Operational probabilistic theories and cellular automata: how I learned to stop worrying and love C* algebras

School on Advanced Topics in Quantum Information and Foundations
Quantum Information Unit and the Yukawa Institute for Theoretical Physics, Kyoto University

UNIVERSITÀ DI PAVIA

INFN Istituto Nazionale di Fisica Nucleare

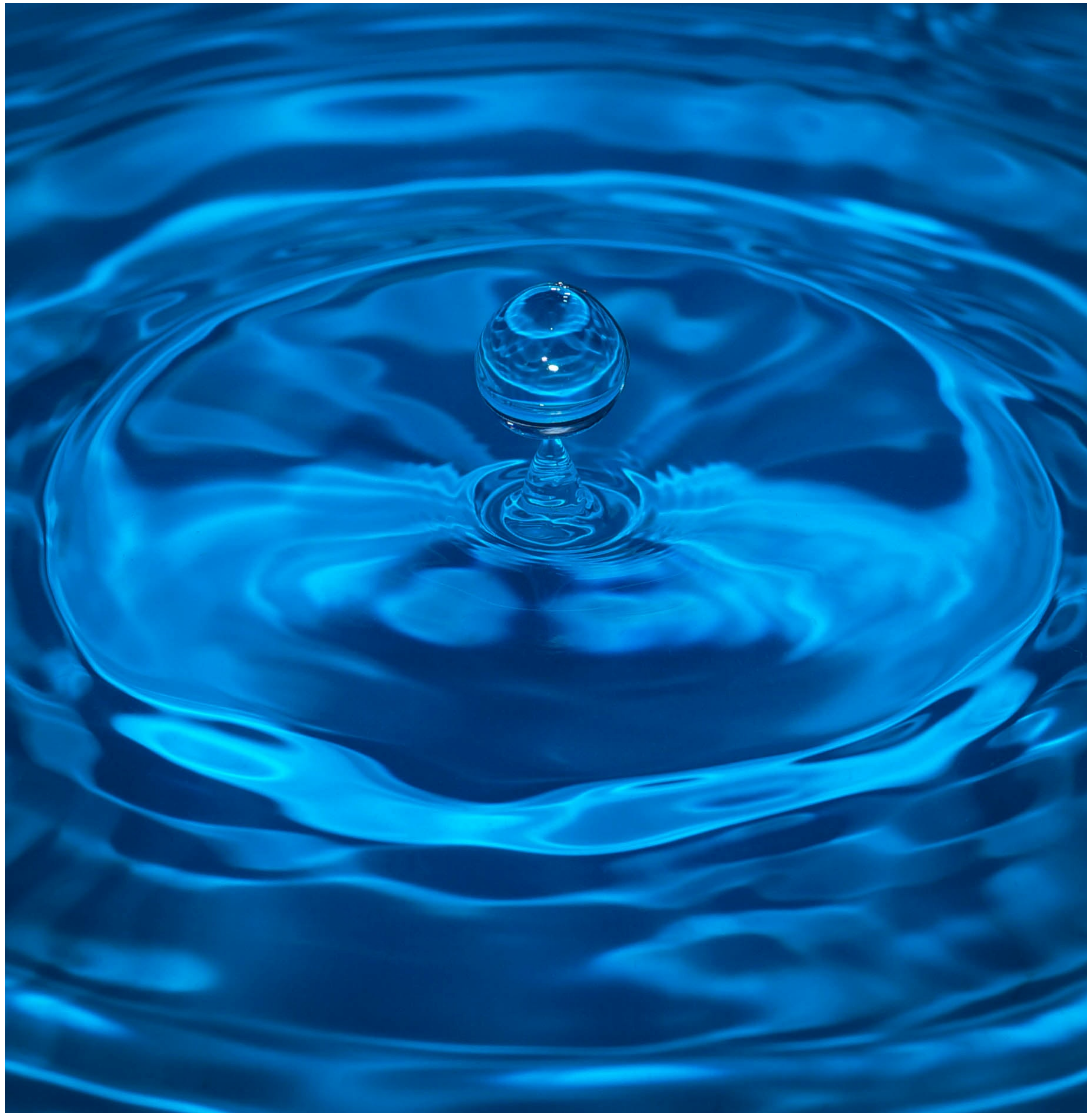
QUit quantum information theory group

Paolo Perinotti - February 8-12 2021

Lecture 4
Causal influence in OPTs

Summary

- Networks and causal cones
- Signalling
- Propagation of interventions
- The comb structure
- Classical and Quantum theory
- No interaction without disturbance



The neighbourhood

Causal influence

- Let us consider the local algebra
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- Let us consider the local algebra $[\![\mathrm{A}_g\mathrm{C} \to \mathrm{A}_g\mathrm{C}]\!]_{Q\mathbb{R}}$

- It is transformed in a subalgebra of some region R

$$\mathscr{V}[\![\mathrm{A}_g\mathrm{C} \to \mathrm{A}_g\mathrm{C}]\!]_{Q\mathbb{R}}\mathscr{V}^{-1} \subseteq [\![\mathrm{A}_R\mathrm{C} \to \mathrm{A}_R\mathrm{C}]\!]_{Q\mathbb{R}}$$

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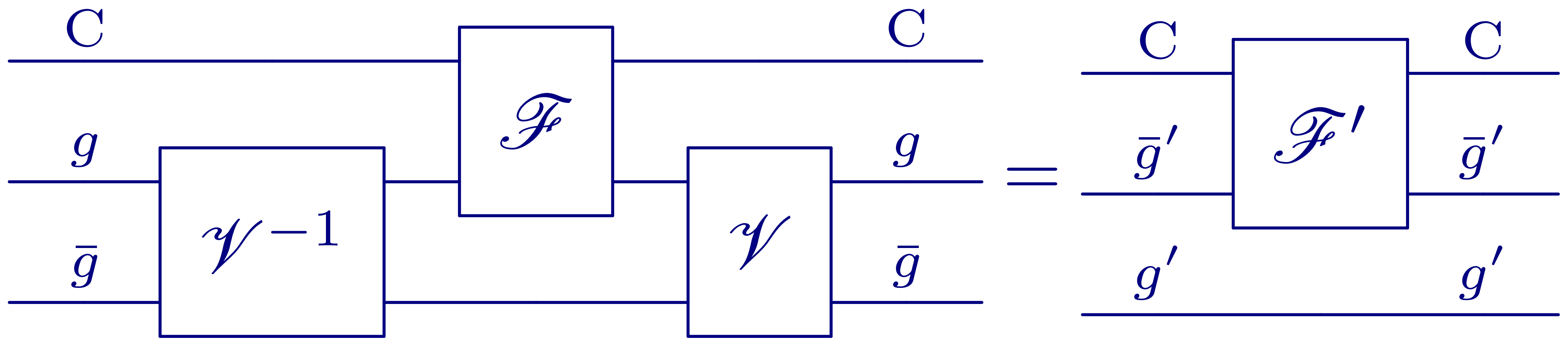

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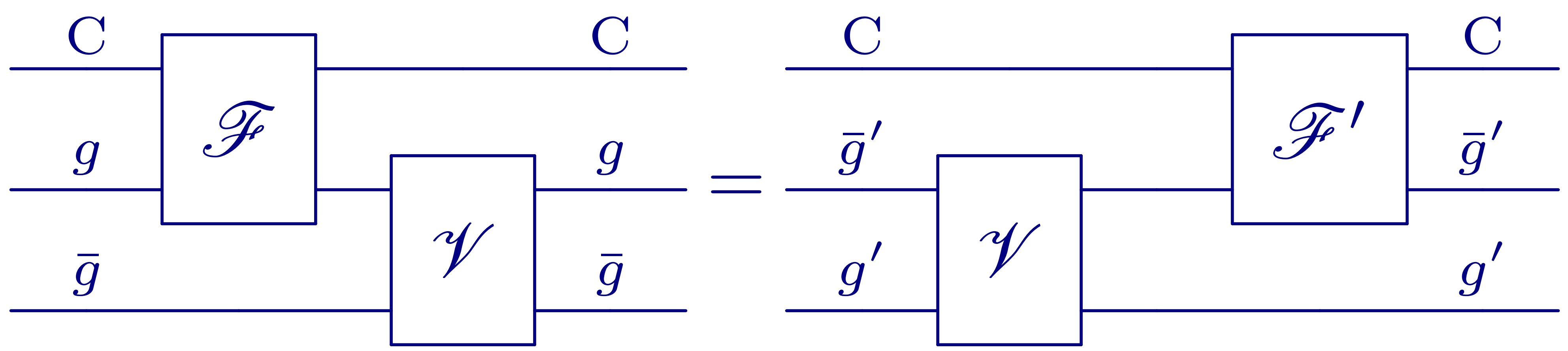

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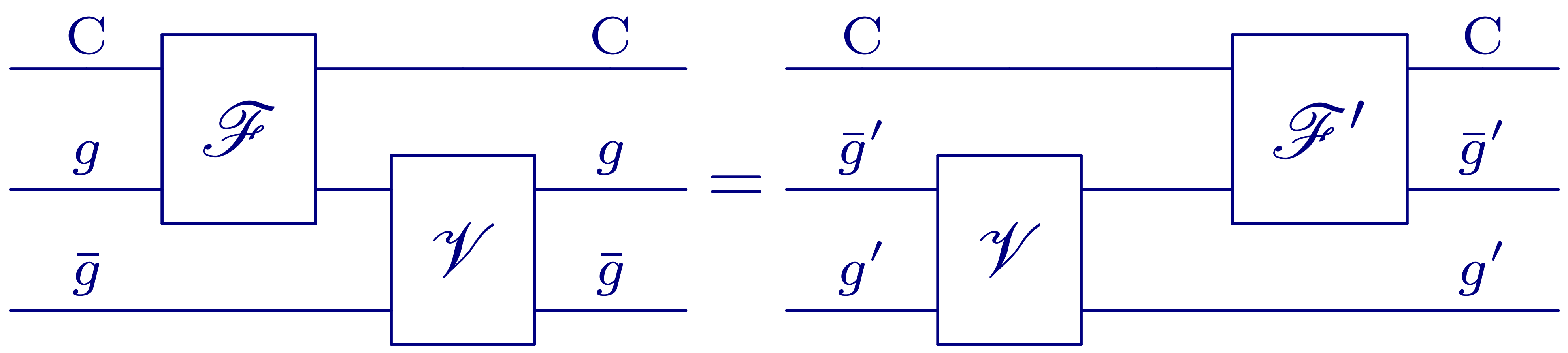

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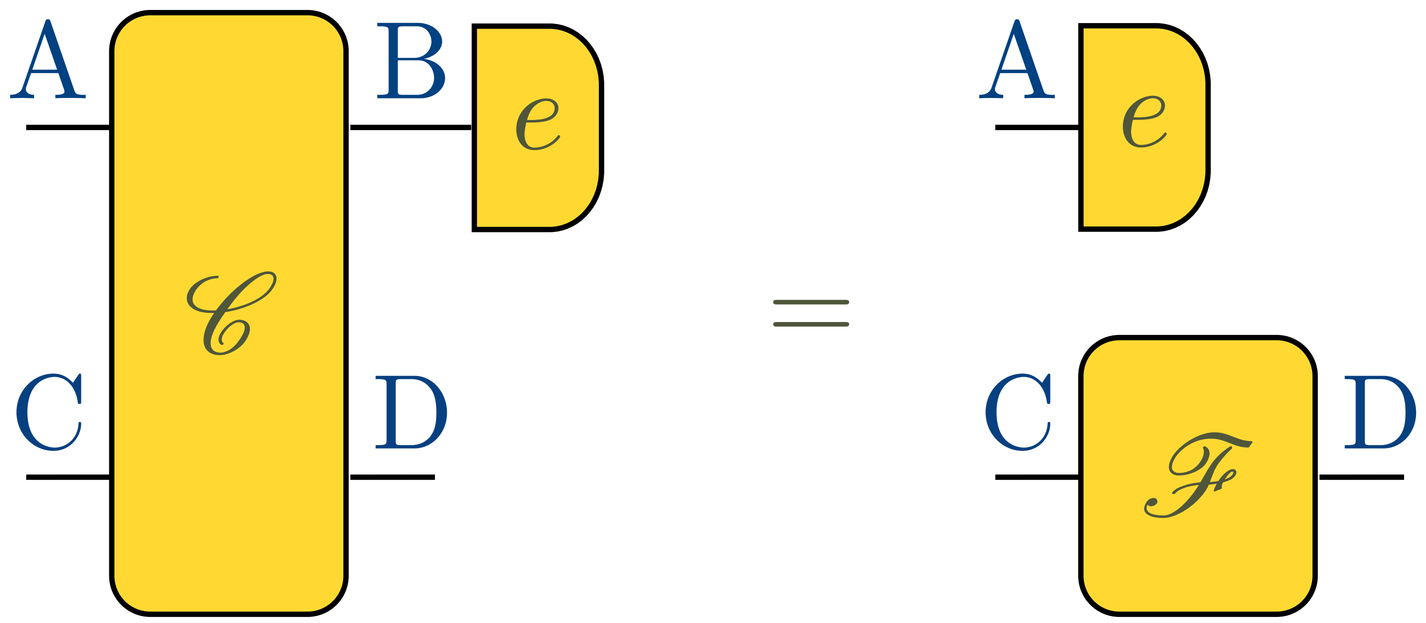
- The forward neighbourhood N_g^+ of g is the set of all g' such that g causally influences g'

Causal influence in quantum literature

The traditional approach: start from no-signalling



$$\mathrm{A} \not\rightsquigarrow \mathrm{D}$$

No intervention on the state of A can influence the state of C

In quantum theory

$$\mathrm{Tr}_{\mathrm{B}}[R_{\mathscr{C}}] = I_{\mathrm{A}} \otimes R_{\mathscr{F}}$$

$R_{\mathscr{E}}$ denoting the Choi operator corresponding to $\mathscr{E}$

D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, Phys Rev. A **64**, 052309 (2001)
T. Eggeling, D. Schlingemann, and R.F. Werner, Europhys. Lett. **57**, pp. 782–788 (2002)

Defining (no) causal influence in OPTs

The raw idea

- The definition is inspired by the notion of neighbourhood in QCAs

- It holds for reversible transformations

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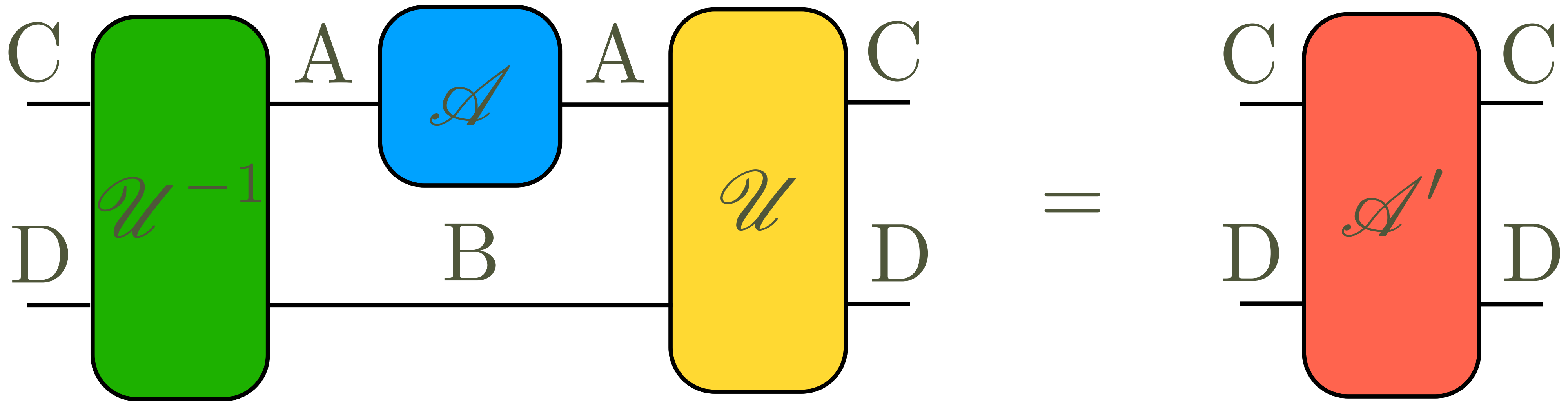

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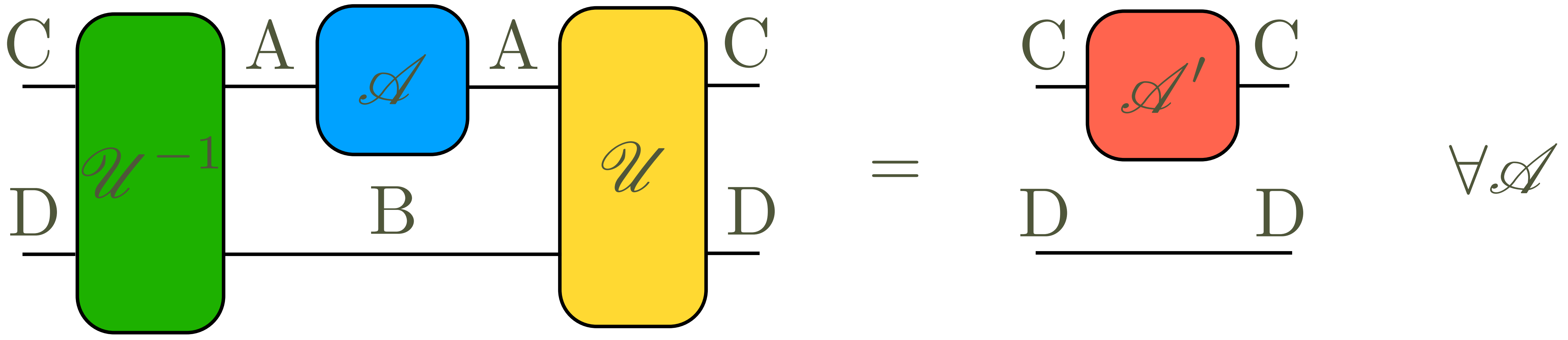

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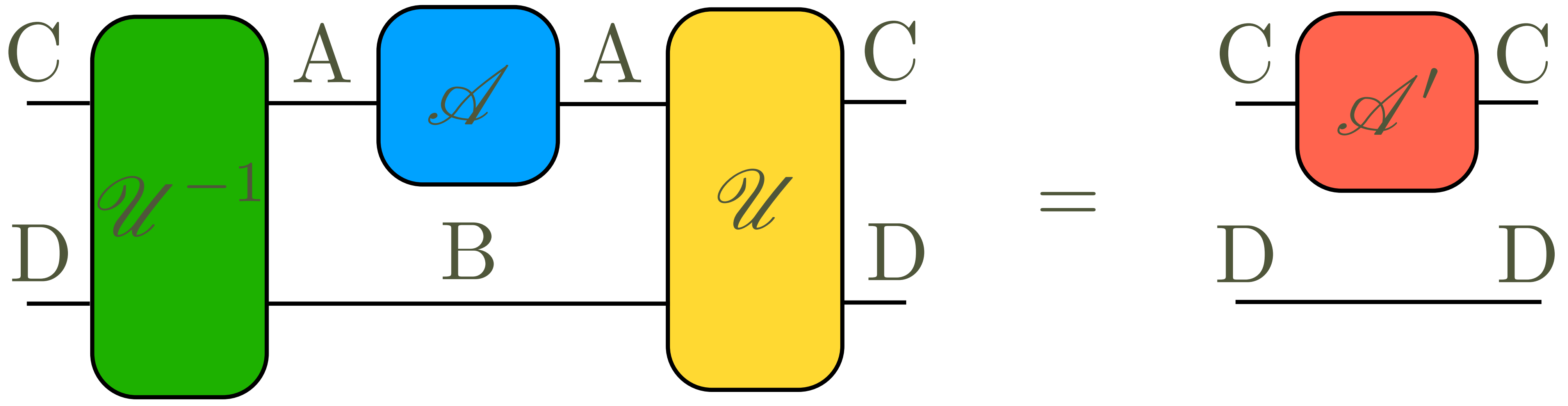

$\forall \mathscr{A} \qquad \mathrm{A} \not\to \mathrm{D}$

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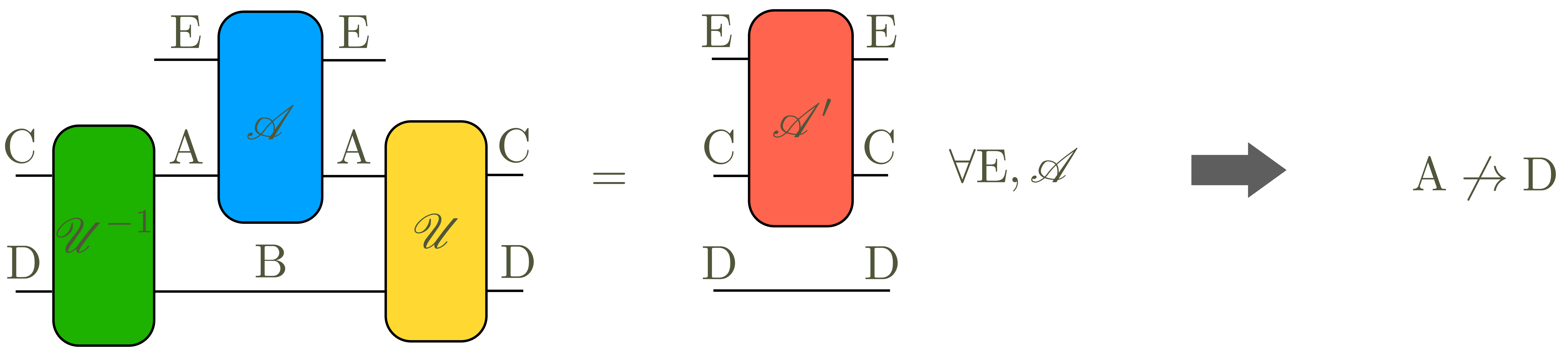
Defining (no) causal influence in OPTs

