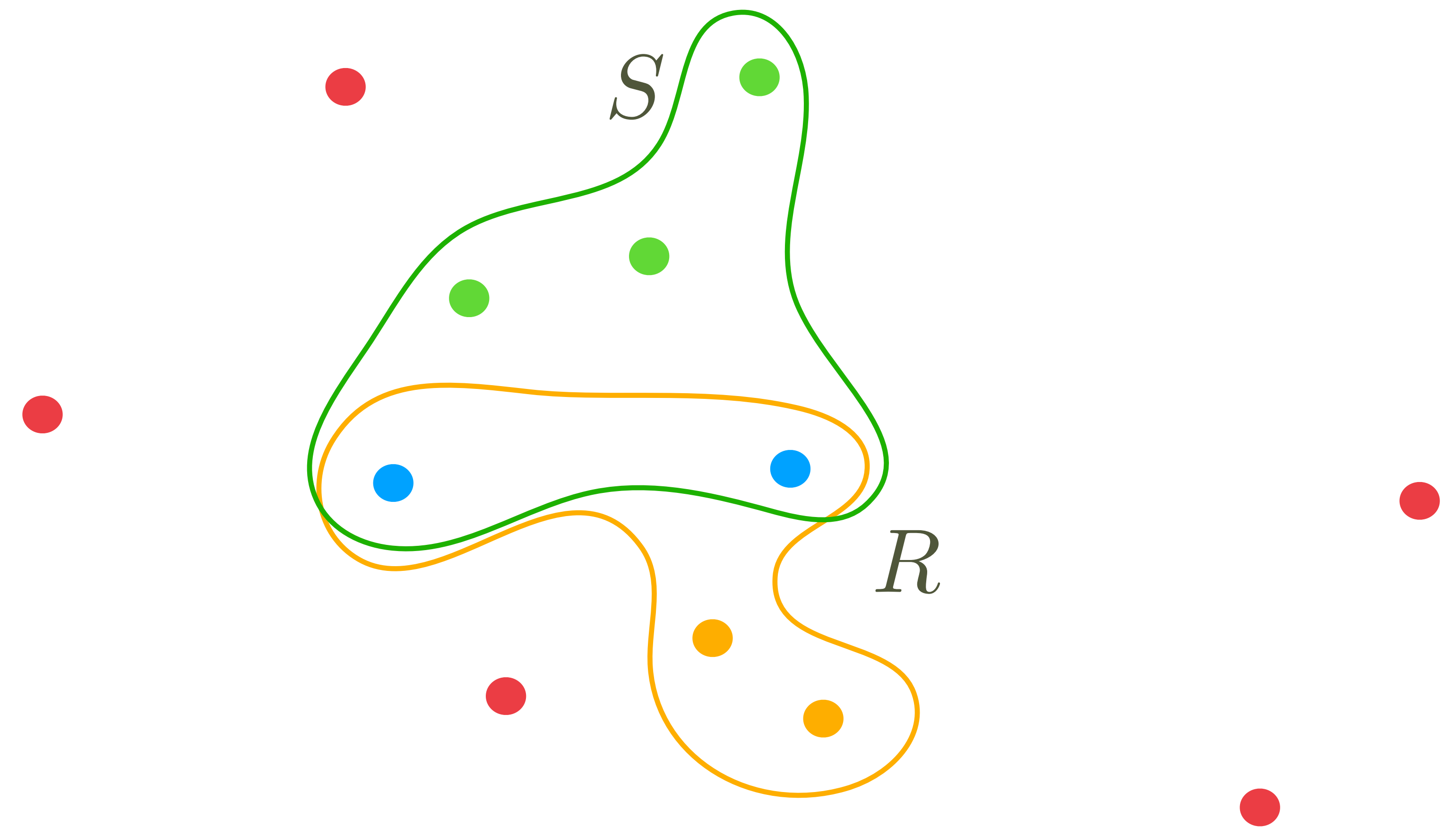
Inductive limit



- Equivalence relation: $a_R \sim b_S$ if

$$\begin{cases} f_{R,R\cup S}(a) = (a_0 \otimes e_{S\setminus R})_{R\cup S} \\ f_{S,R\cup S}(b) = (a_0 \otimes e_{R\setminus S})_{R\cup S} \end{cases}$$

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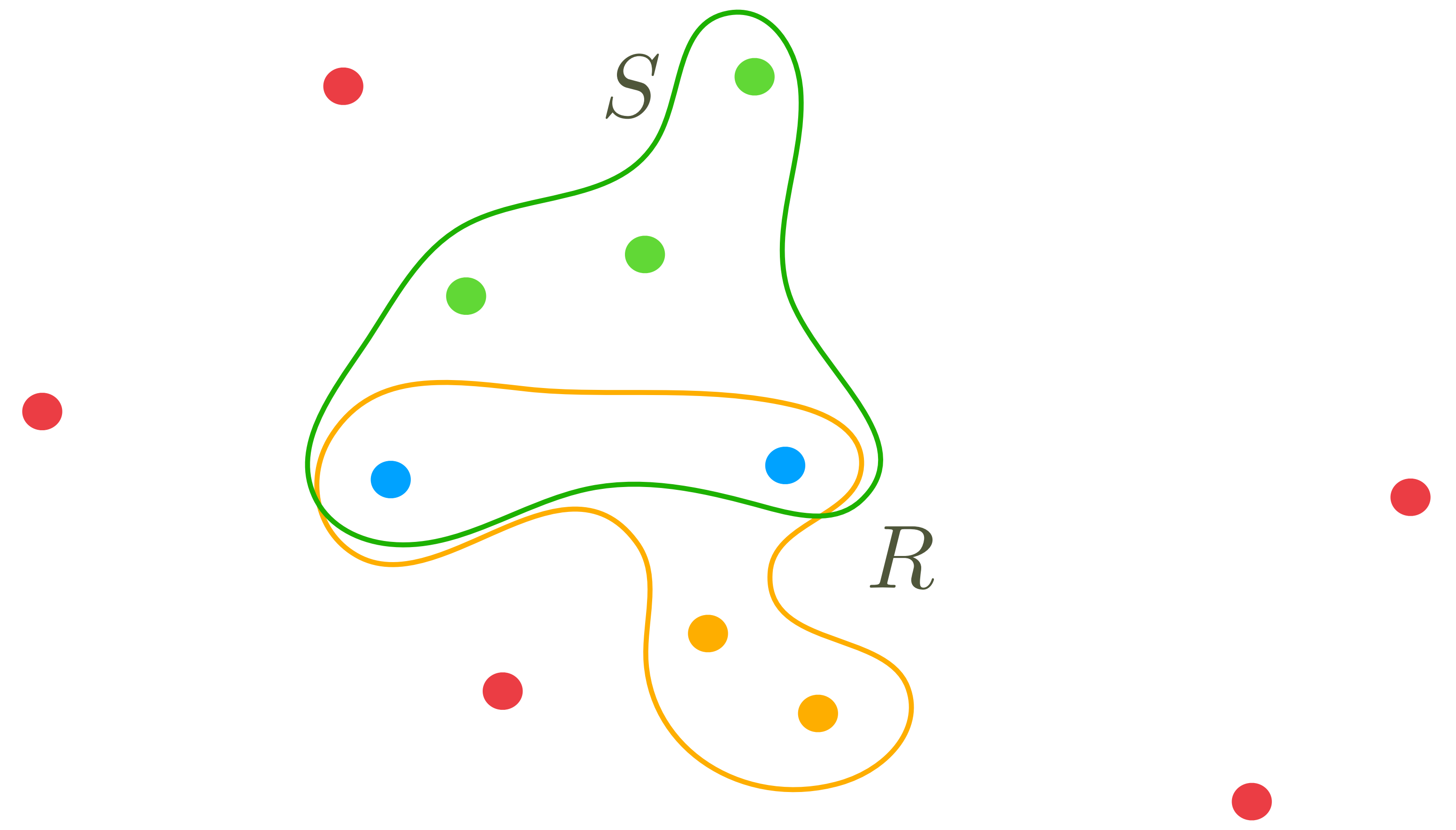