The precise notion

- Without local discriminability (local tomography/tomographic locality) we need to take into account interventions involving ancillary systems



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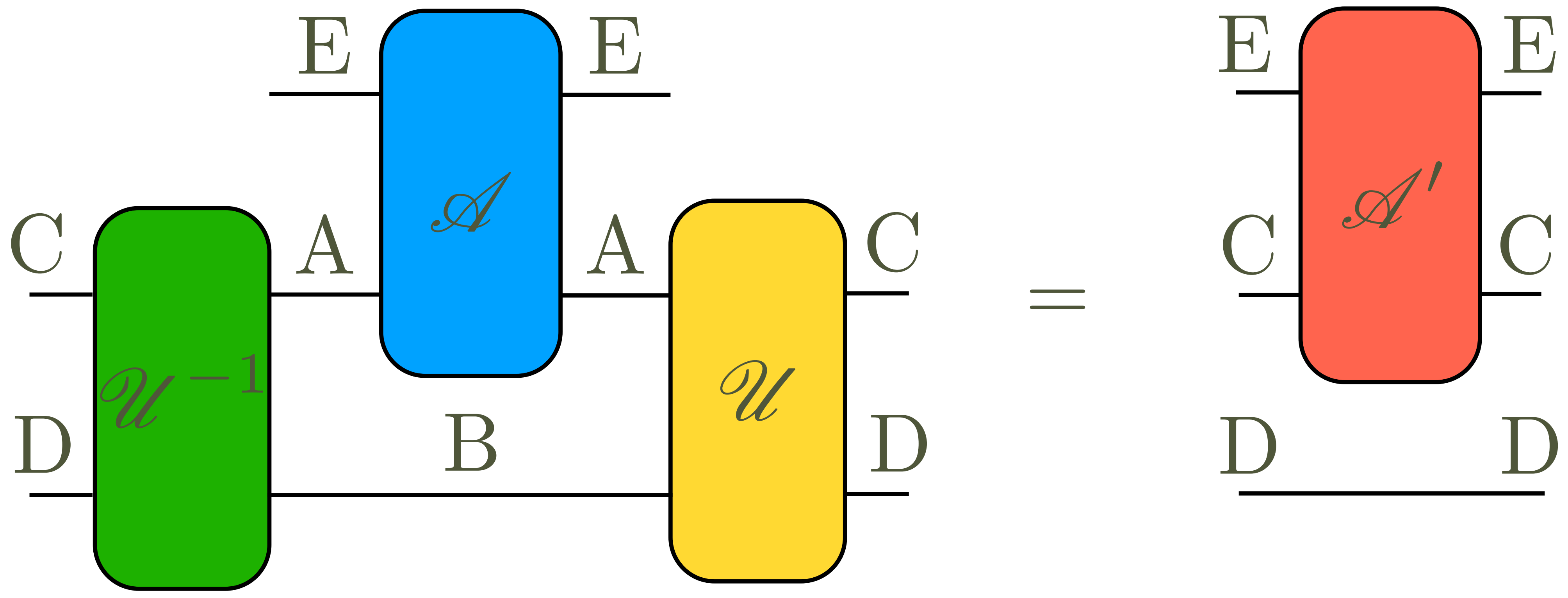
Explanation of the definition

Suppose that under $\mathscr{U}$ one has $\mathrm{A} \not\to \mathrm{D}$



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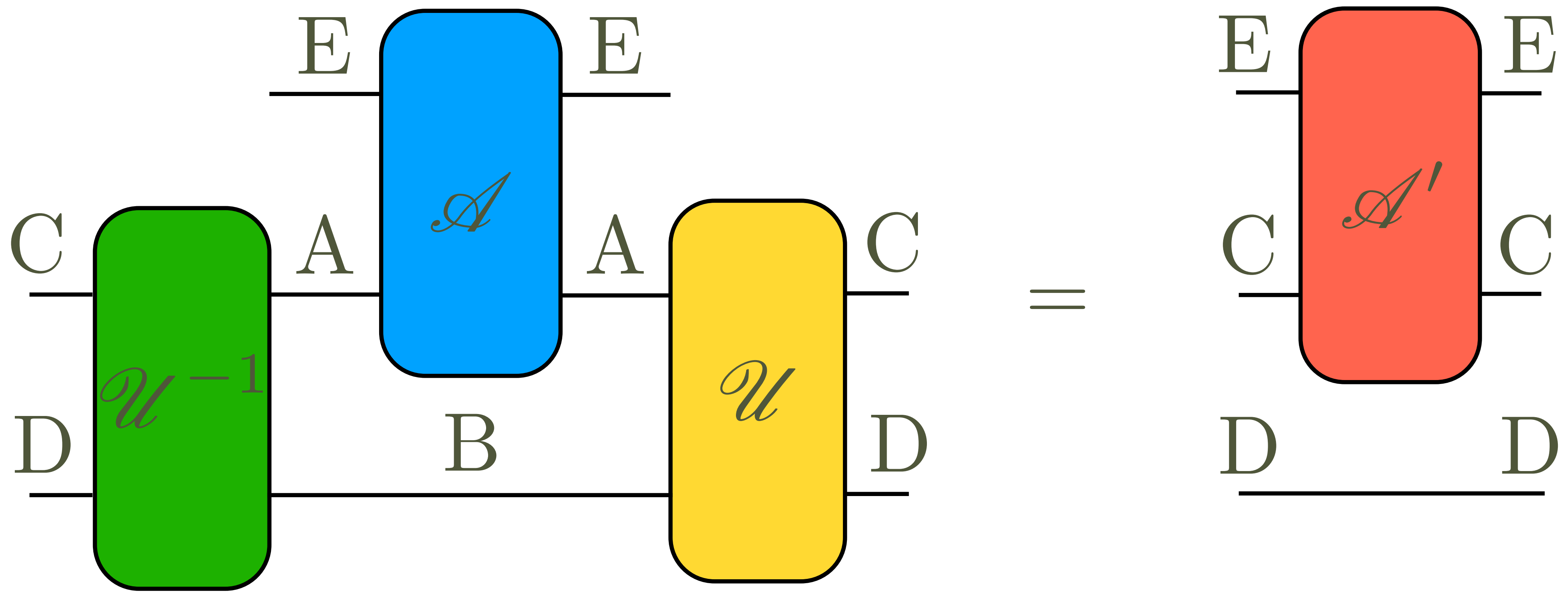
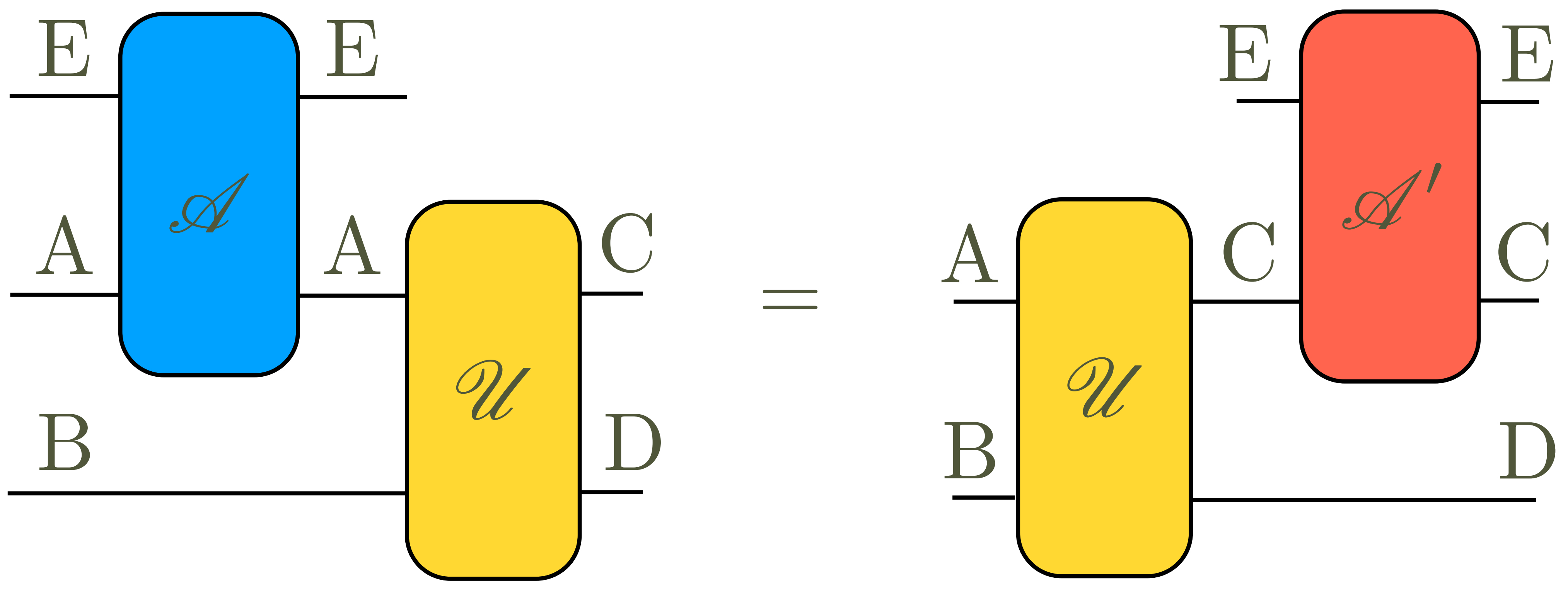


Equivalently:



Necessary condition for no C.I.: no-signalling

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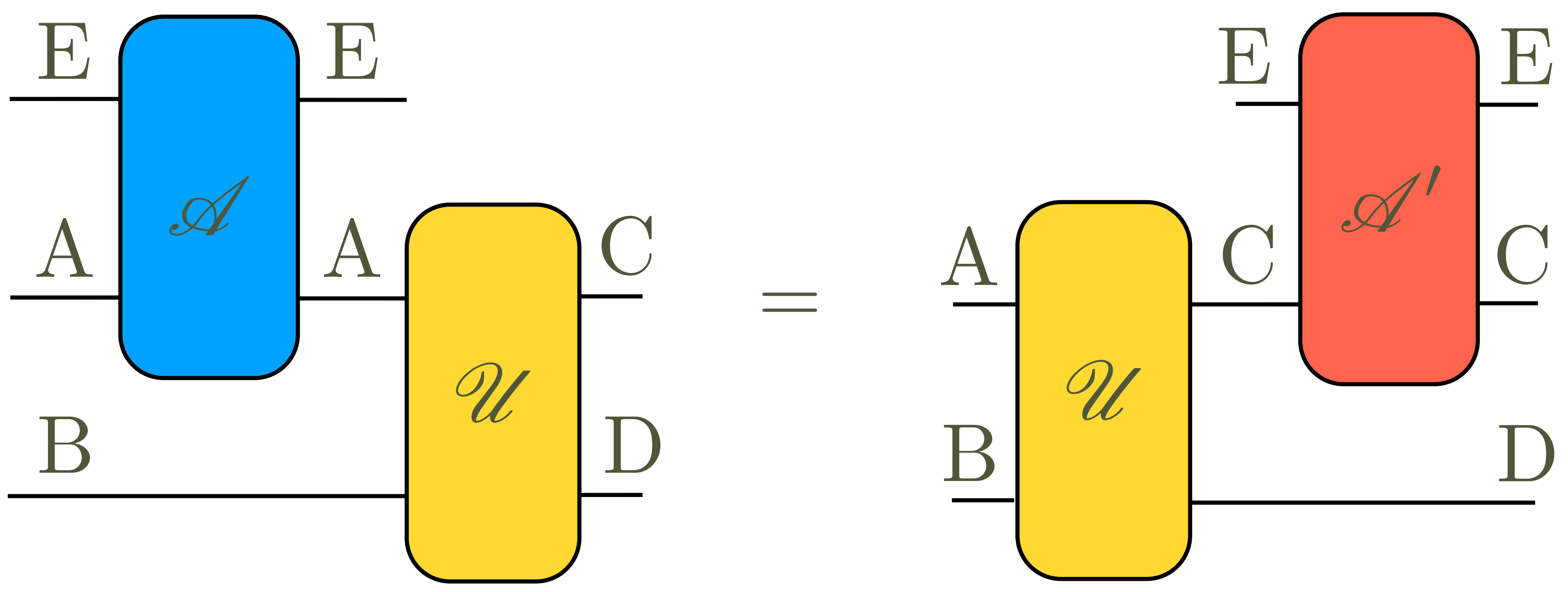

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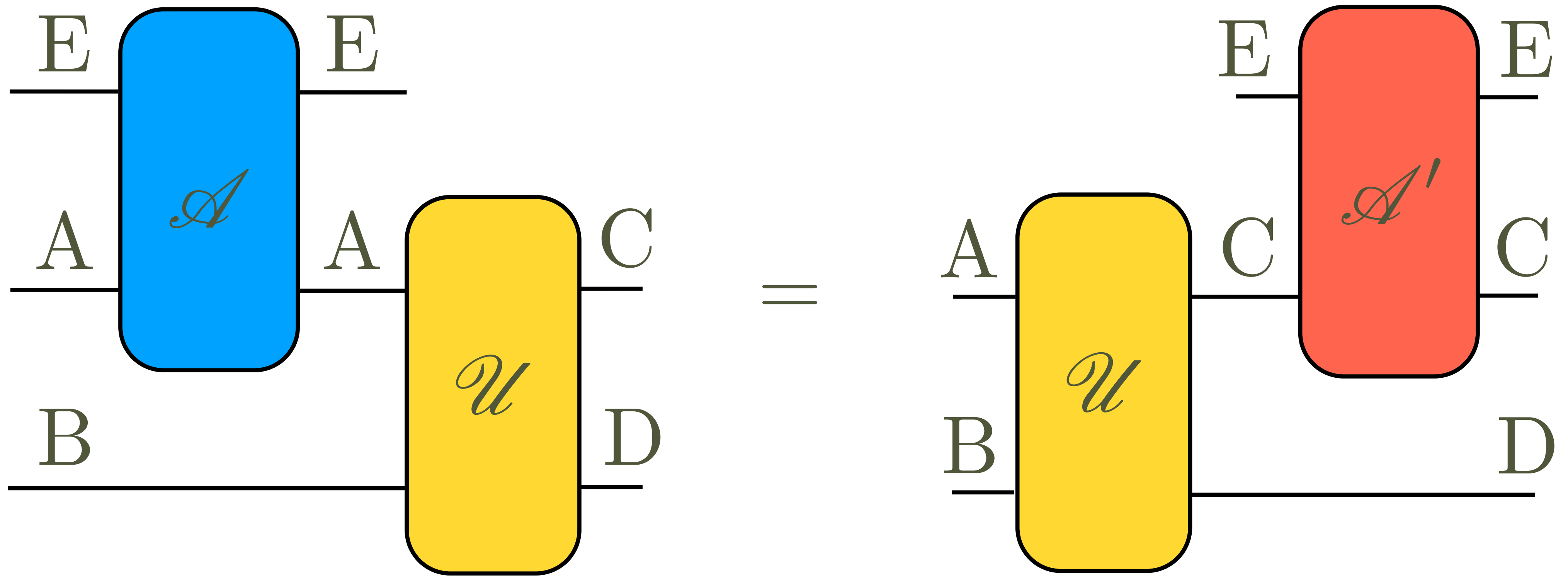

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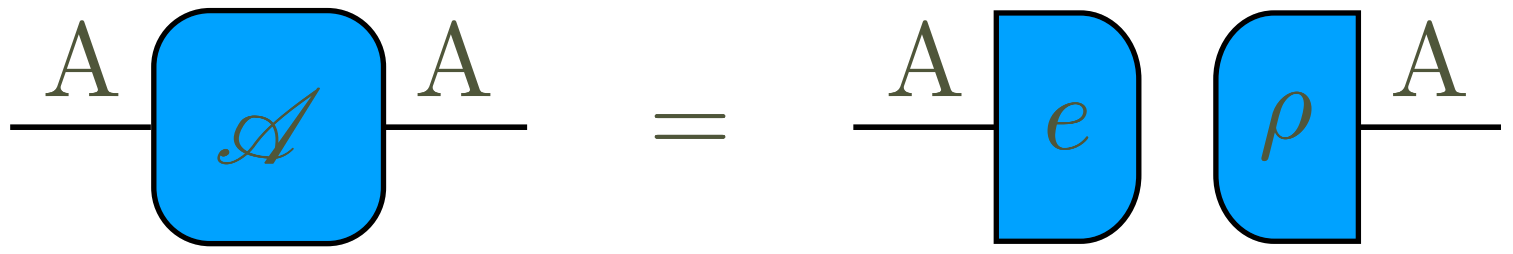

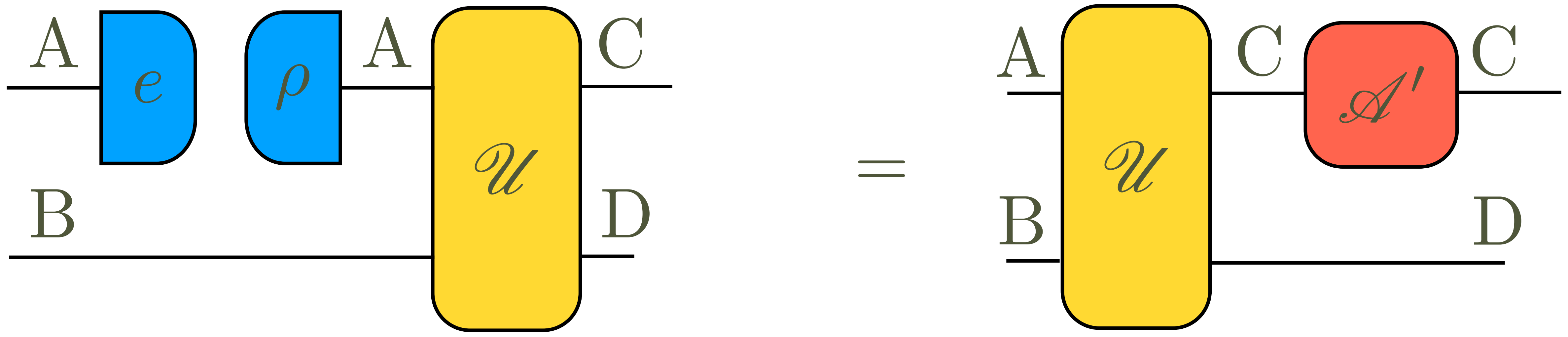

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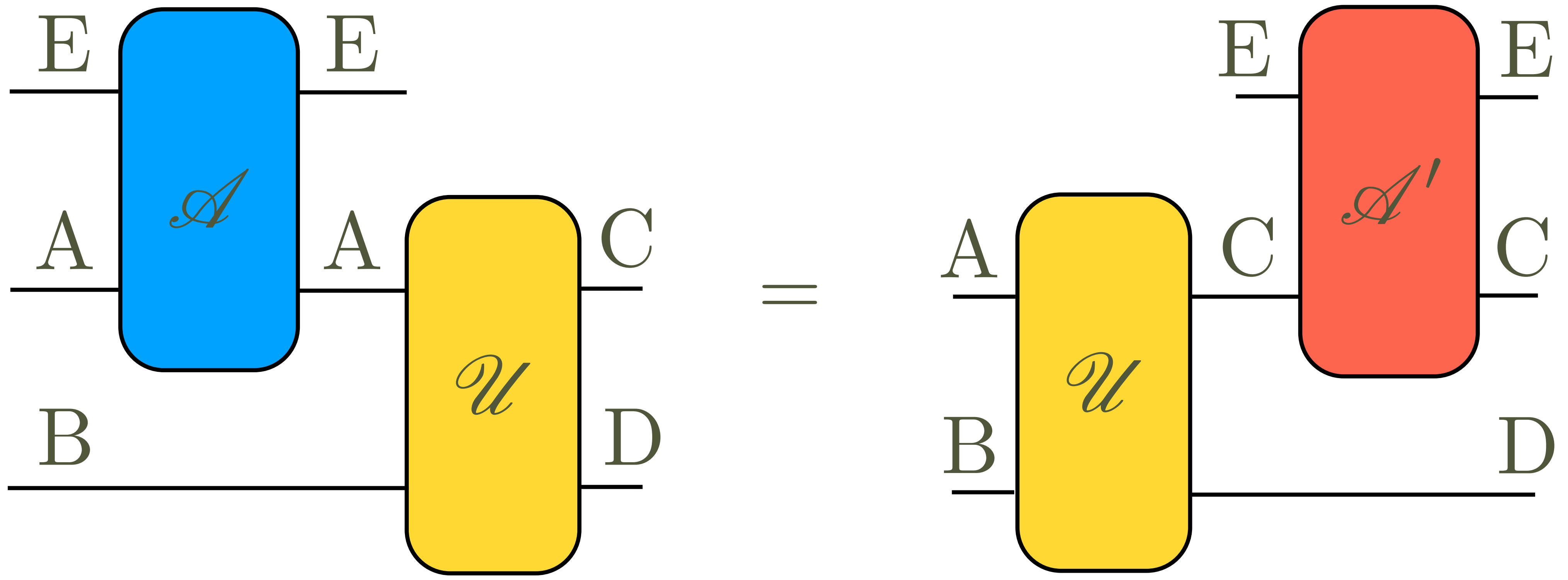