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- Domain set: $[\![\bar{\mathrm{A}}_G]\!]_{L\mathbb{R}} := \bigsqcup_{R\subseteq G} [\![\bar{\mathrm{A}}_R]\!] / \sim$

Inductive limit

Vector space structure

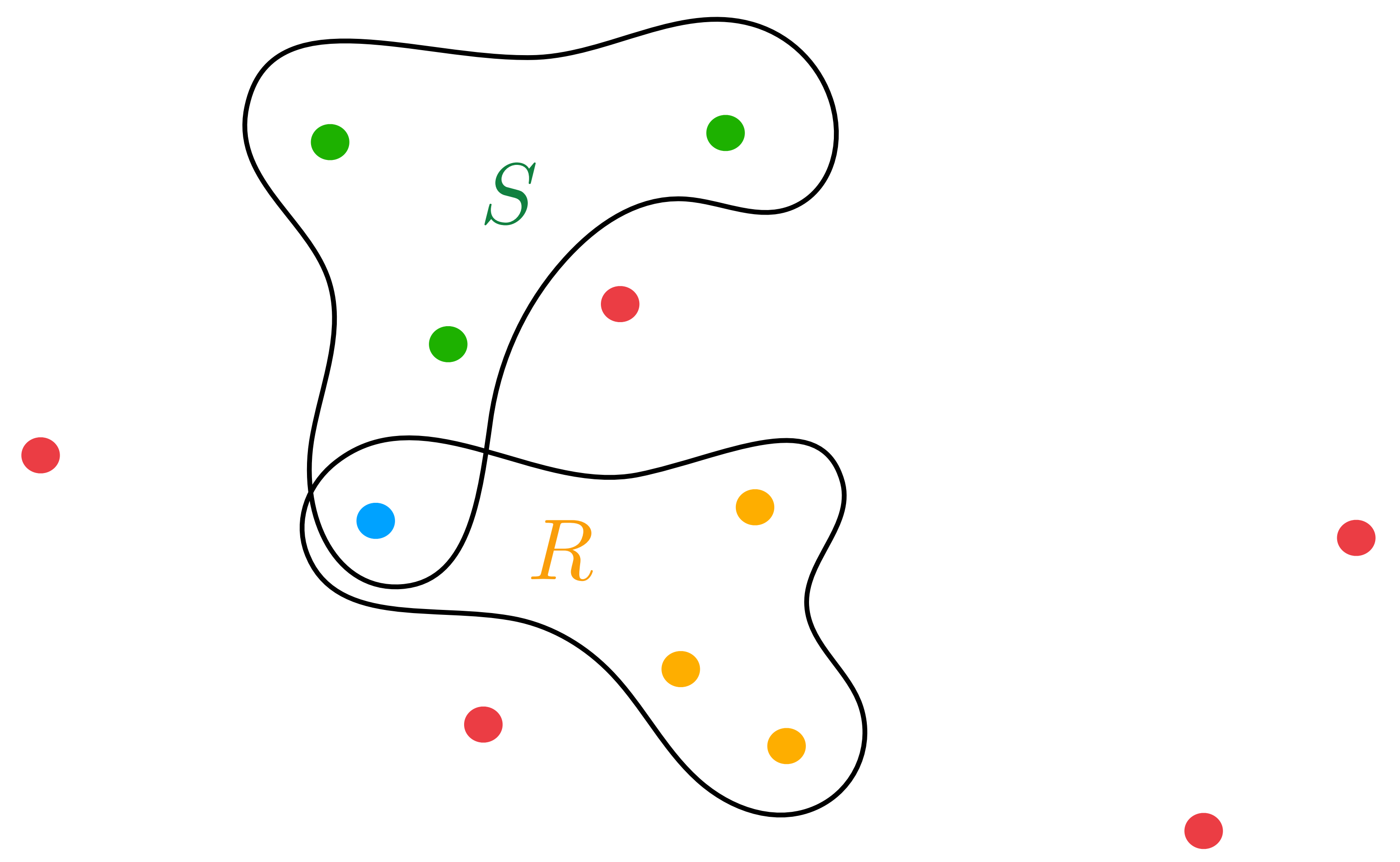
- Sum of local effects

$$a_R + b_S := c_{R\cup S}$$
$$c := a \otimes e_{S\setminus R} + b \otimes e_{R\setminus S}$$

- Multiplication by a real number

$$ha_R := \begin{cases} (ha)_R & h \neq 0 \\ 0_\emptyset & h = 0 \end{cases}$$

- The above operations equip $[\![\bar{\mathrm{A}}]\!]_{L\mathbb{R}}$ with a real vector space structure



Topology of local effects

Operational norm and sup norm

- Operational norm for effects

$$\|a\|_{\mathrm{op}} = \sup_{\rho \in [\![\mathrm{A}]\!]} |(a|\rho)|$$

- We will use the sup-norm

$$J(a) := \{\lambda \in \mathbb{R} \mid \lambda e \pm a \succeq 0\}$$
$$\|a\|_{\sup} := \inf J(a)$$

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- The sup norm is stronger

$$\|a\|_{\mathrm{op}} \leq \|a\|_{\sup}$$

Quasi local effects

Closure of the inductive limit

- Space $[\![\bar{\mathrm{A}}_G]\!]_{C\mathbb{R}}$: Cauchy sequences in $[\![\bar{\mathrm{A}}_G]\!]_{L\mathbb{R}}$

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- We define $[\![\bar{\mathrm{A}}_G]\!]_Q \subseteq [\![\bar{\mathrm{A}}_G]\!]_{Q+}$
- Unique deterministic effect: e_G

Generalized extended States

- Generalised extended state space $[\![\mathrm{A}_G]\!]_{\mathbb{R}}$: space of bounded linear functionals on $[\![\bar{\mathrm{A}}_G]\!]_{Q\mathbb{R}}$
- Norm in $[\![\mathrm{A}_G]\!]_{\mathbb{R}}$: $\|\rho\|_* := \sup_{\|a\|_{\sup}=1} |(a|\rho)|.$
- Criterion for proper extended states: they must locally "look like" states

Extended States

- Local restriction on $R \subseteq G$: given $\rho \in [\![\mathrm{A}_G]\!]_{\mathbb{R}}$ we define a functional on $[\![\bar{\mathrm{A}}_R]\!]_{\mathbb{R}}$ as follows

$$\forall a \in [\![\bar{\mathrm{A}}_R]\!]_{\mathbb{R}}, \ (a|\rho_{|R}) := (a_R|\rho)$$

- A generalised extended state is a proper state if $\forall R \subseteq G \quad \rho_{|R} \in [\![\mathrm{A}_R]\!]$