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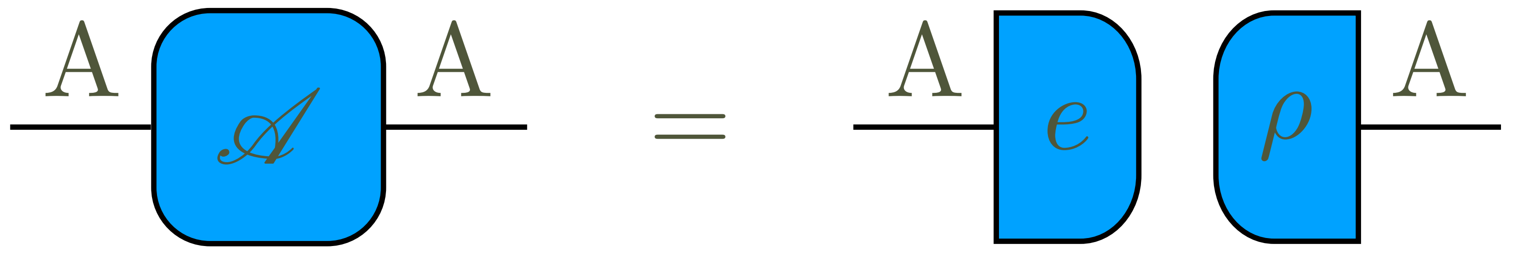

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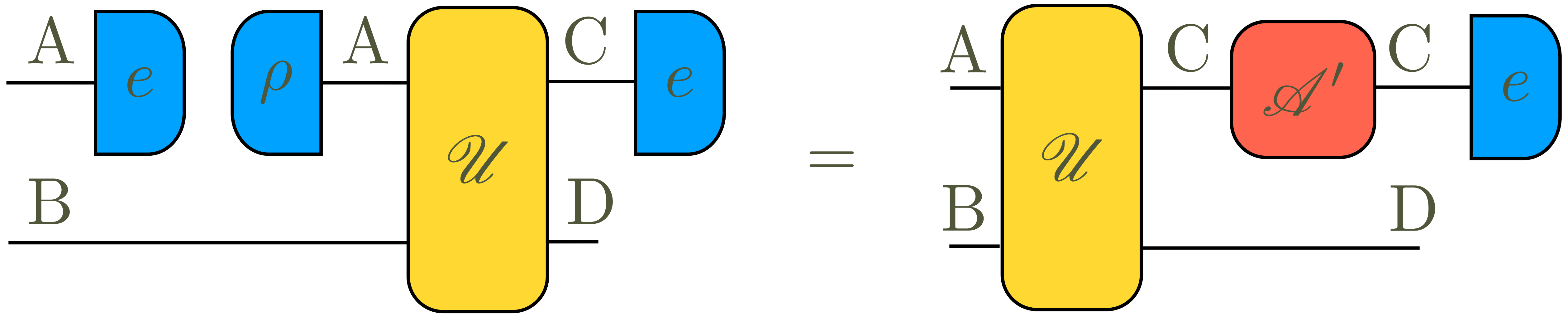

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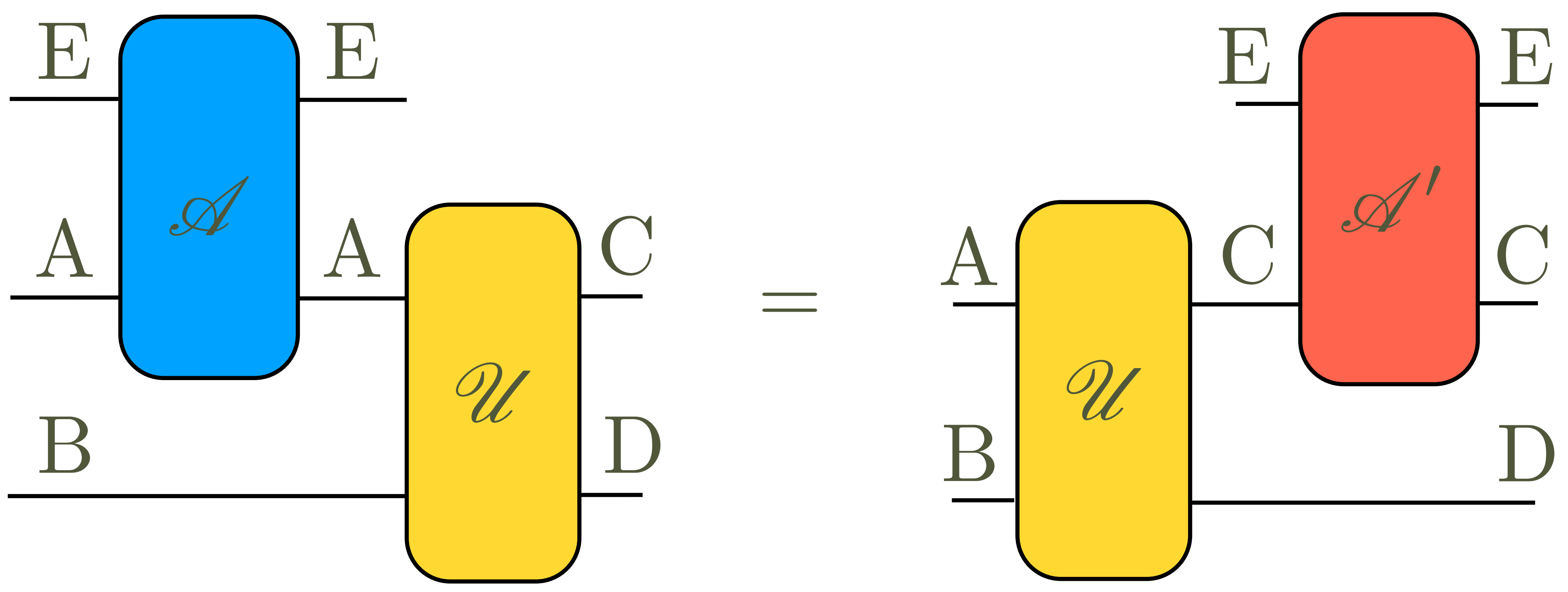

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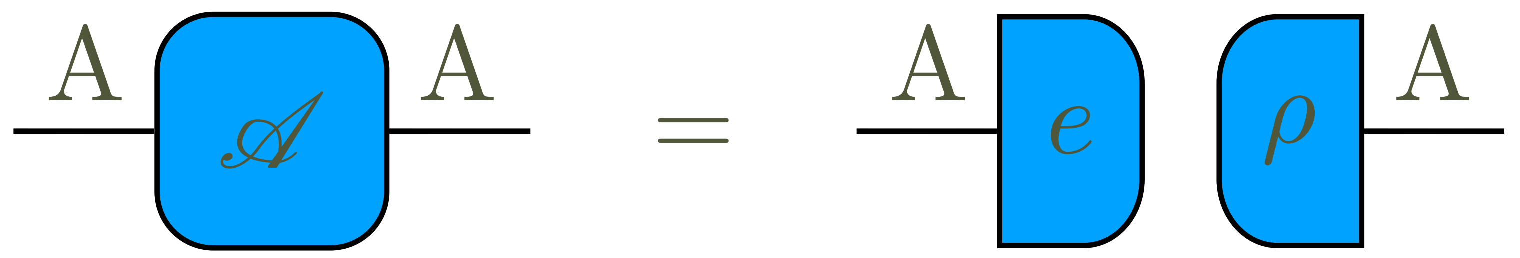

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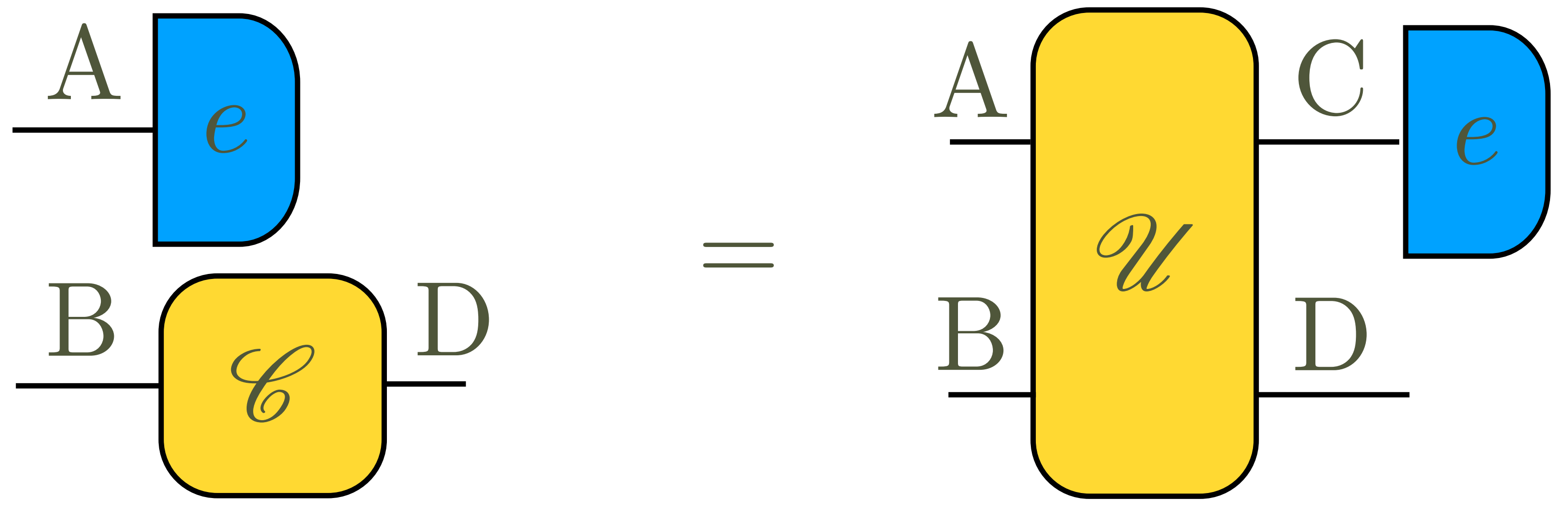

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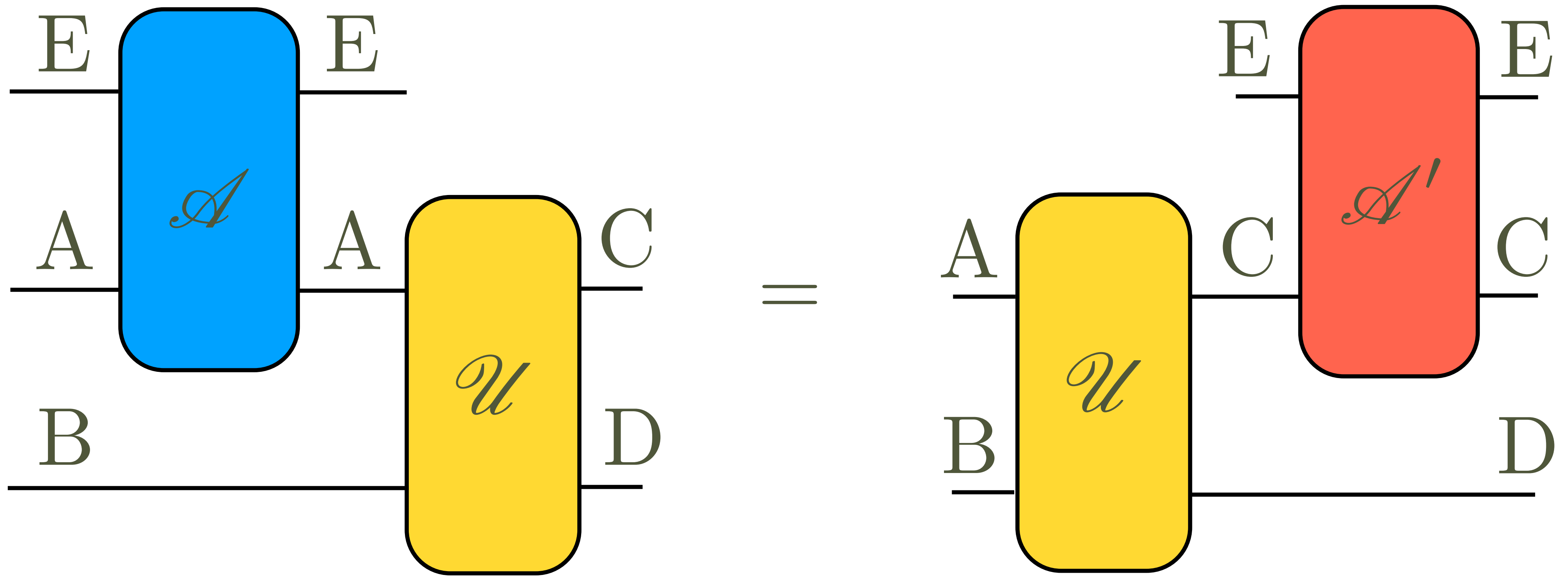

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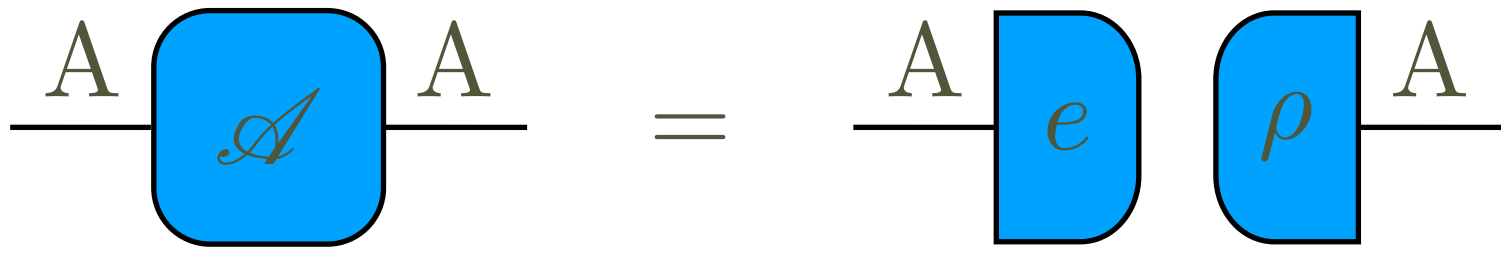

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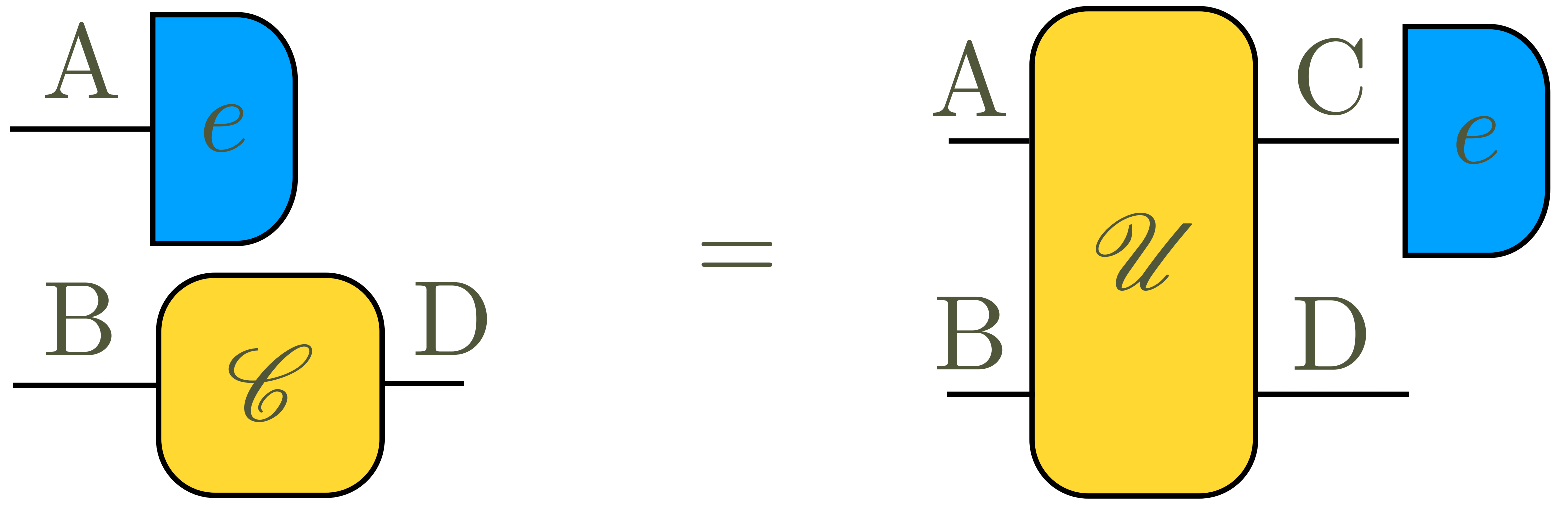

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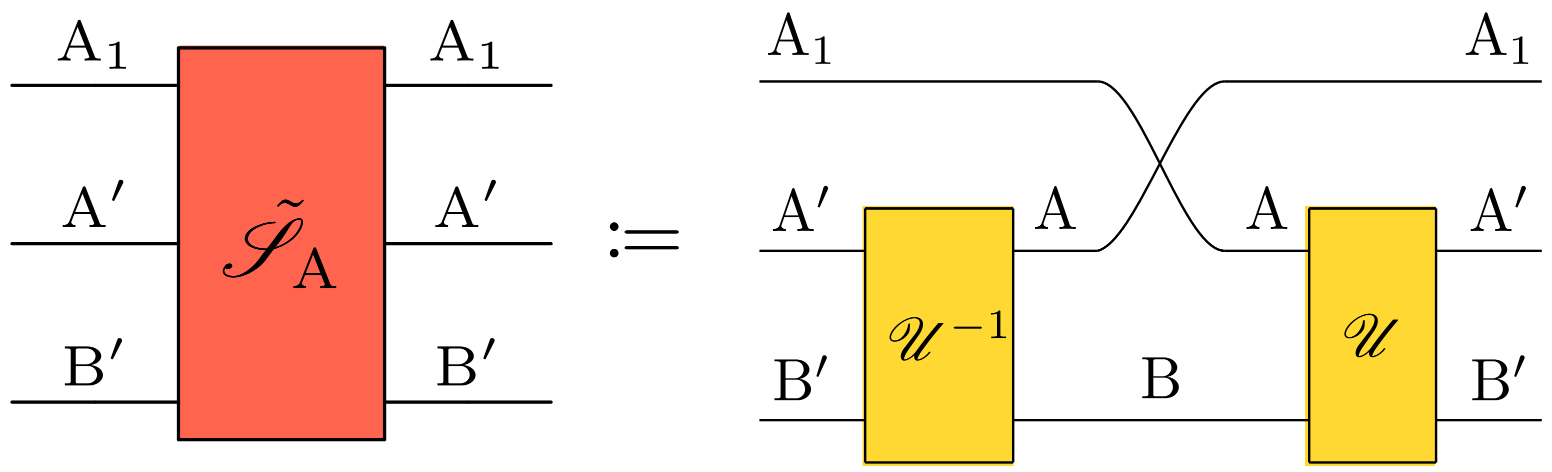
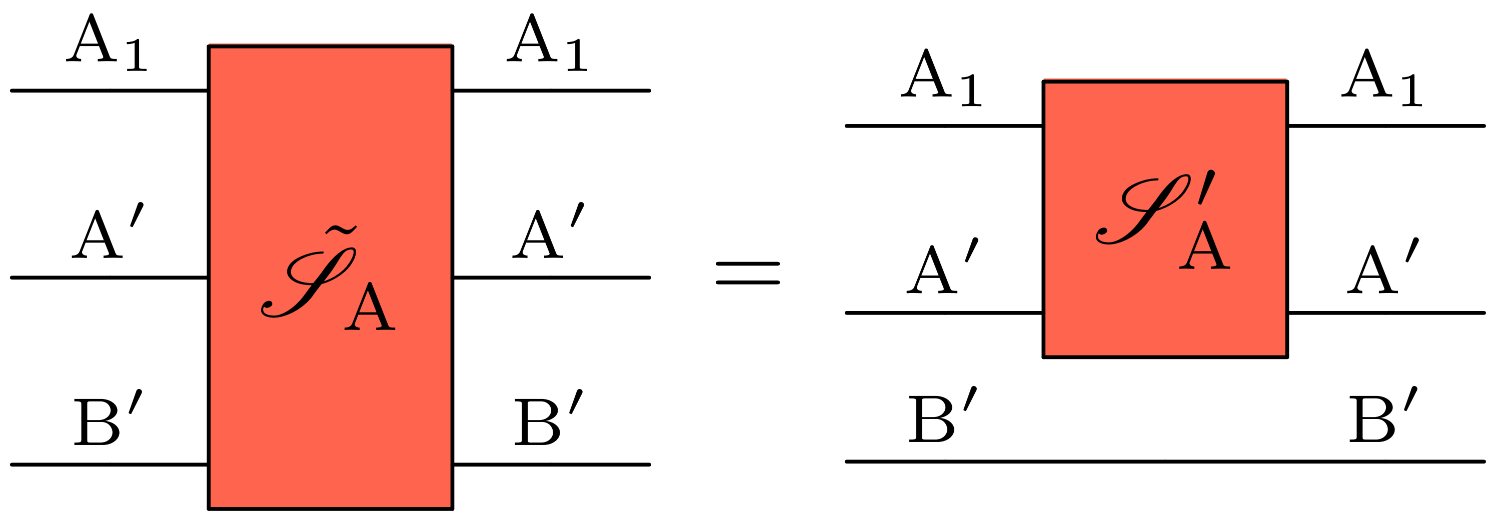
Necessary and sufficient condition