- We can define $[\![\mathrm{A}_G]\!]_+$ and $[\![\mathrm{A}_G]\!]_1$

Quasi local algebra

Transformations instead of effects

- Local transformation: a pair $(\mathscr{A}, R)$ where $R \subseteq G$, $\mathscr{A} \in [\![\mathrm{A}_R \to \mathrm{A}_R]\!]_{\mathbb{R}}$

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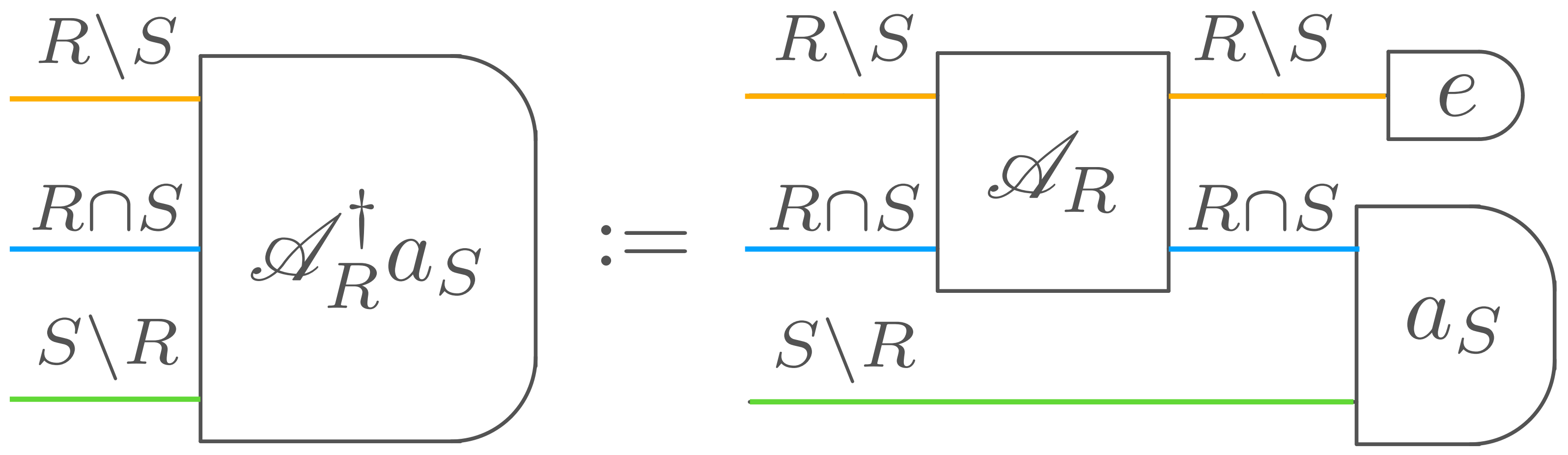

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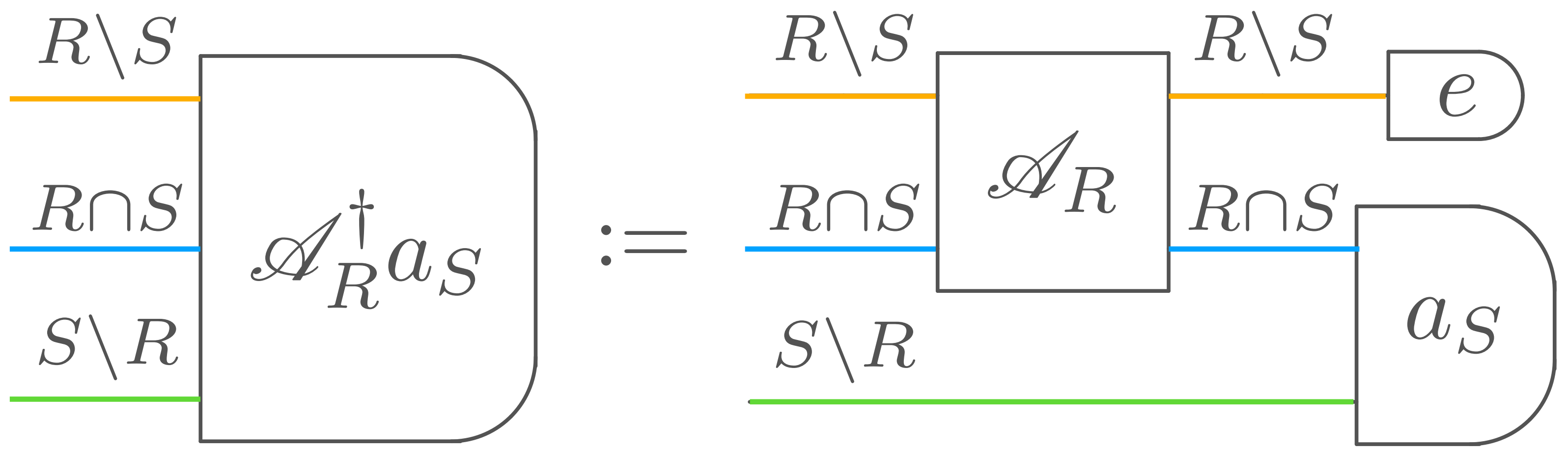
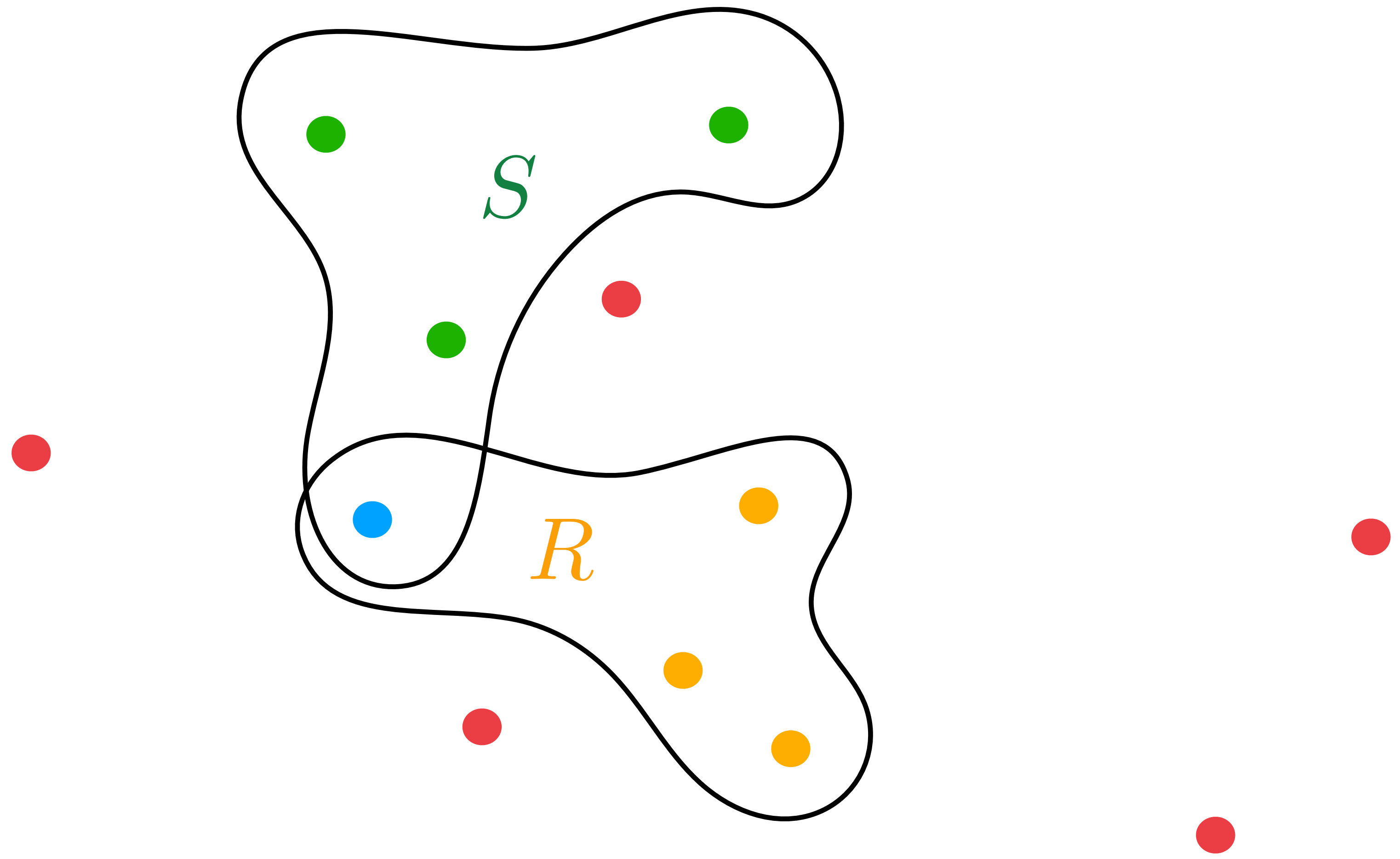
$$\mathscr{A}_R \sim \mathscr{A}'_S \qquad \begin{cases} \mathscr{A} = \mathscr{A}_0 \otimes \mathscr{I}_{S\setminus R} \\ \mathscr{A}' = \mathscr{A}_0 \otimes \mathscr{I}_{R\setminus S} \end{cases}$$