- Definition:



- Condition:

$A \nrightarrow B'$ iff



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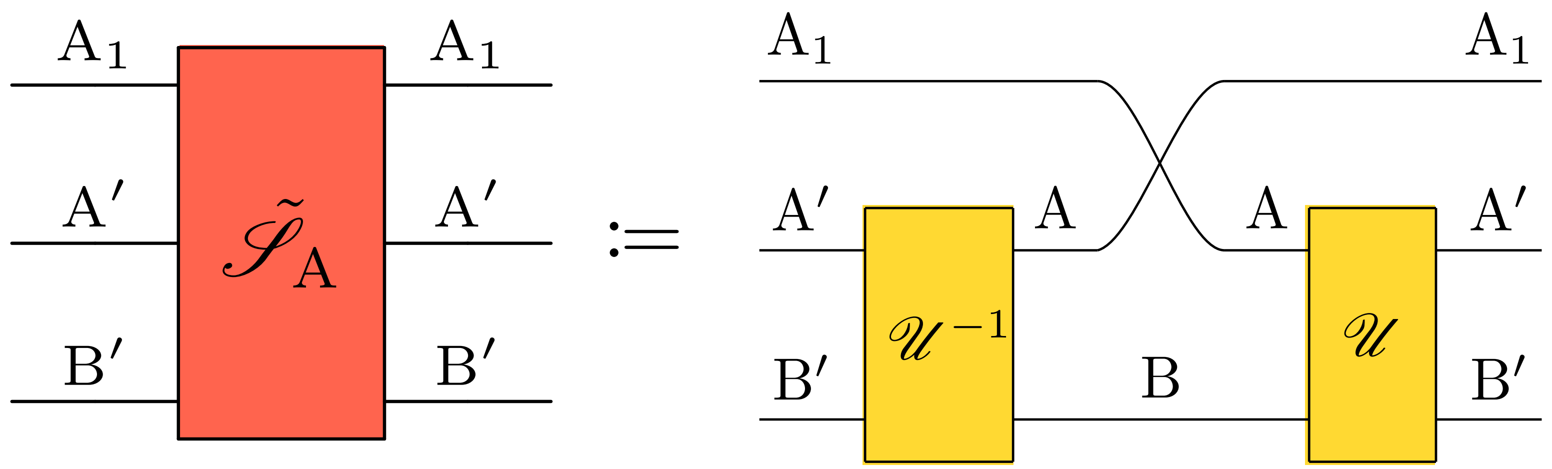
Necessary and sufficient condition

Proof



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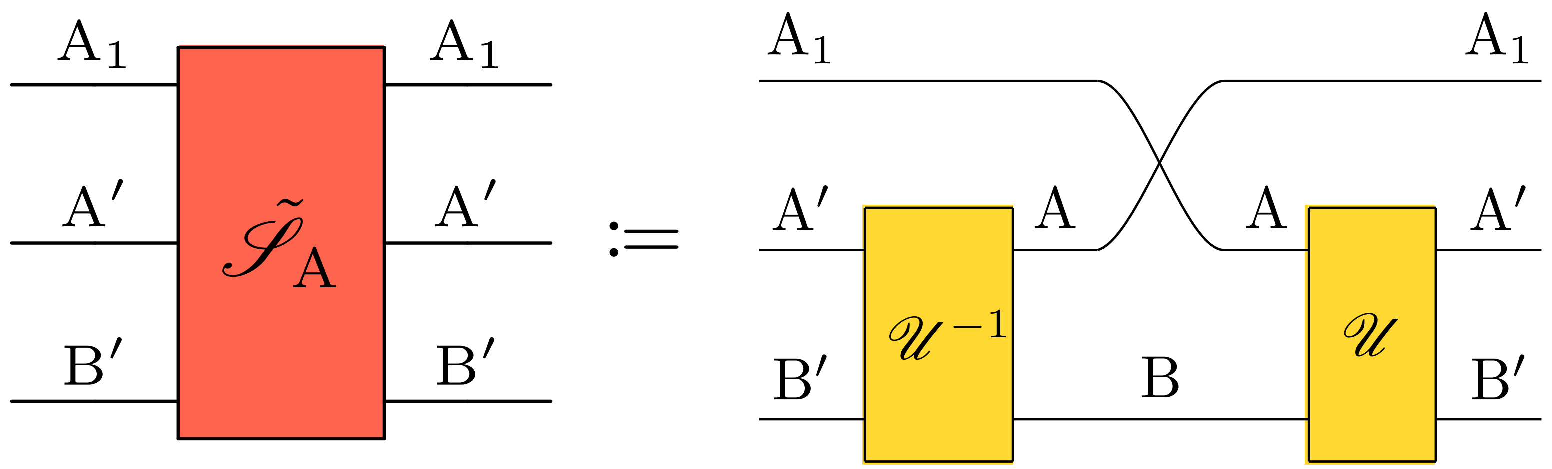
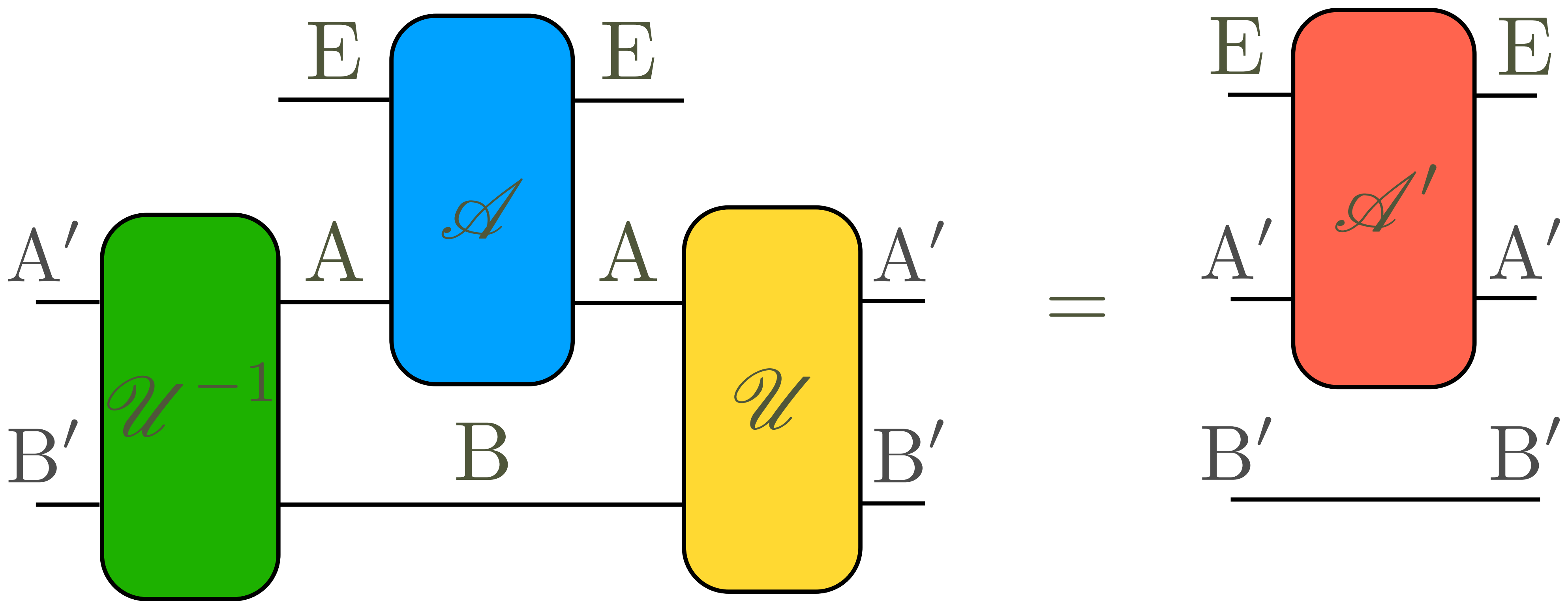


If $\mathrm{A} \not\to \mathrm{B}'$



$\forall \mathrm{E}, \mathscr{A}$

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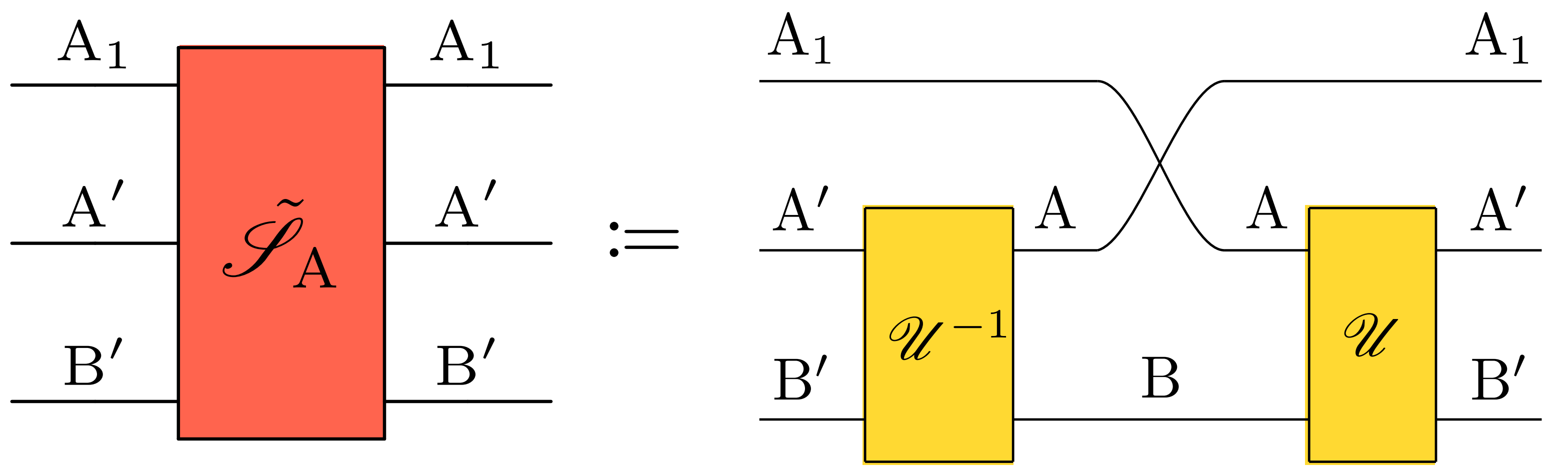

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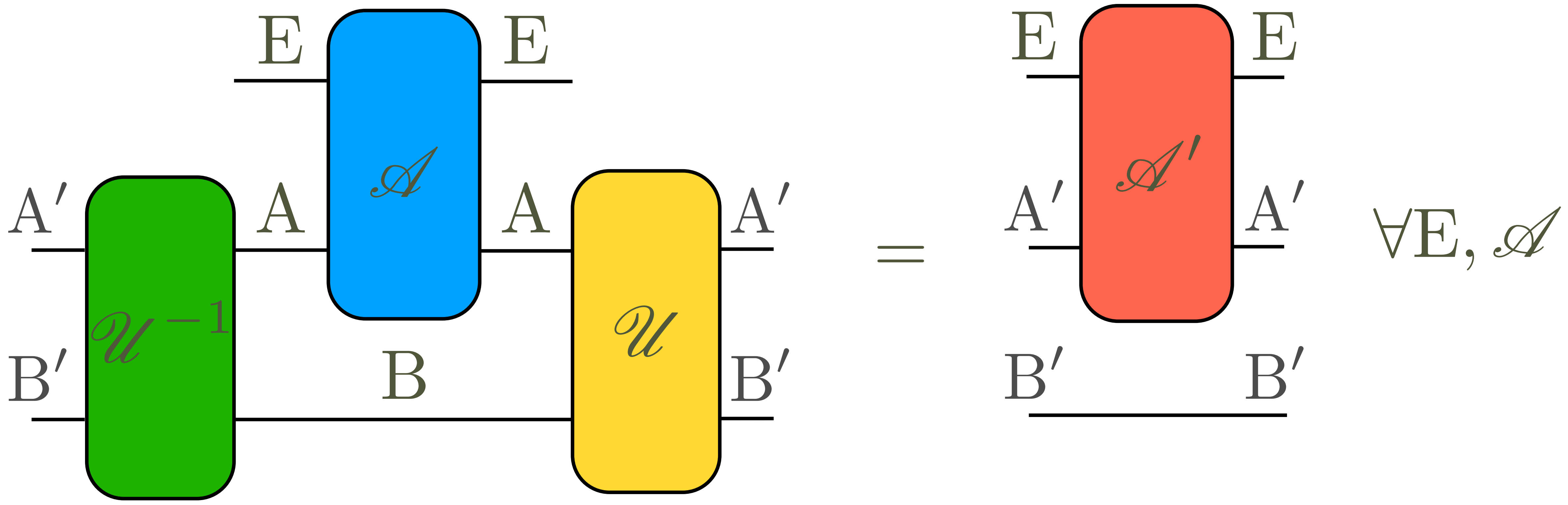

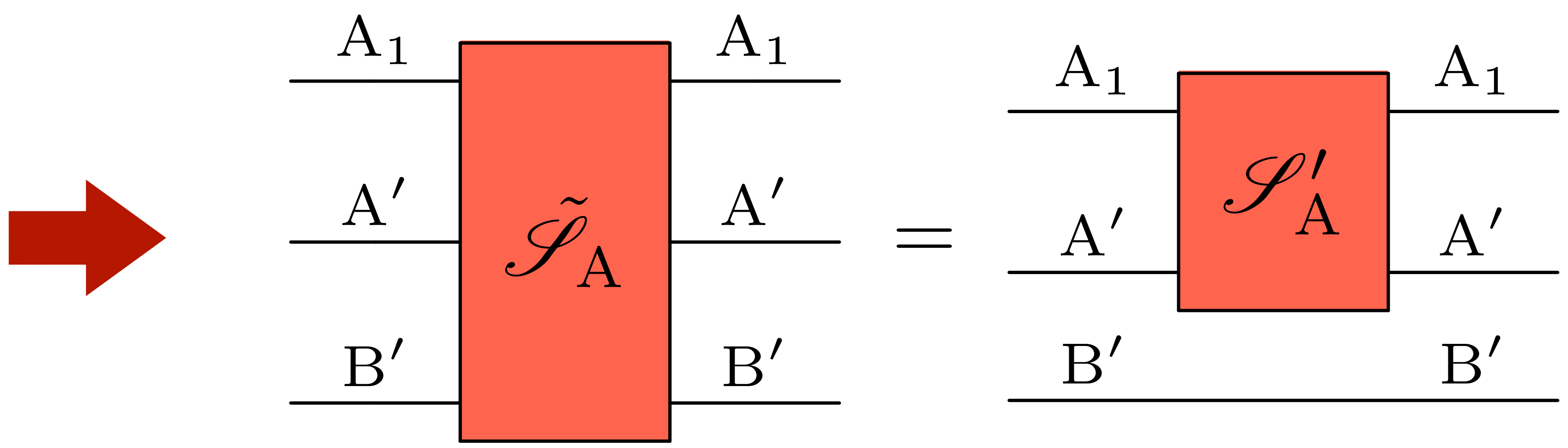

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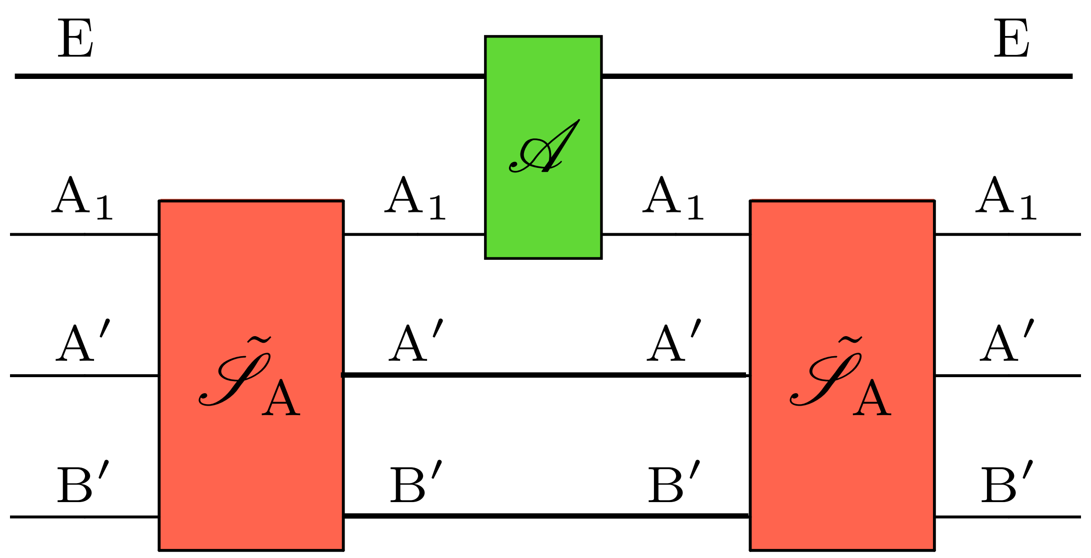

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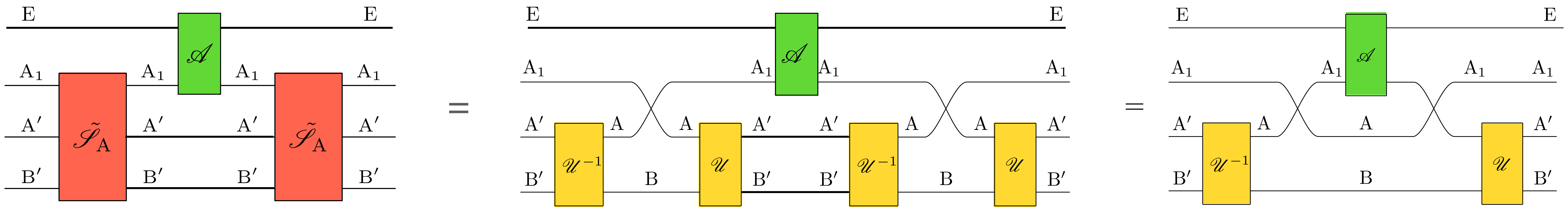

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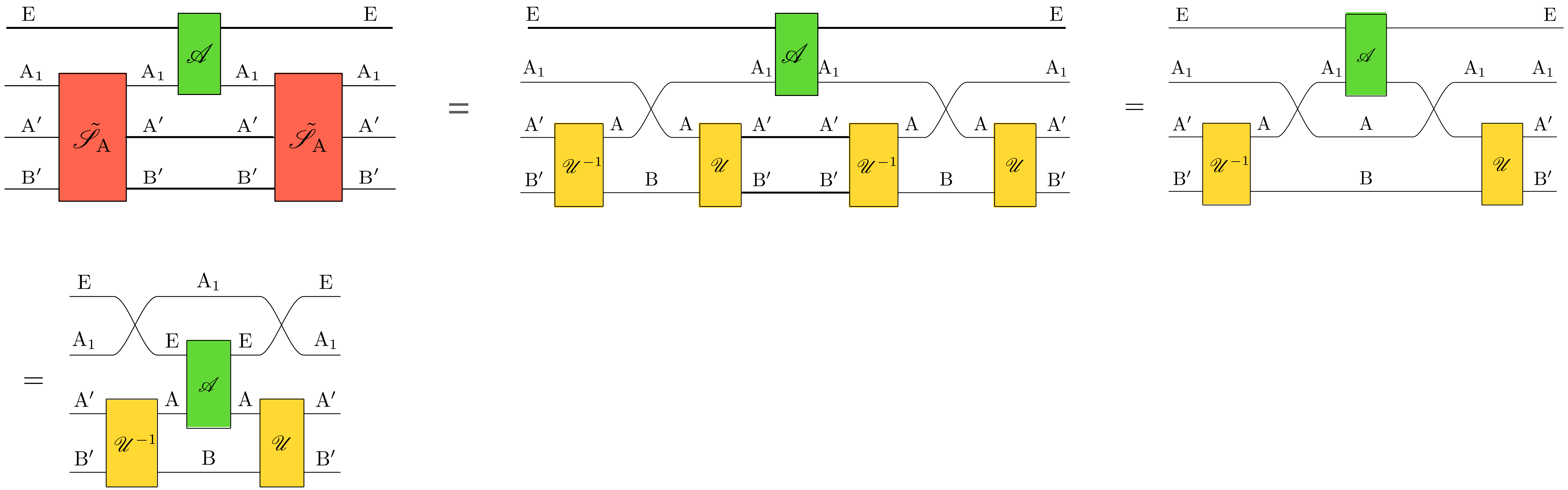