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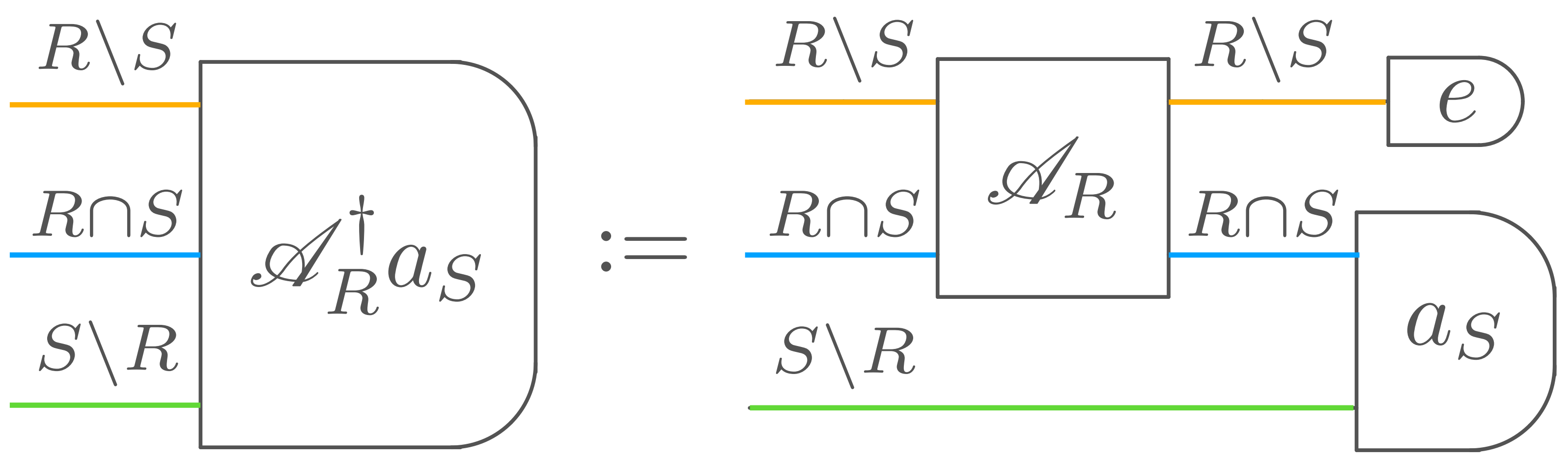

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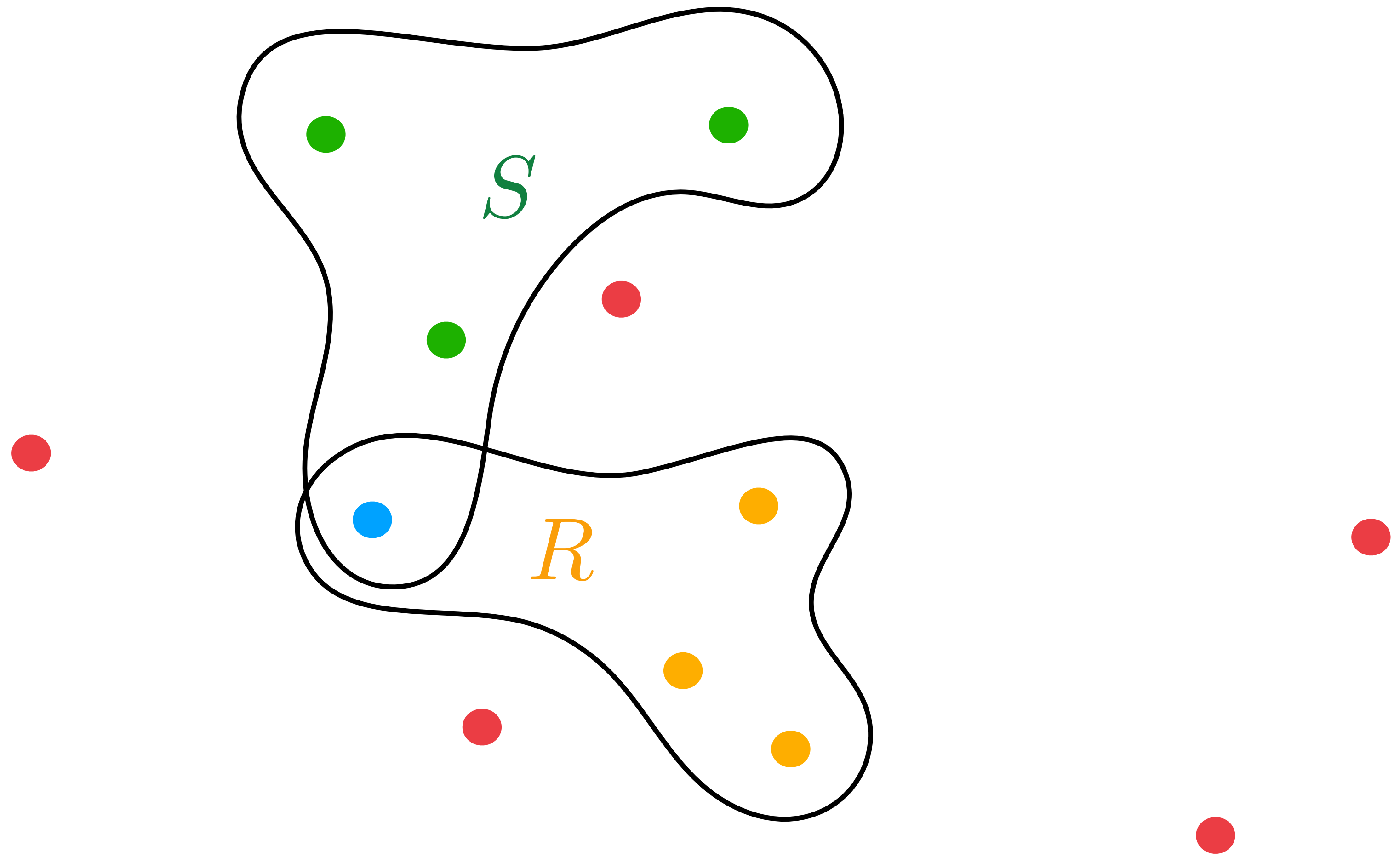

- Domain set:

$$[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{L\mathbb{R}} = \bigsqcup_R \{\mathscr{A}_R \mid \mathscr{A} \in [\![\mathrm{A}_R \to \mathrm{A}_R]\!]\}/\simeq$$

Inductive limit

Algebra structure

- Sum of local transformations and multiplication by a real number

$$h\mathscr{A}_R := \begin{cases} (h\mathscr{A})_R & h \neq 0, \\ 0_\emptyset & h = 0, \end{cases}$$

$$\mathscr{A}_R + \mathscr{B}_S := \mathscr{C}_{R\cup S}$$

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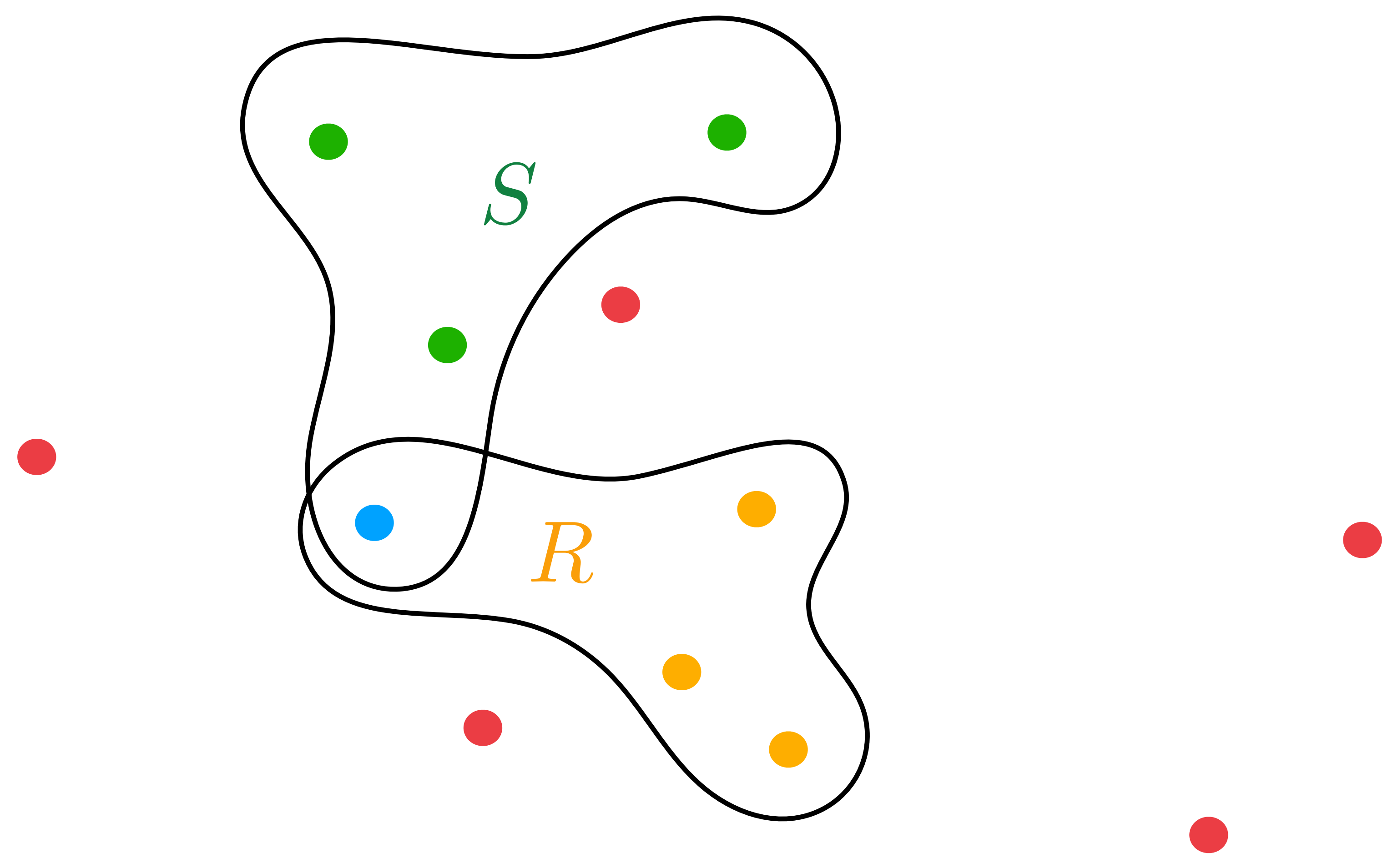