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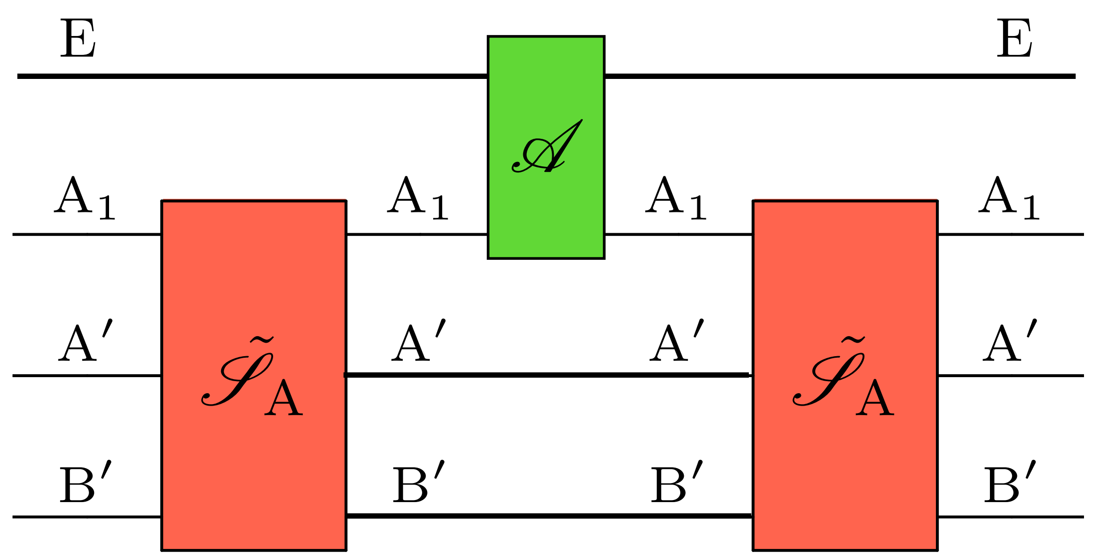

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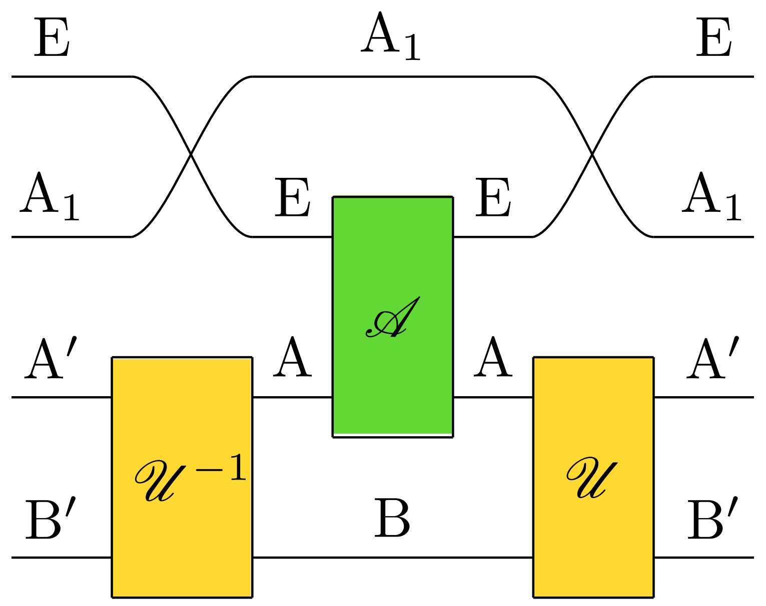

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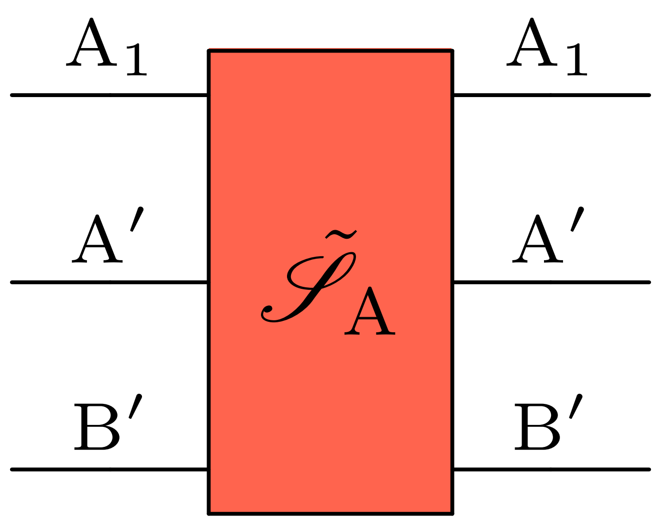

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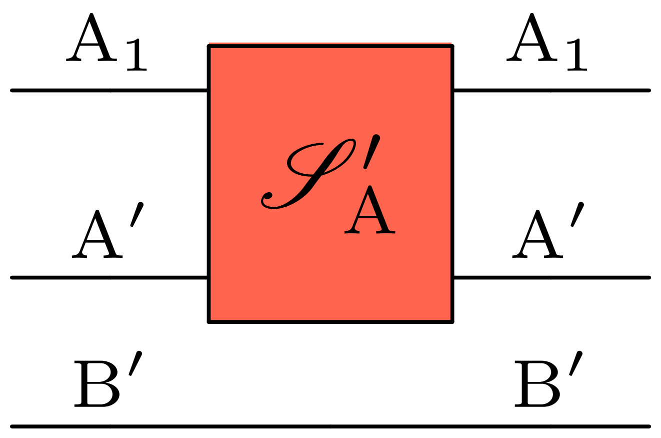

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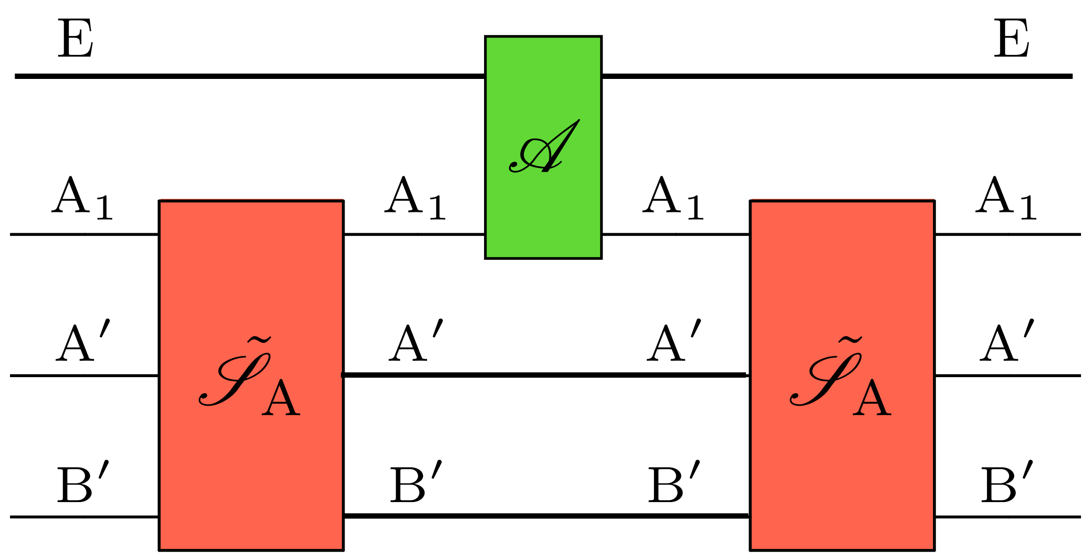

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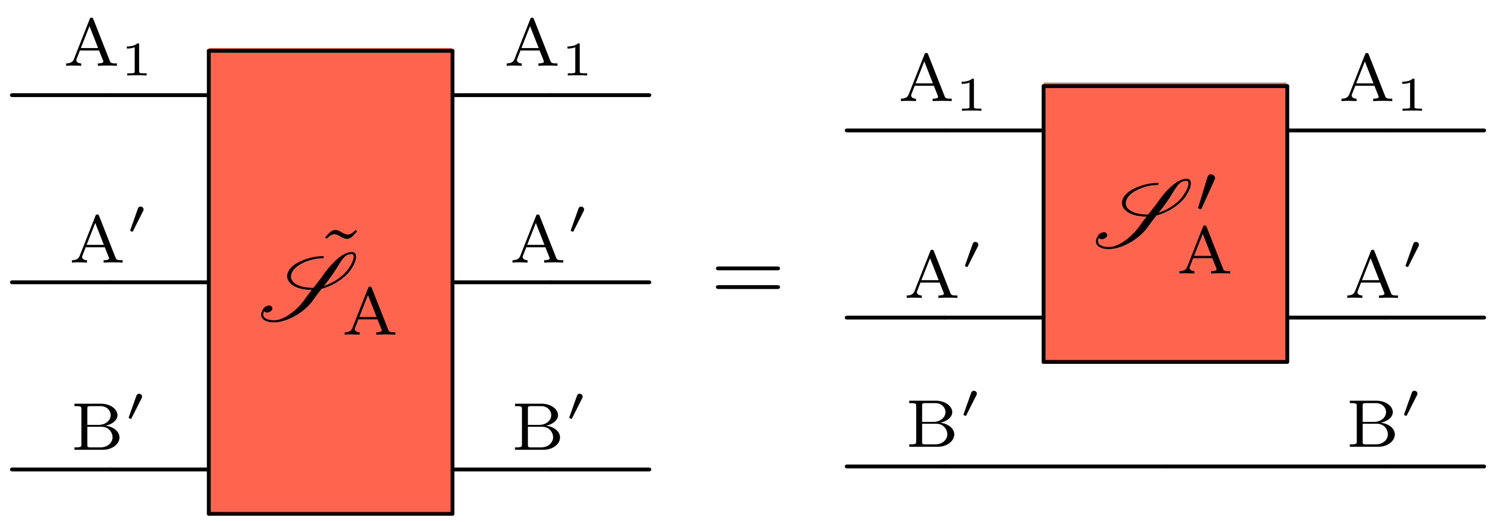

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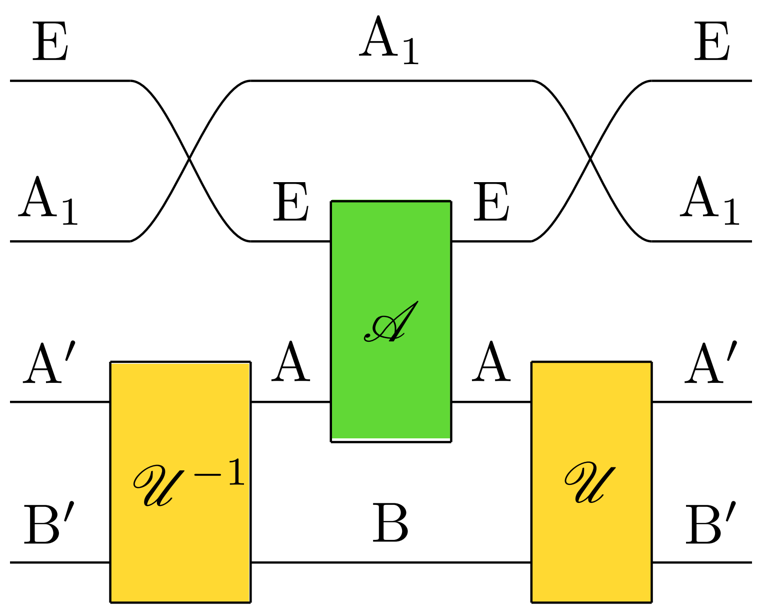

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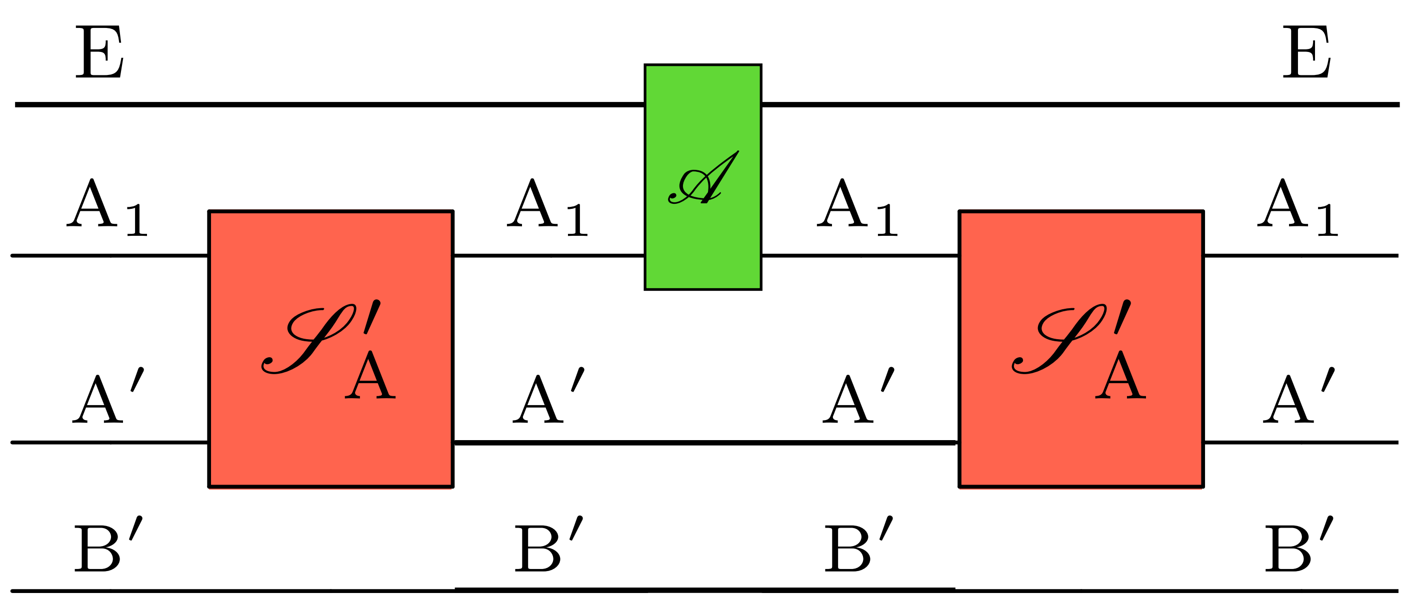

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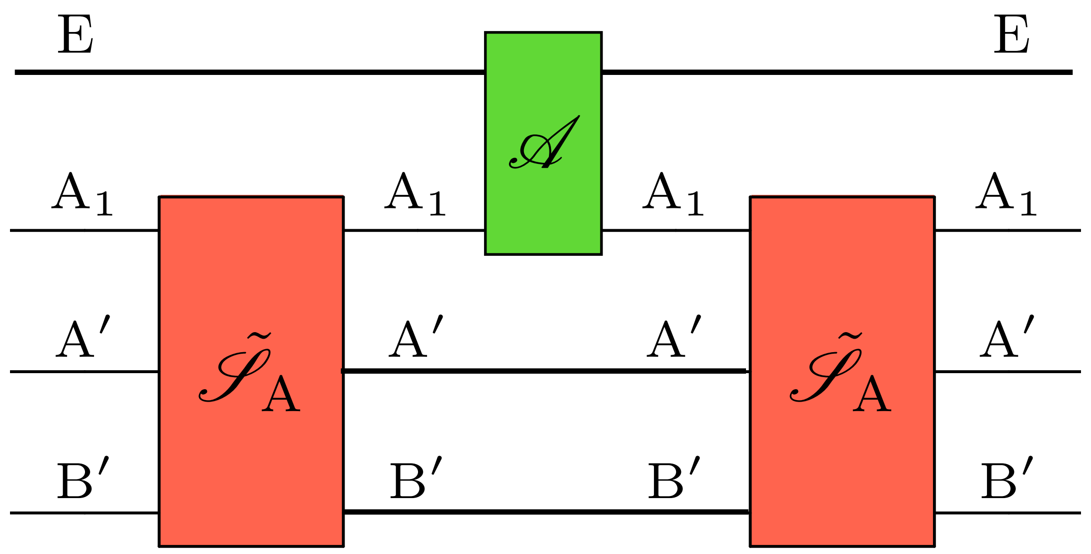

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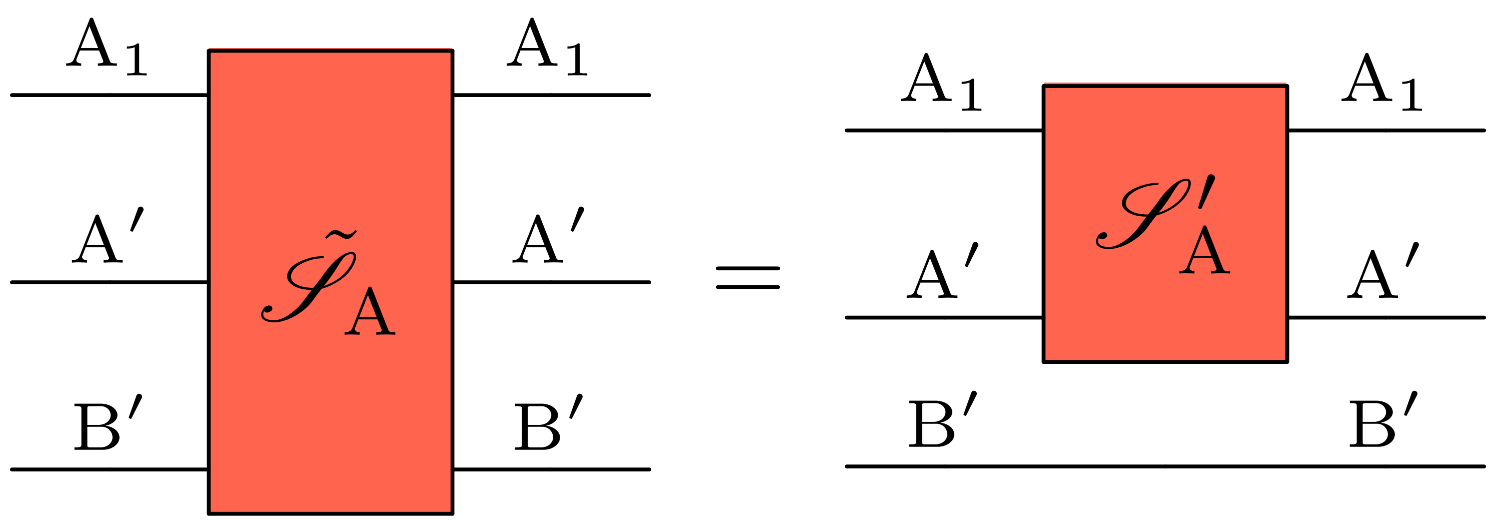

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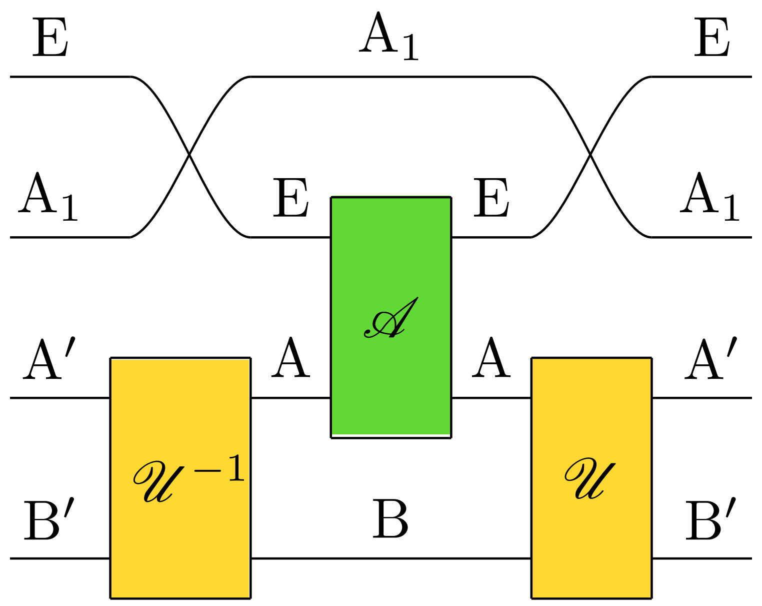