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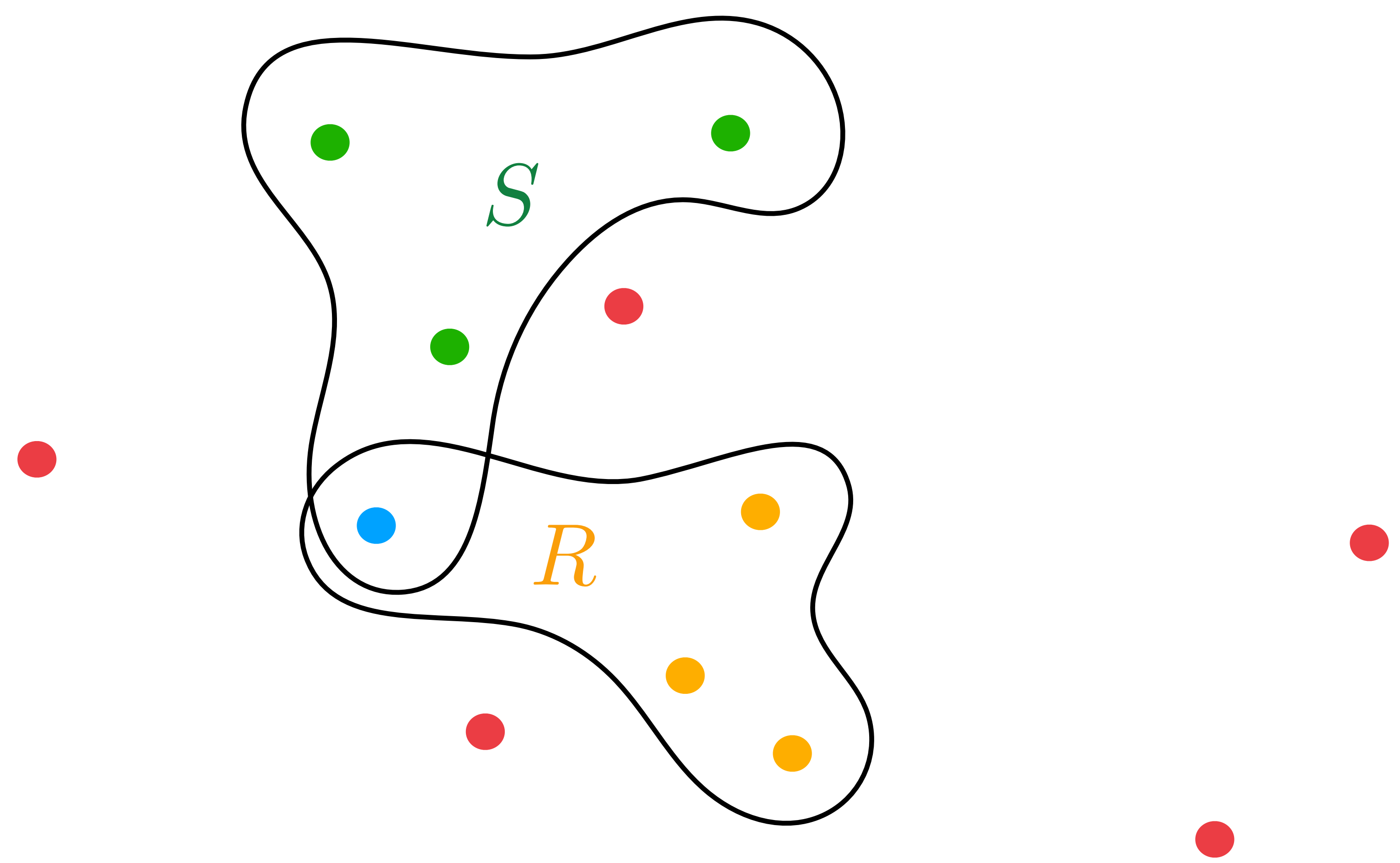
- Composition of local transformations

$$\mathscr{A}_R\mathscr{B}_S := (\{\mathscr{A} \otimes \mathscr{I}_{S\setminus R}\}\{\mathscr{B} \otimes \mathscr{I}_{R\setminus S}\})_{R\cup S}.$$



Topology

Operational norm and sup norm

- Topology given by the sup-norm

$$J(\mathscr{A}) := \{\lambda \in \mathbb{R} \mid \exists \mathscr{C} \in [\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{L1},\ \lambda\mathscr{C} \pm \mathscr{A} \geq 0\}$$
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- The limit of product sequences is the product of limits

- Closure in the operational norm would not be an algebra

Quasi local algebra

Closure of the inductive limit

- Algebra $[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{C\mathbb{R}}$ of Cauchy sequences
- Equivalence relation: $\mathscr{A}_n \simeq \mathscr{B}_n$ if $\lim\limits_{n\to\infty} \|\mathscr{A}_n - \mathscr{B}_n\|_{\sup} = 0$
- Quasi-local algebra $[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q\mathbb{R}}$: $[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{C\mathbb{R}}/ \simeq$
- Local transformations make up a subalgebra
- We define $[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q1} \subseteq [\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q} \subseteq [\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q+}$

Action on quasi-local effects

- Action of $[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q\mathbb{R}}$ on $[\![\bar{\mathrm{A}}_G]\!]_{Q\mathbb{R}}$

  $\mathscr{A}^\dagger a = \lim_{m,n\to\infty} \mathscr{A}_m^\dagger a_n$

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- Main result

  $\forall a \in [\![\bar{\mathrm{A}}_G]\!]_{Q*}\ \exists \mathscr{A} \in [\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q*},\ \text{s.t. } a = \mathscr{A}^\dagger e_G$

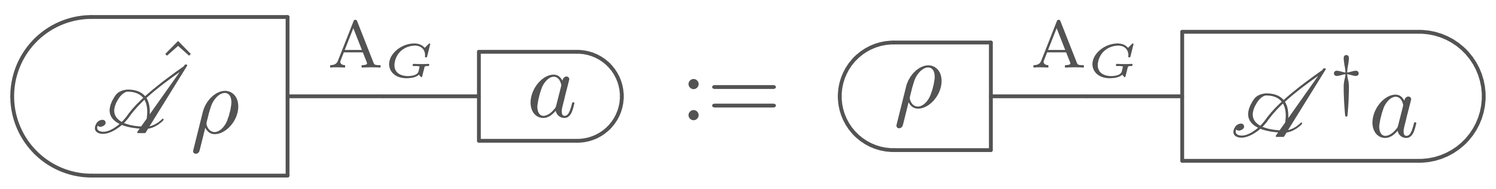
  $* = \text{nothing}, +, 1, \mathbb{R}$

Action on quasi-local effects

- Action of $[\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q\mathbb{R}}$ on $[\![\bar{\mathrm{A}}_G]\!]_{Q\mathbb{R}}$
  $\mathscr{A}^\dagger a = \lim_{m,n\to\infty} \mathscr{A}_m^\dagger a_n$

- Main result
  $\forall a \in [\![\bar{\mathrm{A}}_G]\!]_{Q*}\ \exists \mathscr{A} \in [\![\mathrm{A}_G \to \mathrm{A}_G]\!]_{Q*},\ \text{s.t. } a = \mathscr{A}^\dagger e_G$
  $* = \text{nothing}, +, 1, \mathbb{R}$

- Dual action on $[\![\mathrm{A}_G]\!]_{\mathbb{R}}$: $\forall a \in [\![\bar{\mathrm{A}}_G]\!]_{Q\mathbb{R}}$



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- We need to build consistent families also for CAs

Automorphic families

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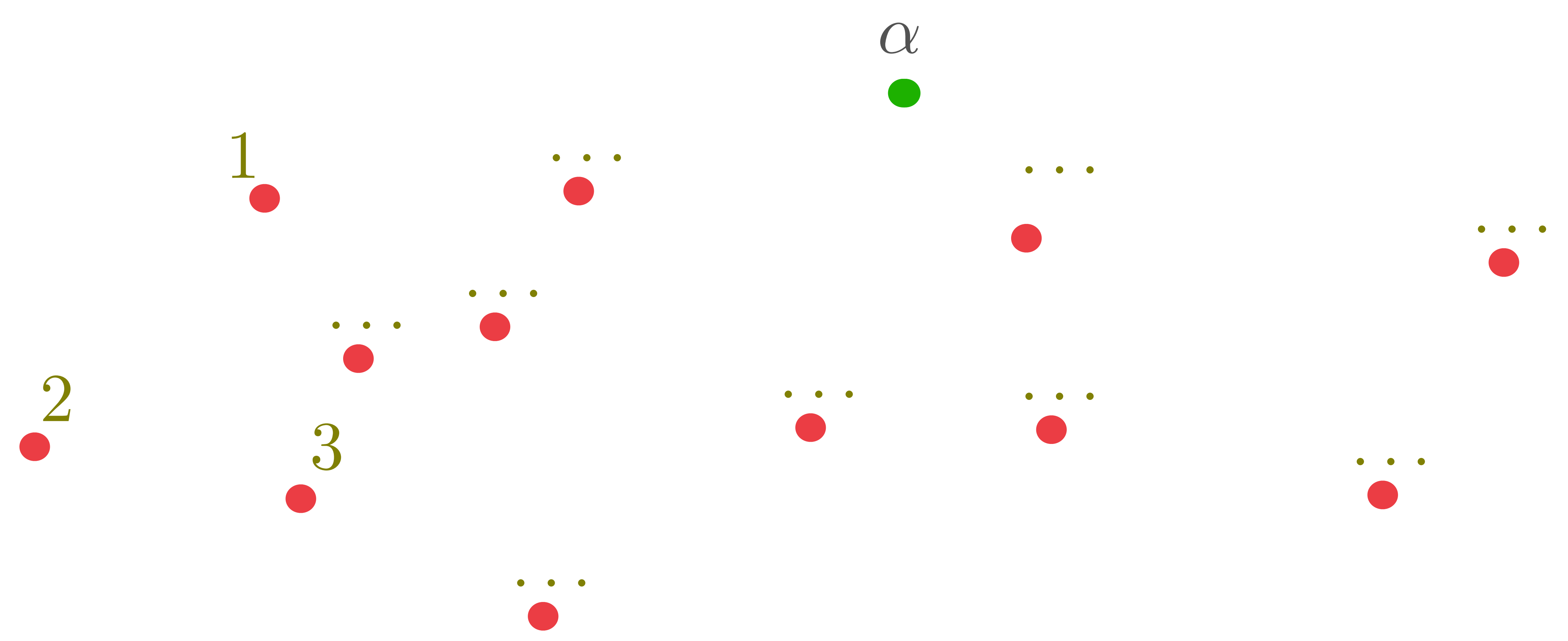

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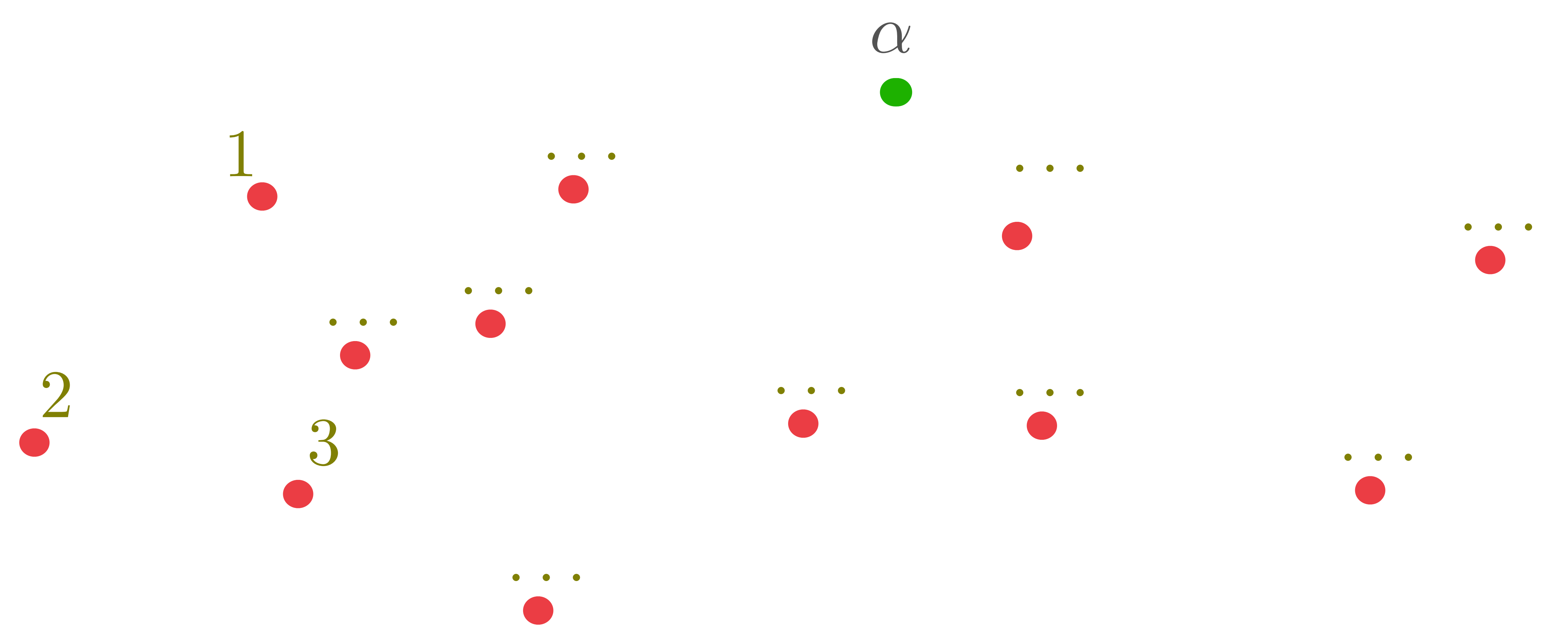

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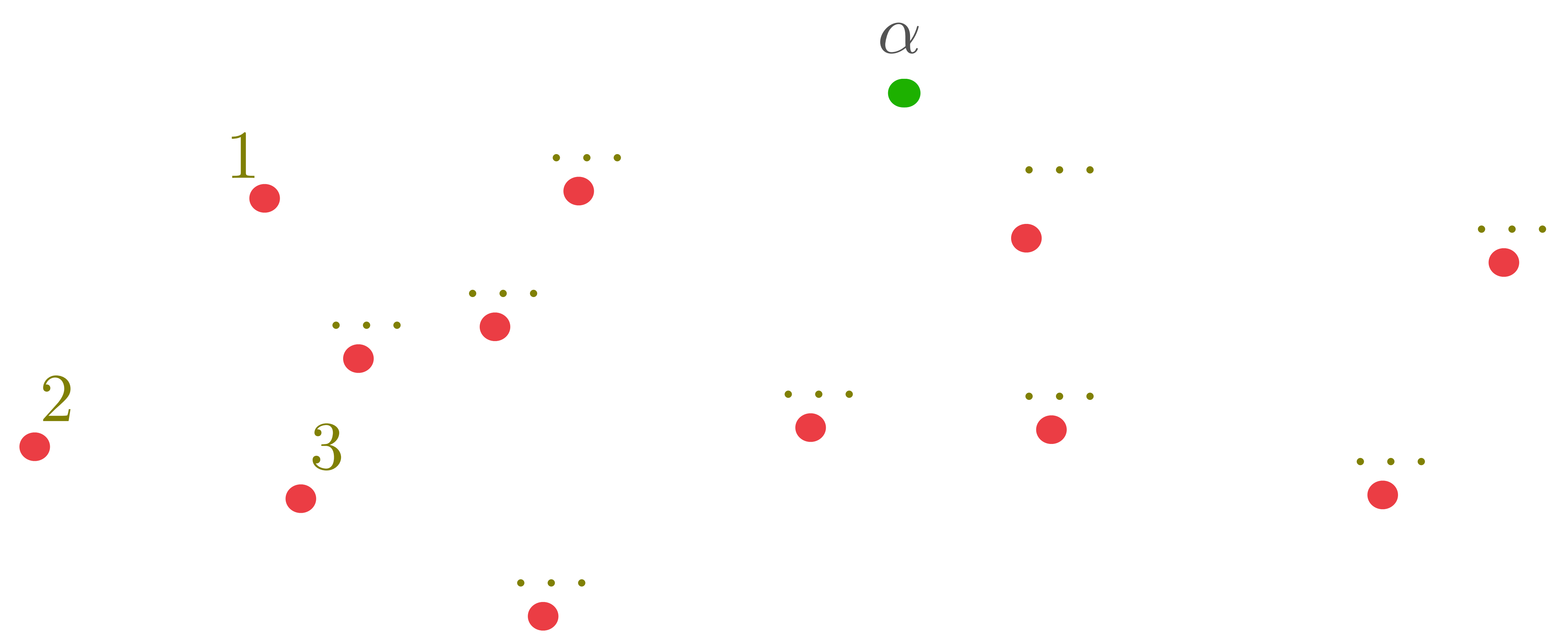