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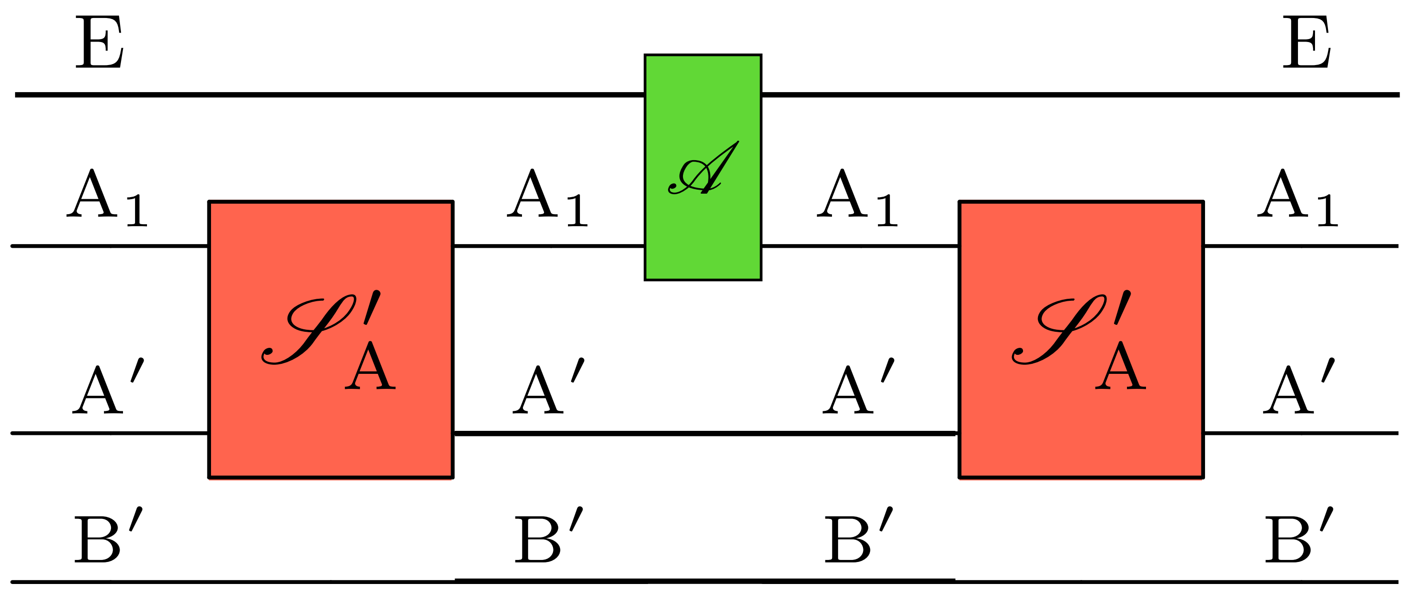

$\Rightarrow$ $A \not\to B'$

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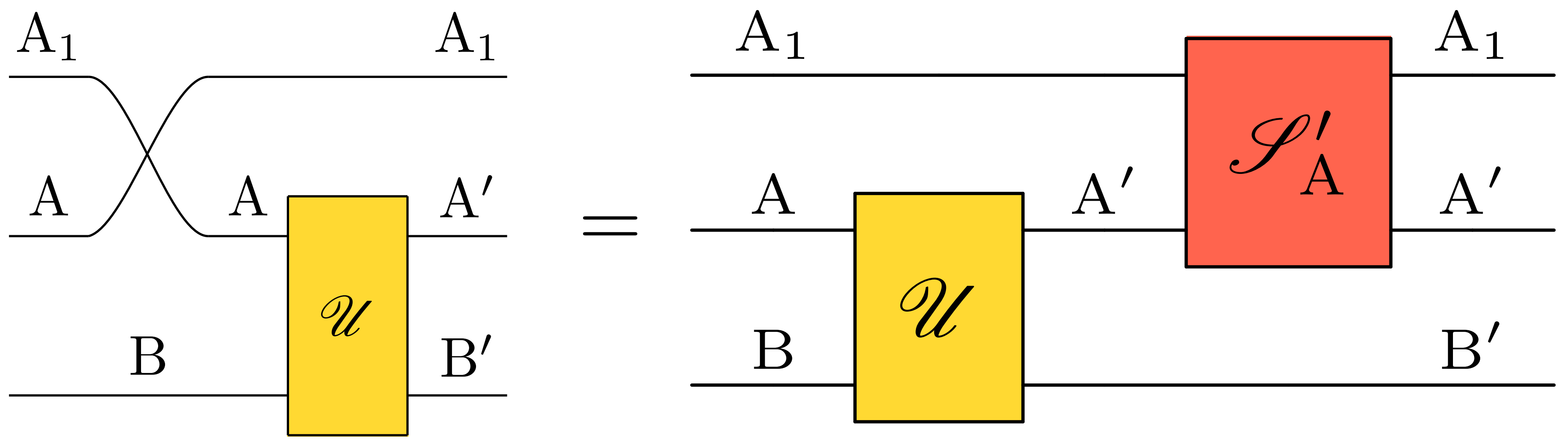
Necessary condition: comb structure

- Suppose that $\mathrm{A} \nrightarrow \mathrm{B}'$. Then it must be



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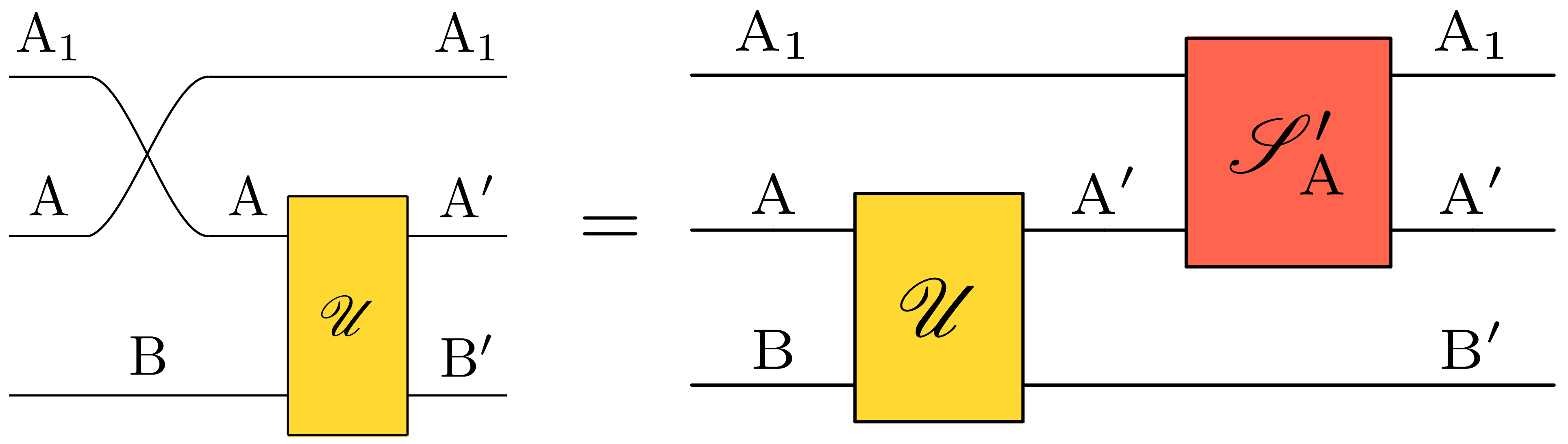
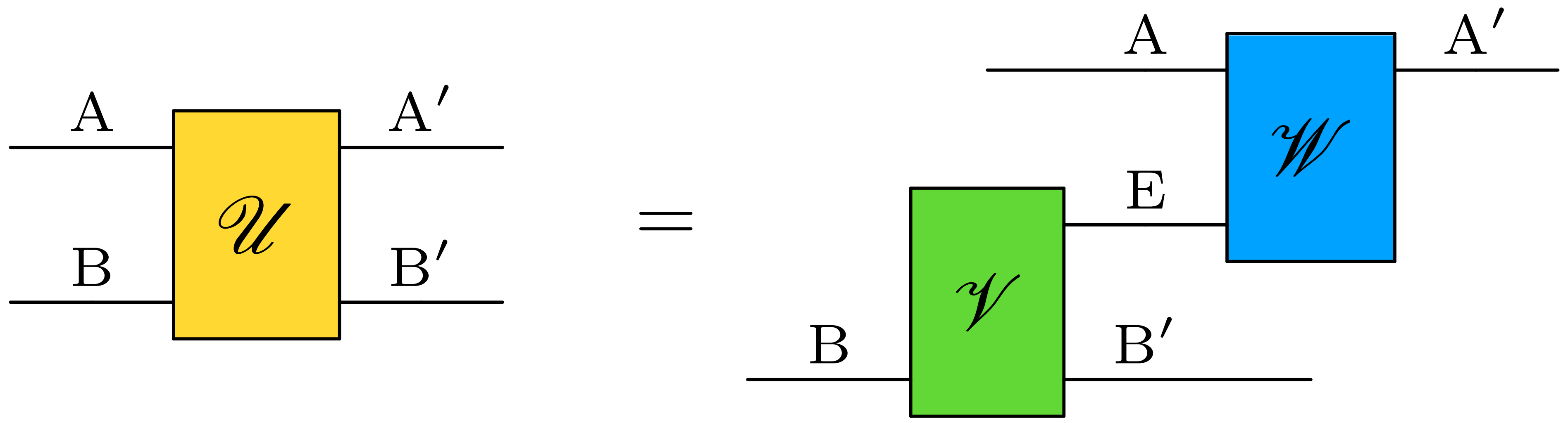


- Preparing a state of A and discarding A_1 we obtain that



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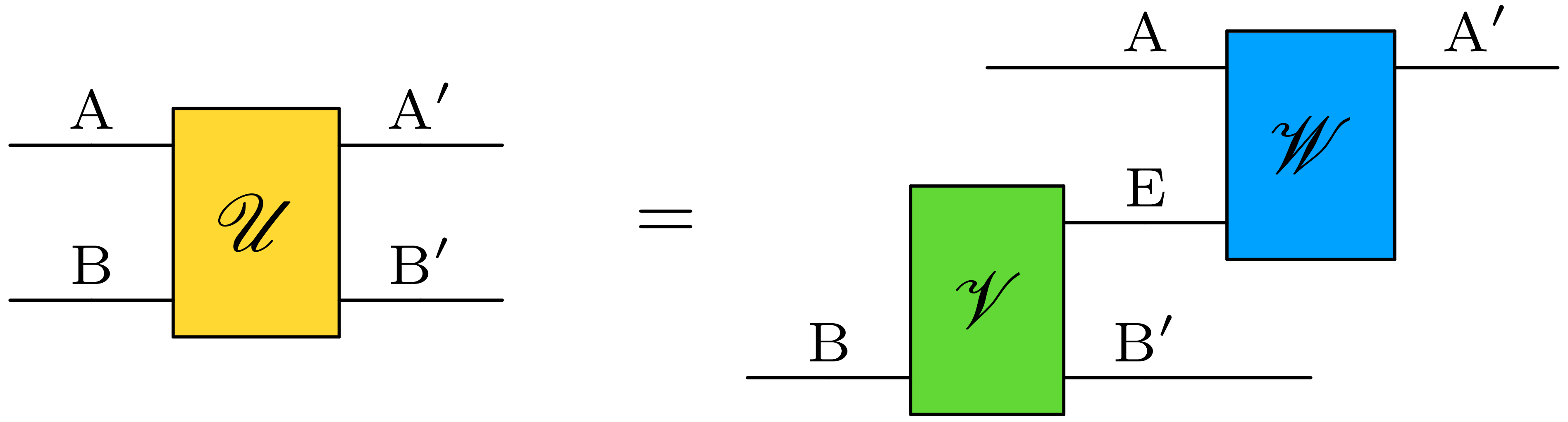
Chain of conditions

$\mathrm{A} \not\to \mathrm{B}'$ ⇒



⇒ $\mathrm{A} \not\rightsquigarrow \mathrm{B}'$

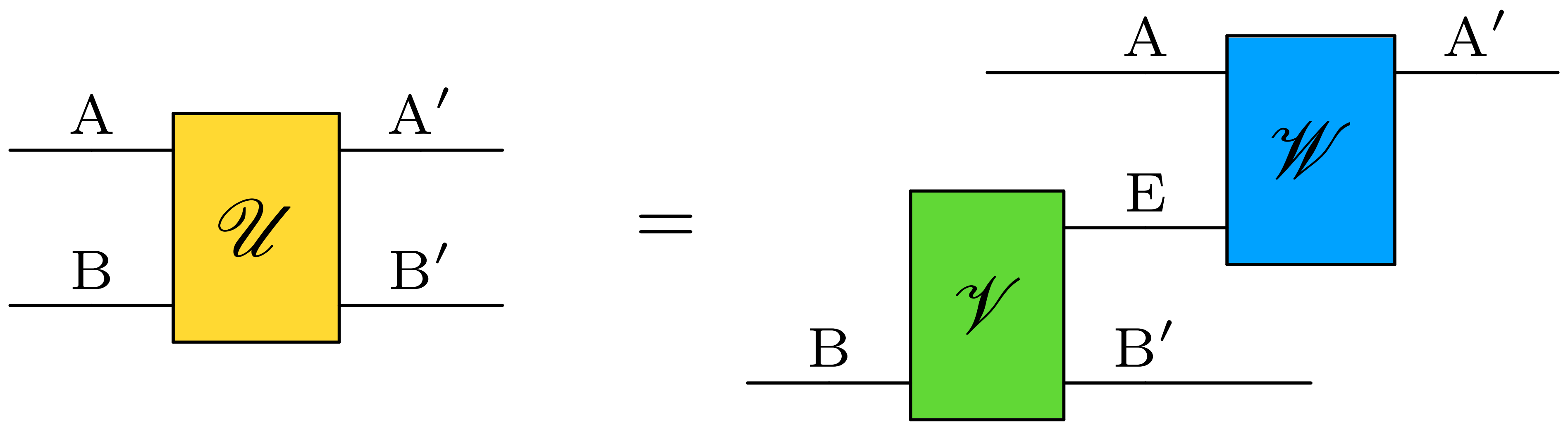
PP, arXiv:2012.15213

Classical theory

Example 1

- One can prove that in classical theory

$$\mathrm{A} \not\rightsquigarrow \mathrm{B}'$$



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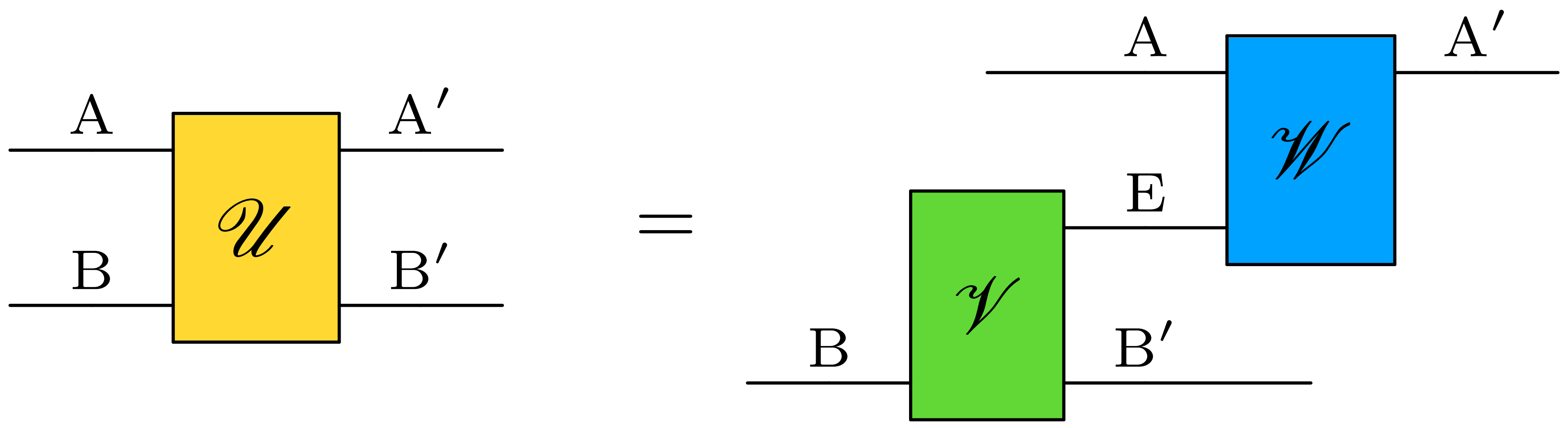
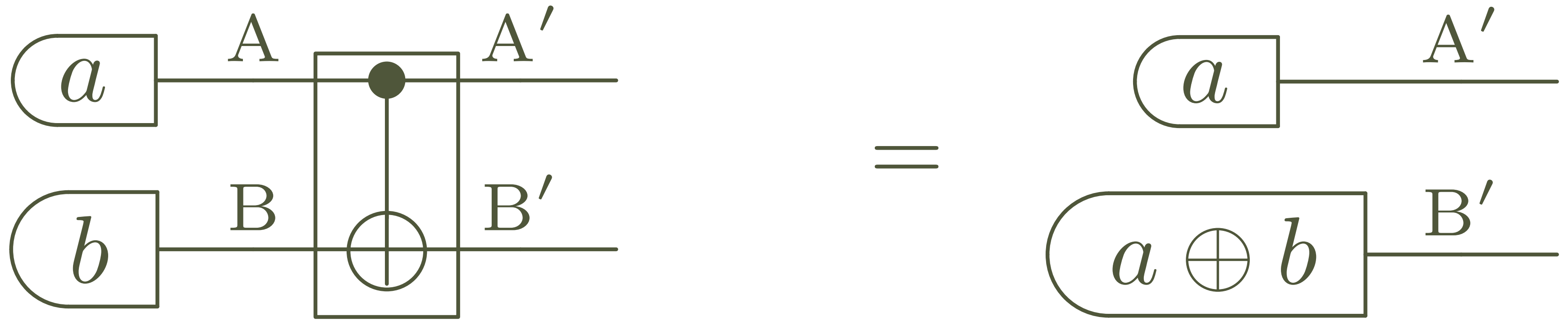


- Classical C-not

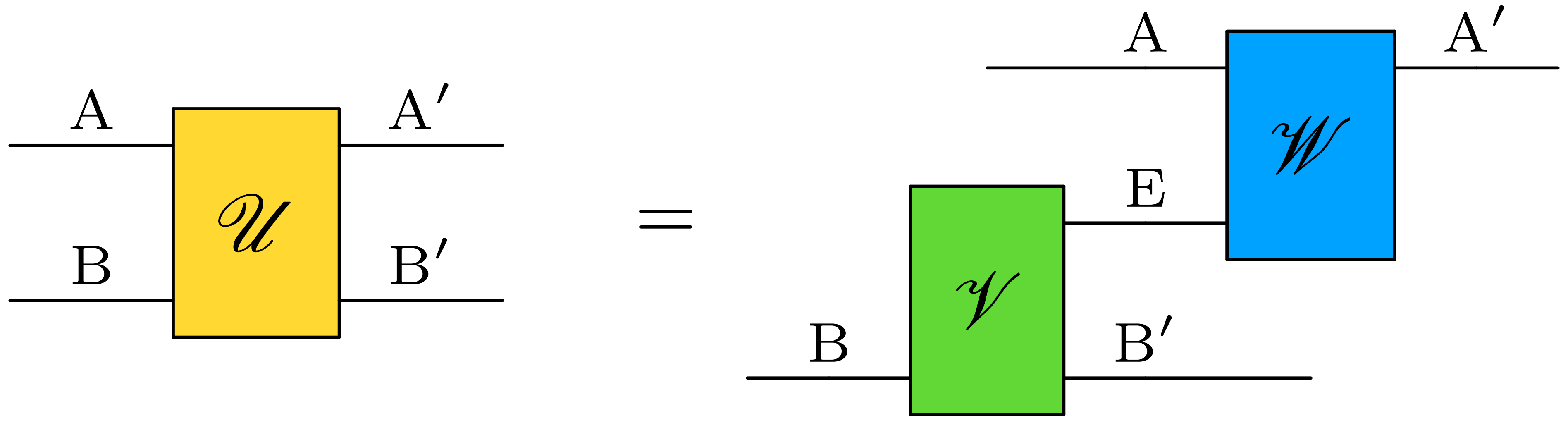


Classical theory

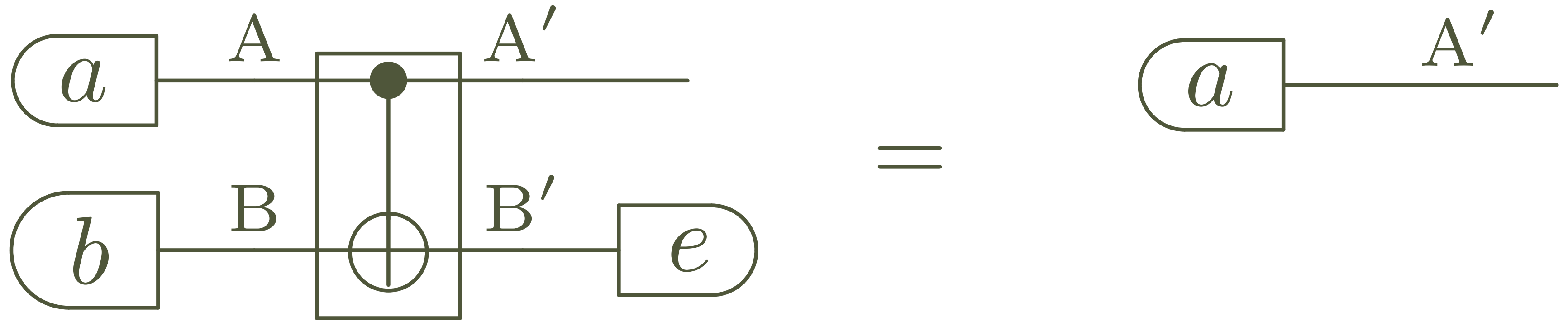
Example 1

- One can prove that in classical theory

$A \not\rightsquigarrow B'$



- Classical C-not



Classical theory

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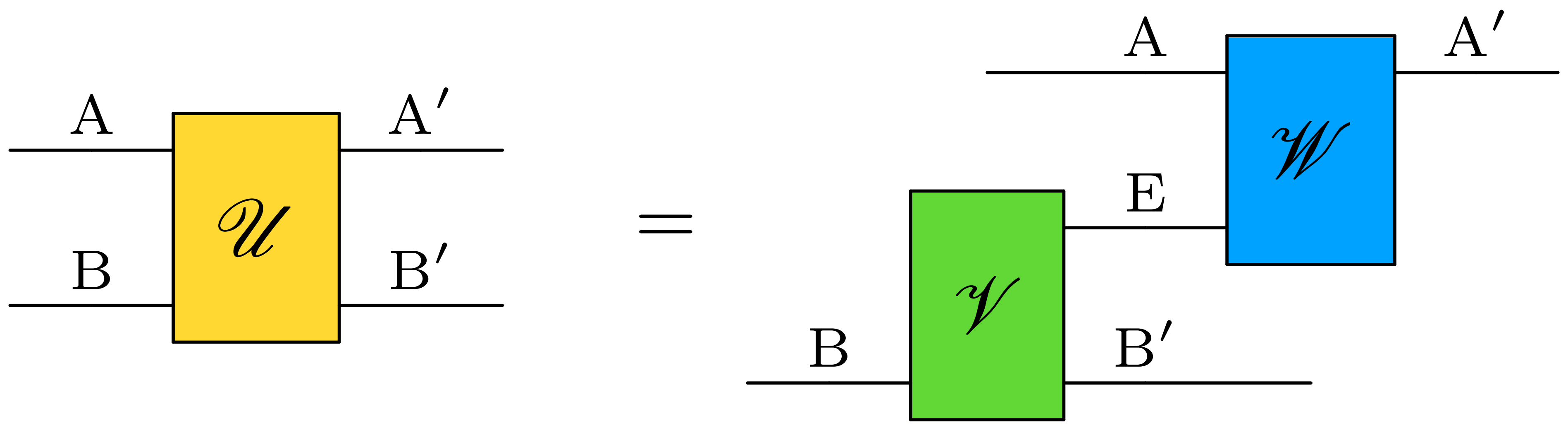

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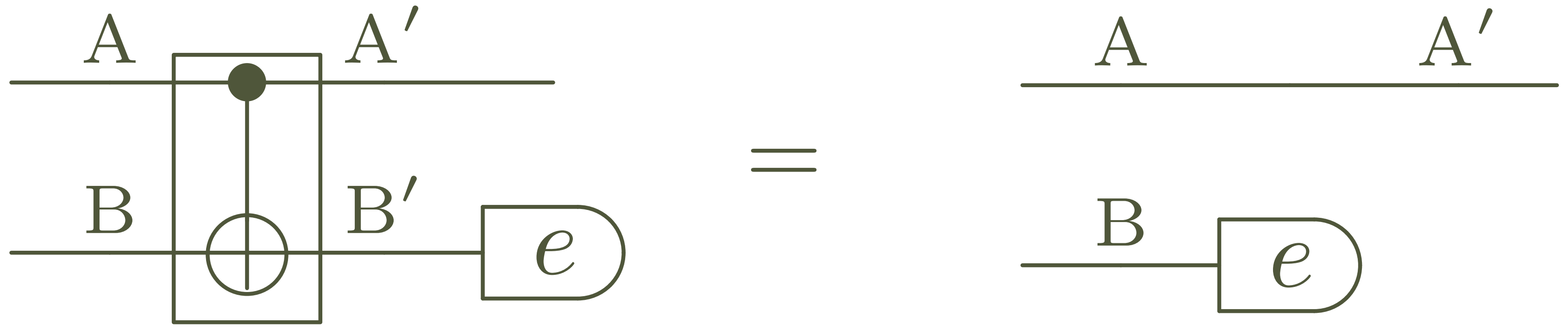

Classical theory

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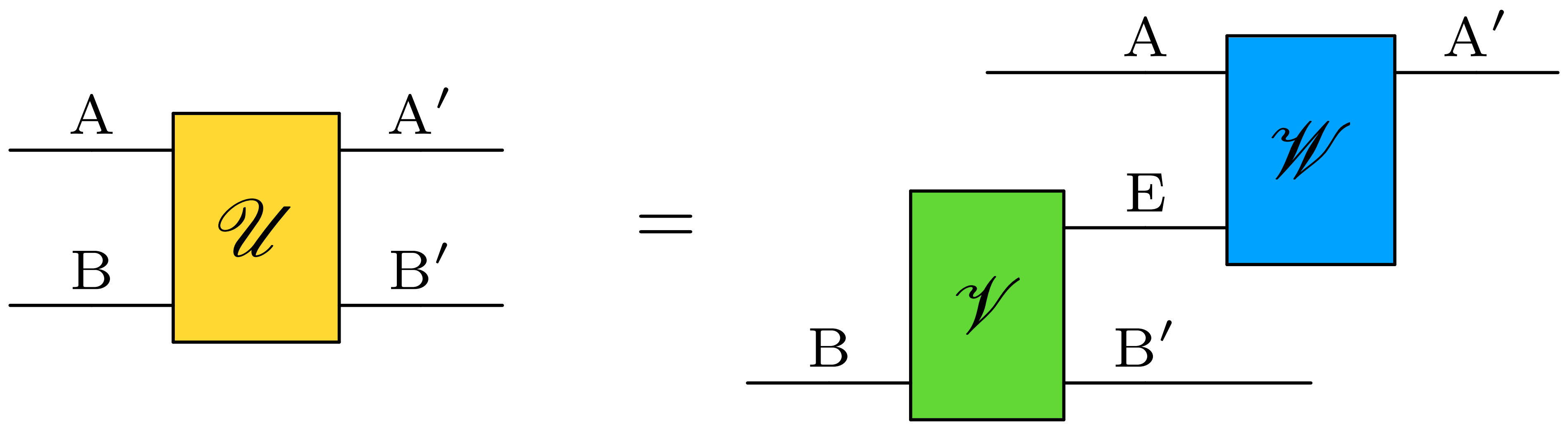

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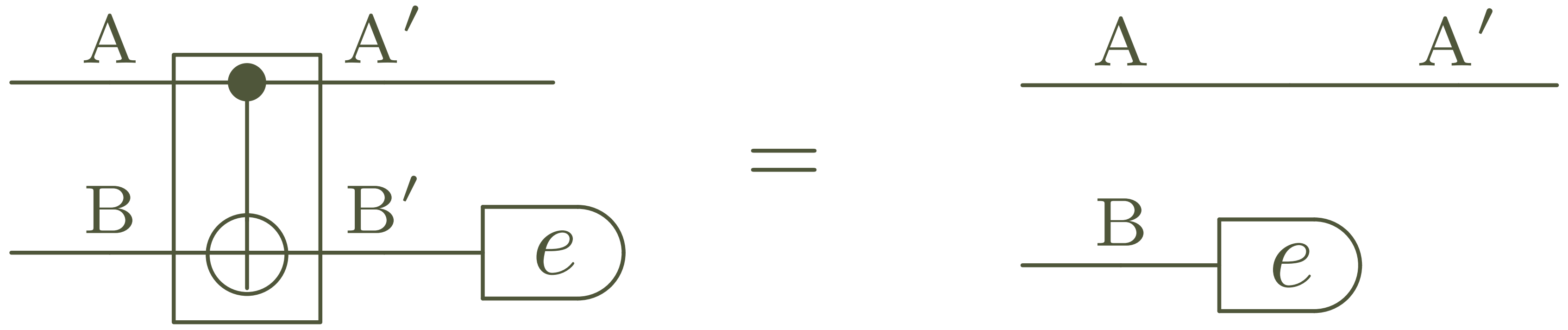
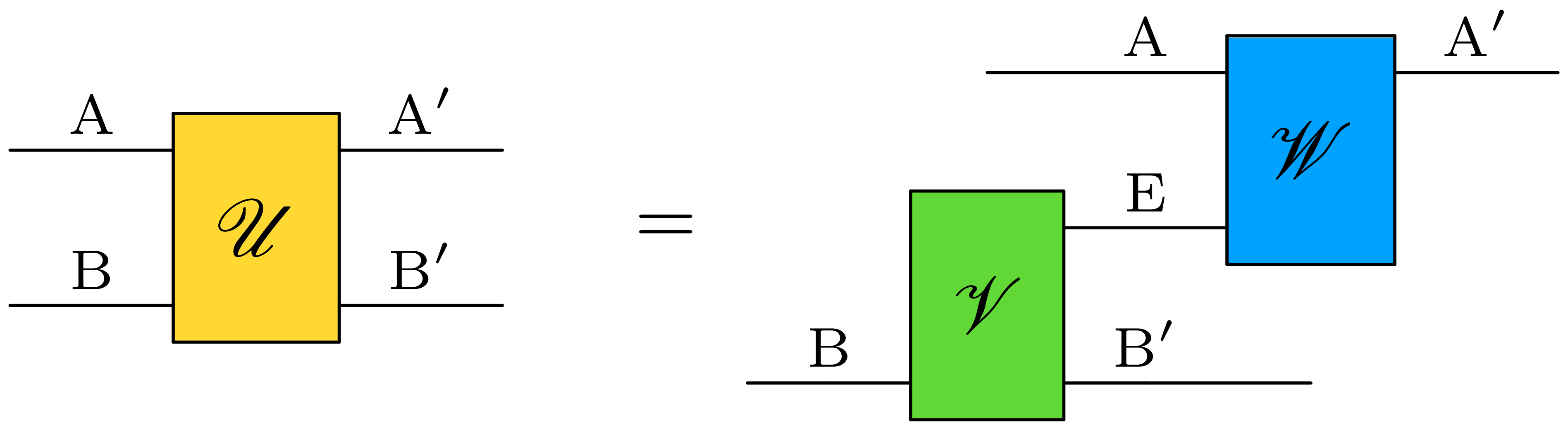


$B \not\rightsquigarrow A'$

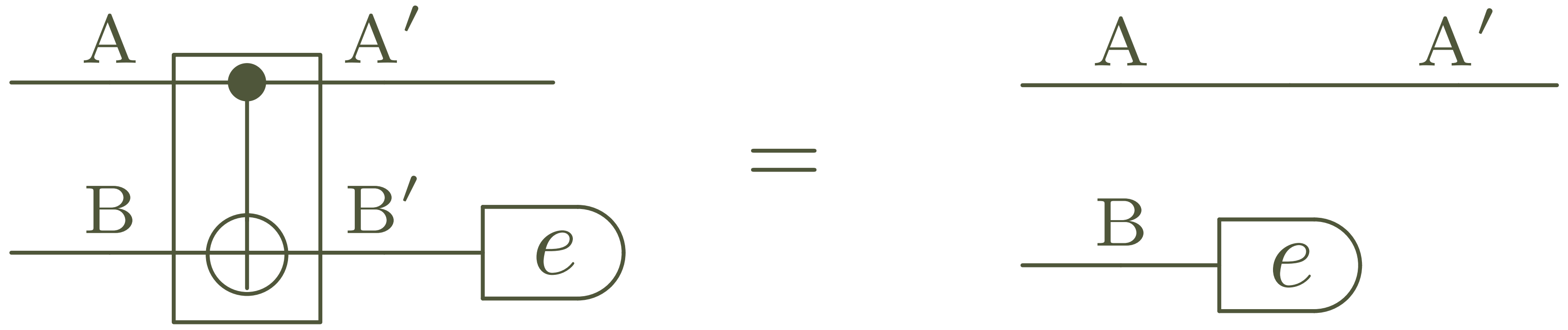
Classical theory

Example 1

- One can prove that in classical theory

$A \nrightarrow B'$ ➡ 

- Classical C-not



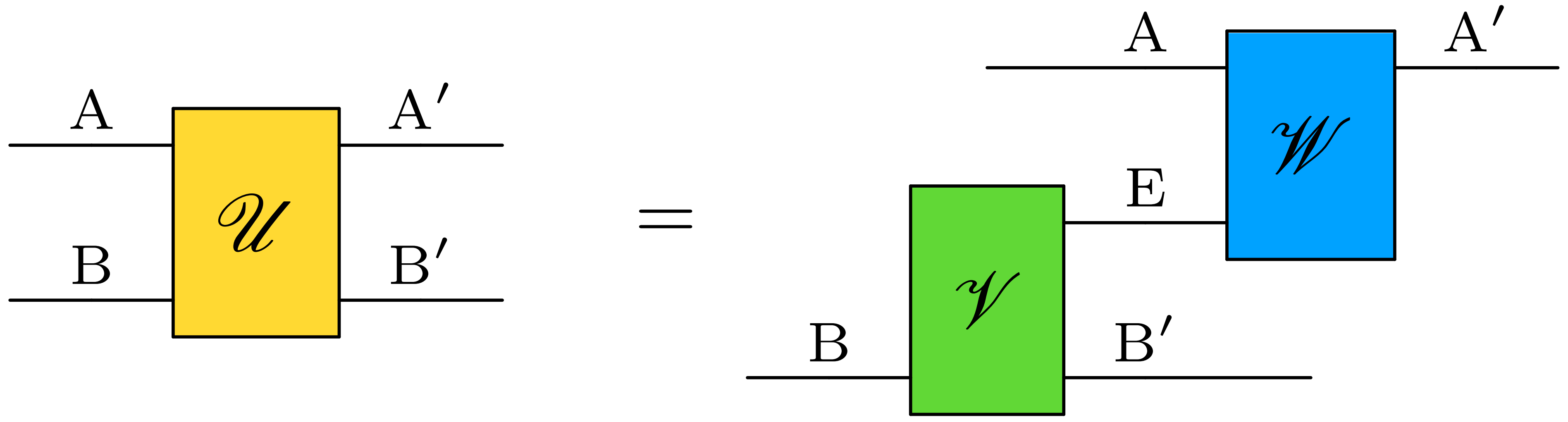
➡ $B \nrightarrow A'$

  - However, $\tilde{\mathscr{S}}_{\mathrm{B}} \neq \mathscr{S}'_{\mathrm{B}} \otimes \mathscr{I}_{\mathrm{A}'}$, thus $\mathrm{B} \rightarrow \mathrm{A}'$