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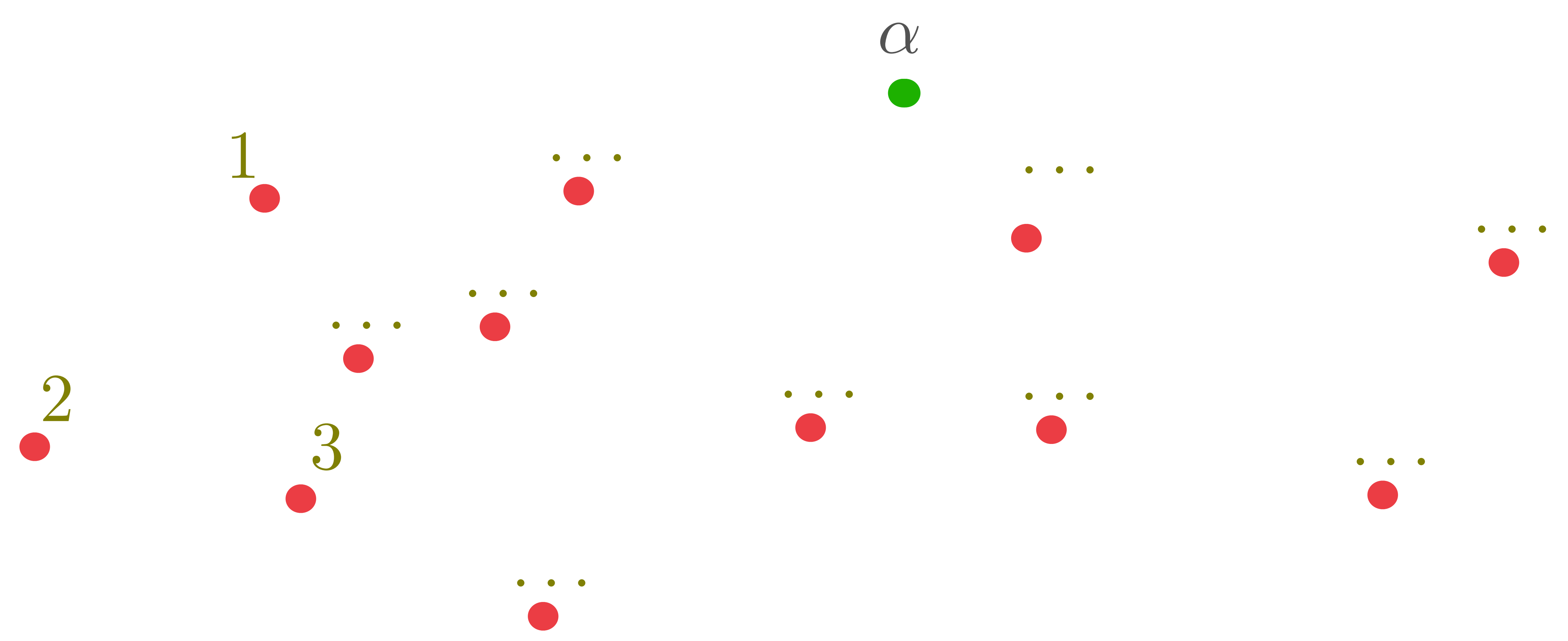

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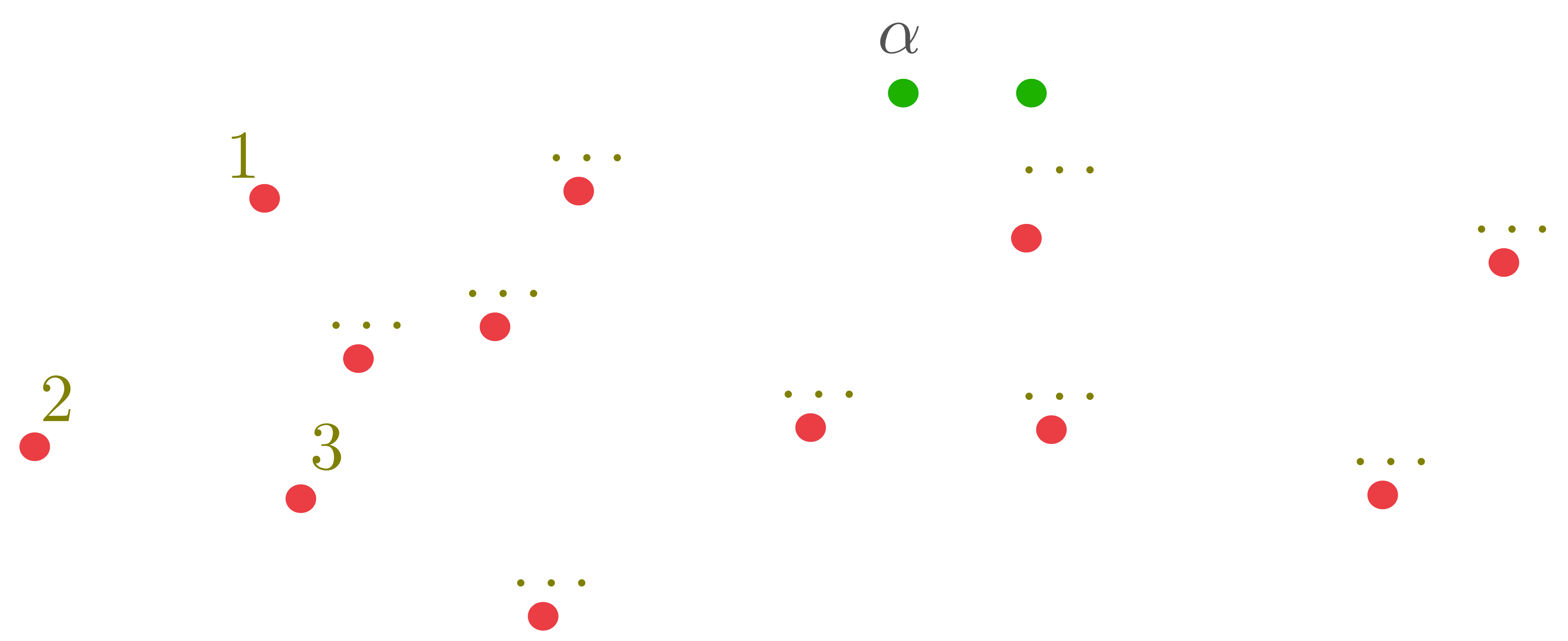

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- A Global Update Rule (GUR) is a UR $(G, \mathrm{A}, \mathscr{V}^{\dagger})$ such that $(G, \mathrm{A}, \mathscr{V}^{\dagger})$, $(G, \mathrm{A}, \mathscr{V}^{-1\dagger})$ are admissible