Chain of conditions

In classical theory

$A \nrightarrow B'$ ➡



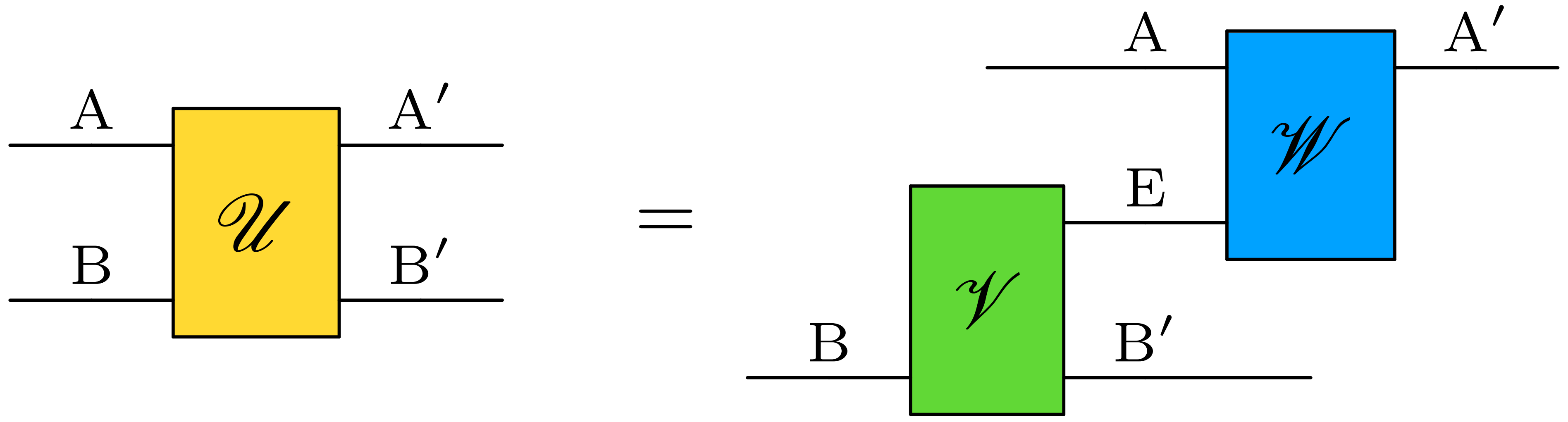
➡ $A \not\rightsquigarrow B'$

PP, arXiv:2012.15213

Chain of conditions

In classical theory

$A \not\to B'$

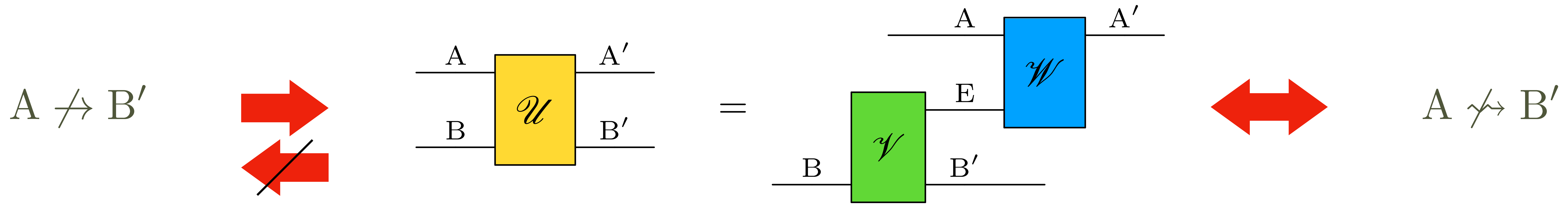


$A \not\rightsquigarrow B'$

PP, arXiv:2012.15213

Chain of conditions

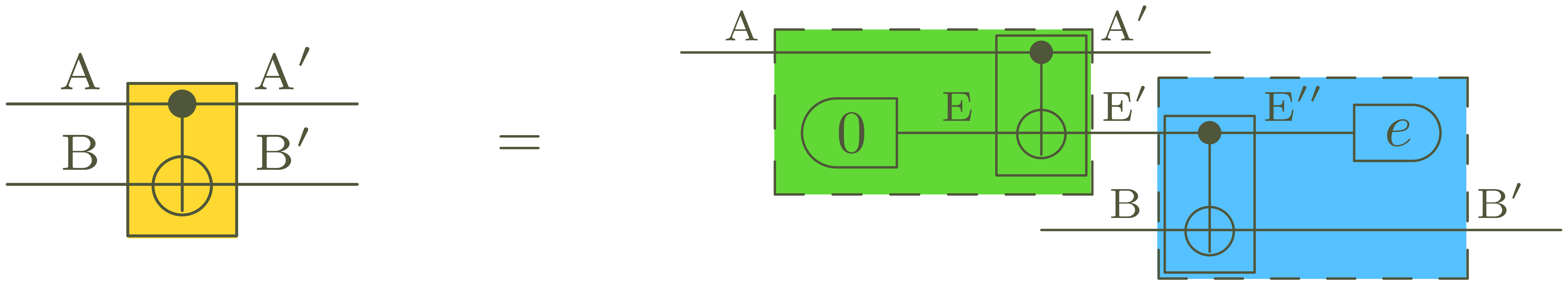
In classical theory



PP, arXiv:2012.15213

Again on the classical C-not

Comb structure

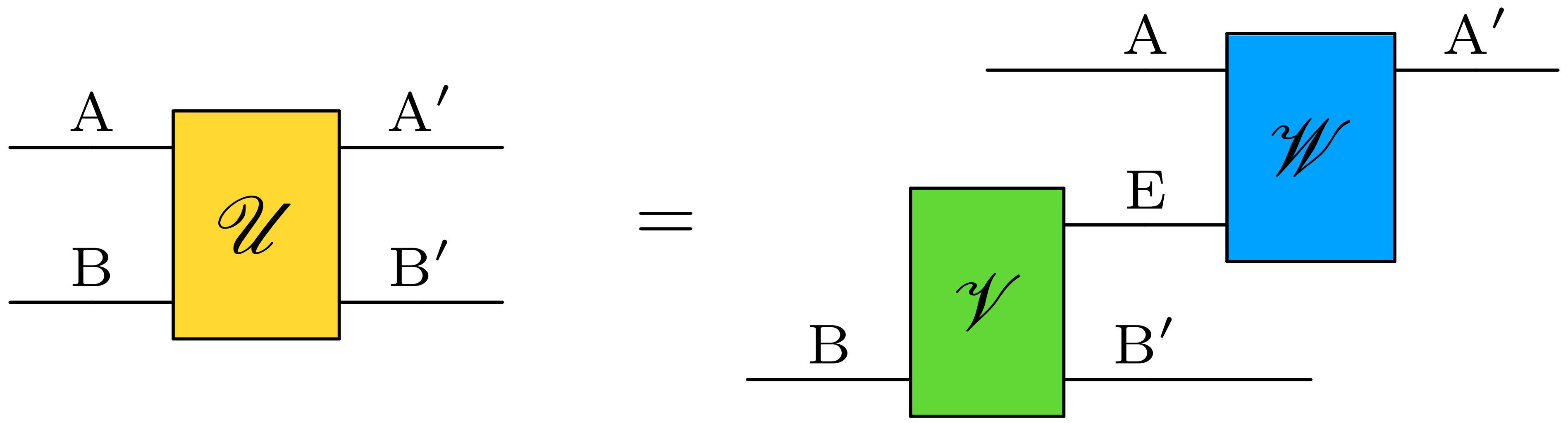


Quantum theory

Example 2

- Also in quantum theory

$$A \not\rightsquigarrow B'$$



T. Eggeling, D. Schlingemann, and R.F. Werner, Europhys. Lett. **57**, pp. 782–788 (2002)

Quantum theory

Example 2

- Also in quantum theory

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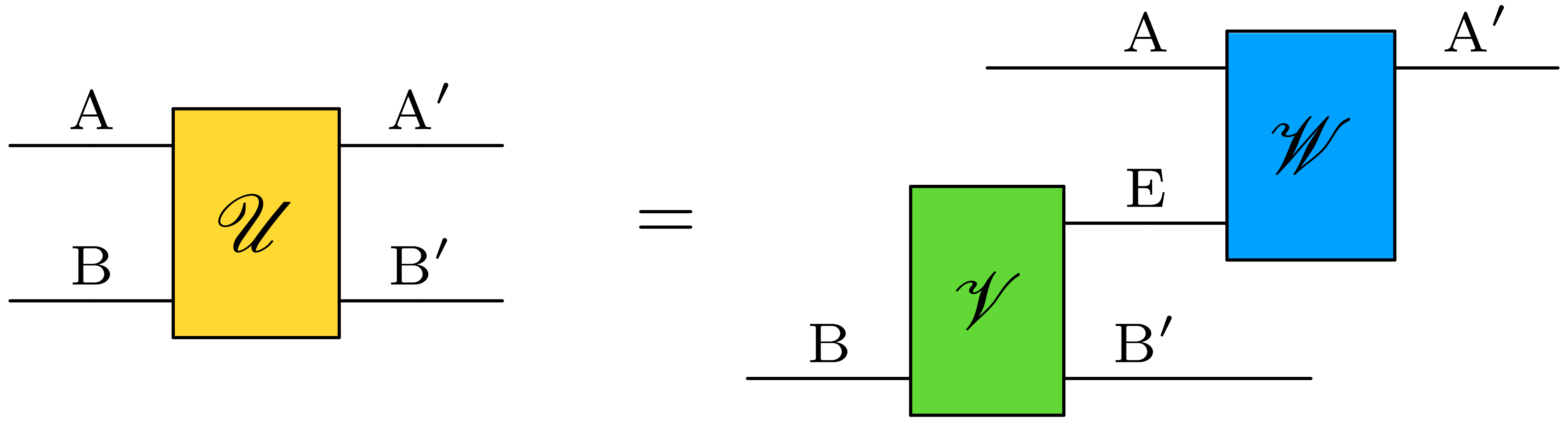
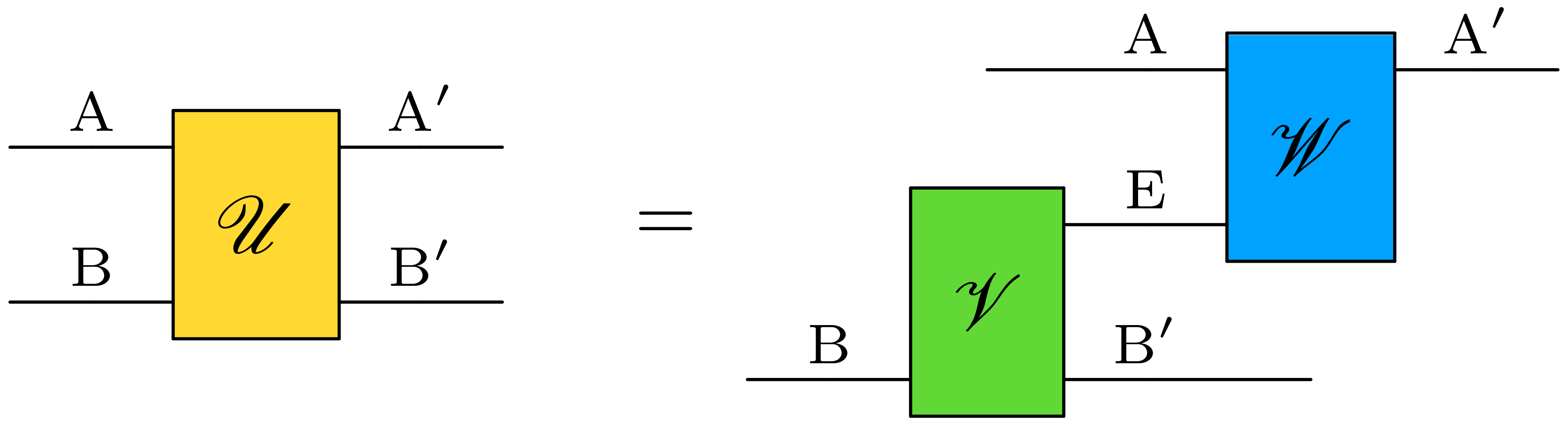


- Thus

$A \not\to B'$



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Quantum theory

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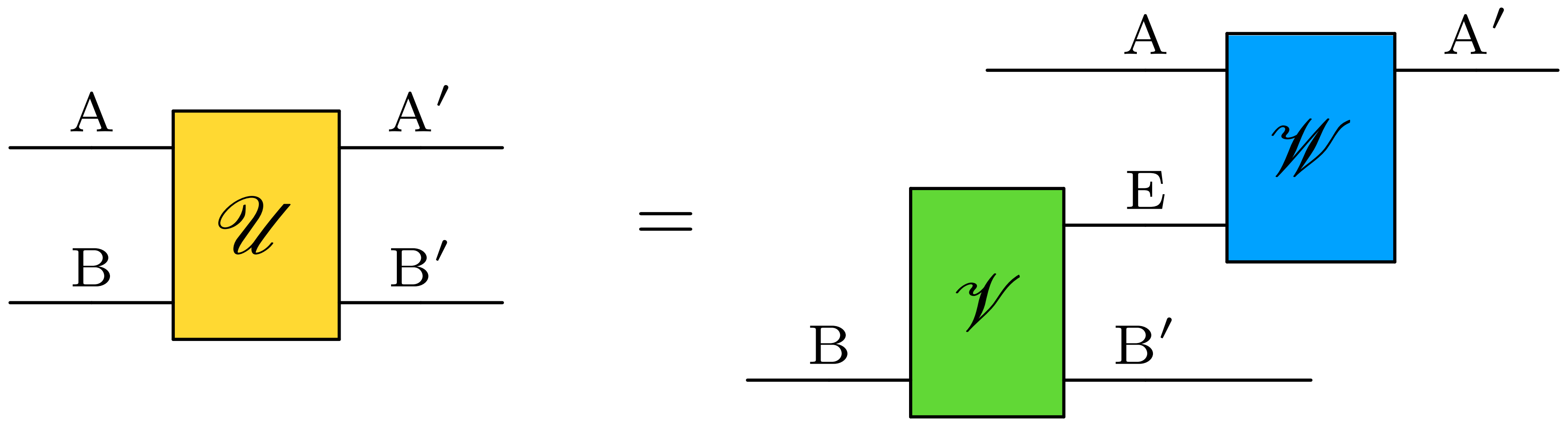

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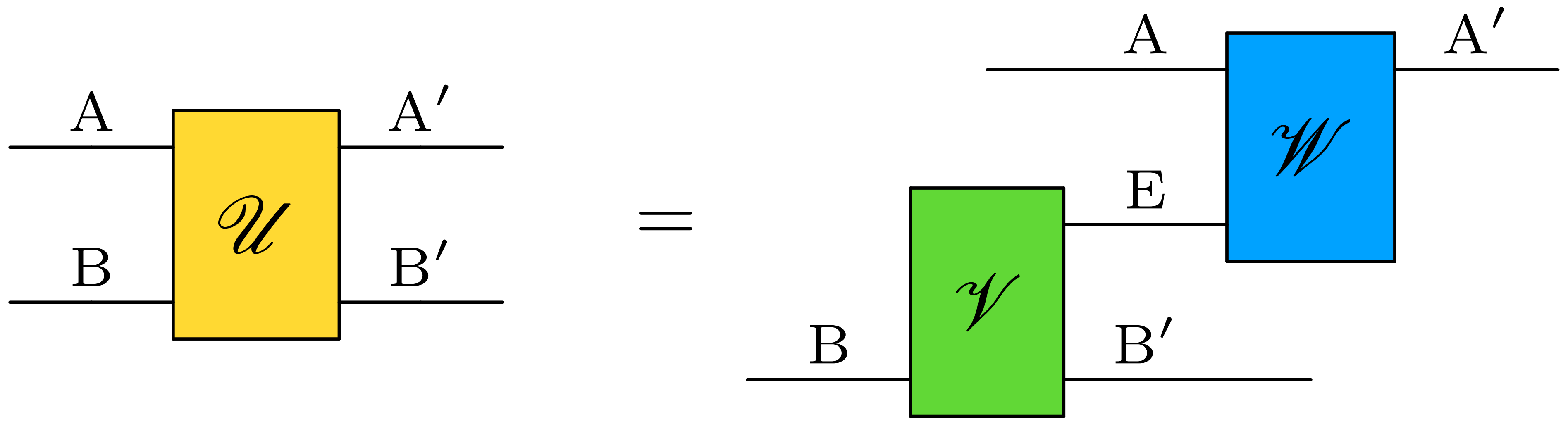


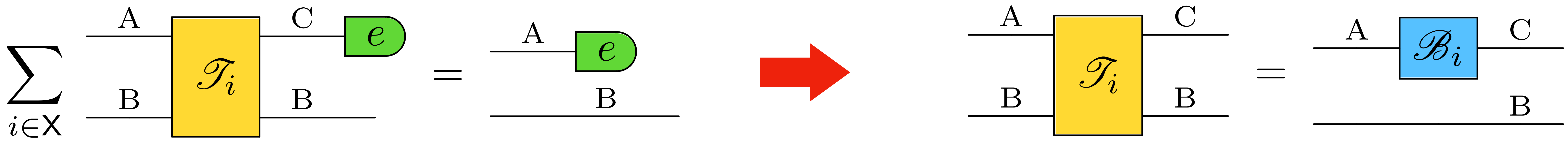
$A \not\rightsquigarrow B'$

- What about the first implication?

T. Eggeling, D. Schlingemann, and R.F. Werner, Europhys. Lett. **57**, pp. 782–788 (2002)

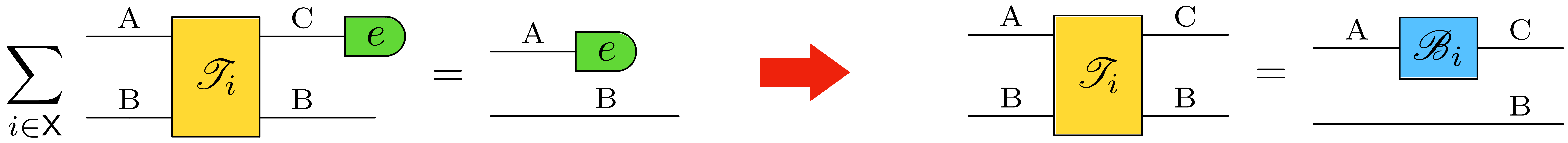
Quantum theory

- From the characterisation of Kraus decompositions of a given channel



Quantum theory

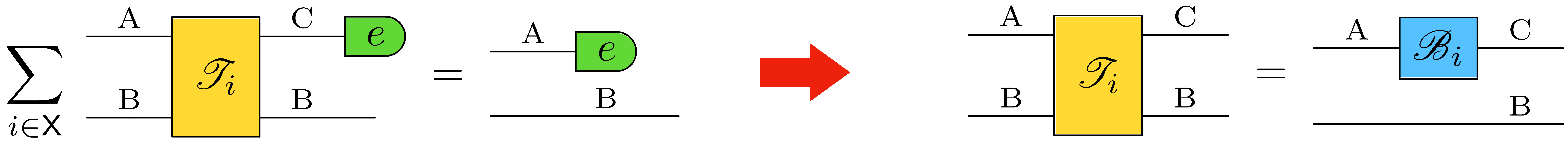
- From the characterisation of Kraus decompositions of a given channel



- Also from purification

Quantum theory

- From the characterisation of Kraus decompositions of a given channel



- Also from purification

➡ The above result holds also in Fermionic theory and Real Quantum theory

Quantum theory

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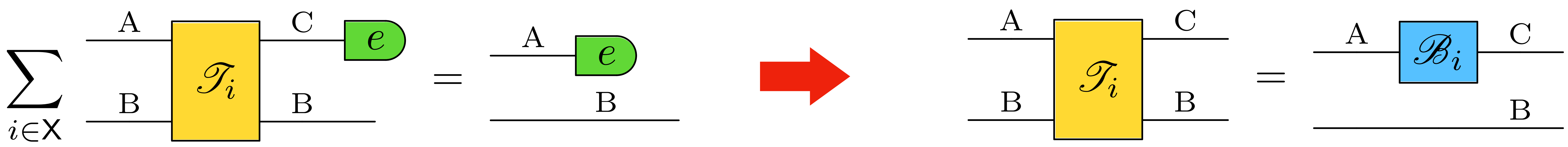


- Also from purification

➡ The above result holds also in Fermionic theory and Real Quantum theory

No interaction without disturbance

Quantum theory

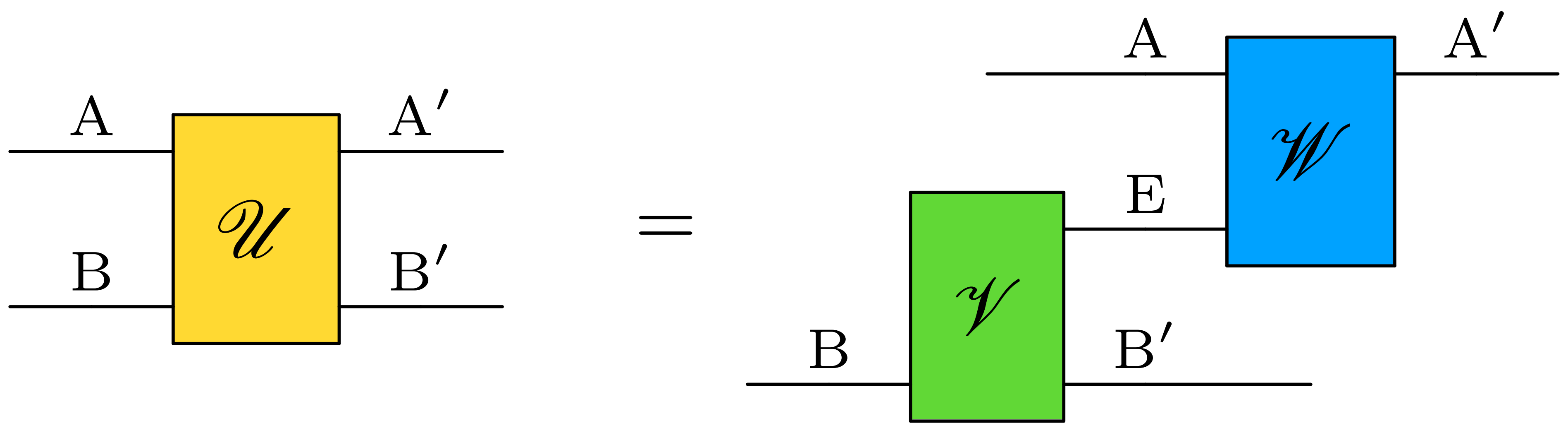
- From no interaction without disturbance one has $\mathrm{A} \not\rightsquigarrow \mathrm{B} \;\Rightarrow\; \mathrm{A} \nrightarrow \mathrm{B}$

PP, arXiv:2012.15213

Quantum theory

- From no interaction without disturbance one has $\mathrm{A} \not\rightsquigarrow \mathrm{B} \Rightarrow \mathrm{A} \not\to \mathrm{B}$
- Thus

$\mathrm{A} \not\to \mathrm{B}' \Leftrightarrow$



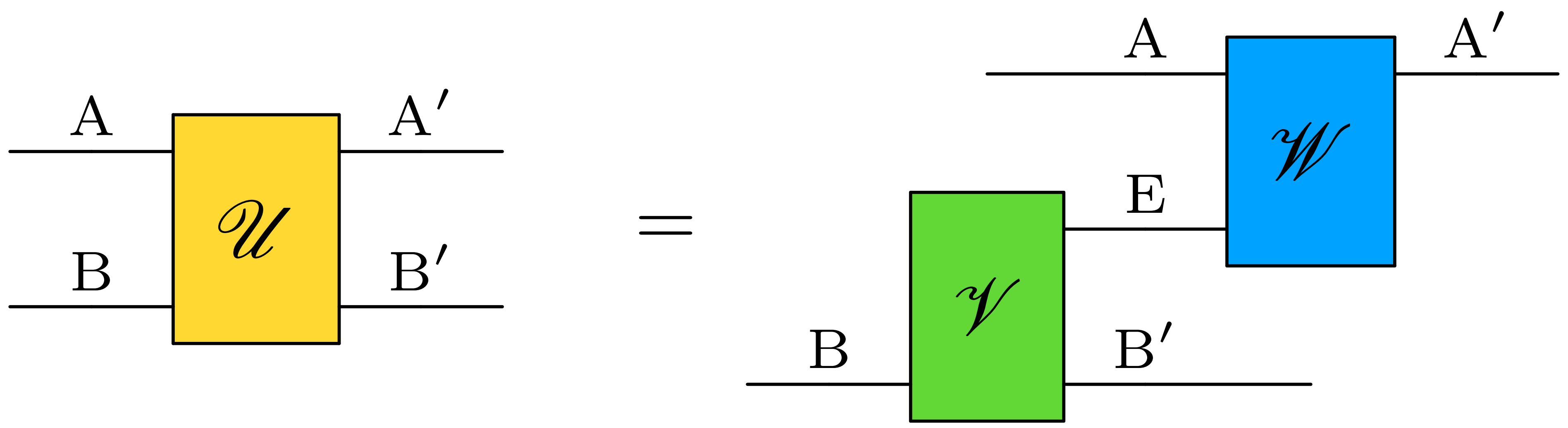
$\Leftrightarrow \mathrm{A} \not\rightsquigarrow \mathrm{B}'$

PP, arXiv:2012.15213

Quantum theory

- From no interaction without disturbance one has $A \not\rightsquigarrow B$ ➡ $A \not\to B$
- Thus

$A \not\to B'$ ⬌



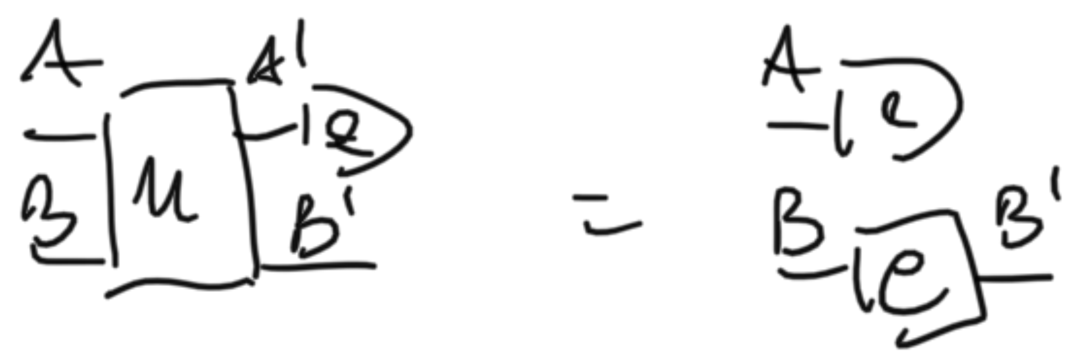
⬌ $A \not\rightsquigarrow B'$

- True in every theory with purification or just no interaction without disturbance

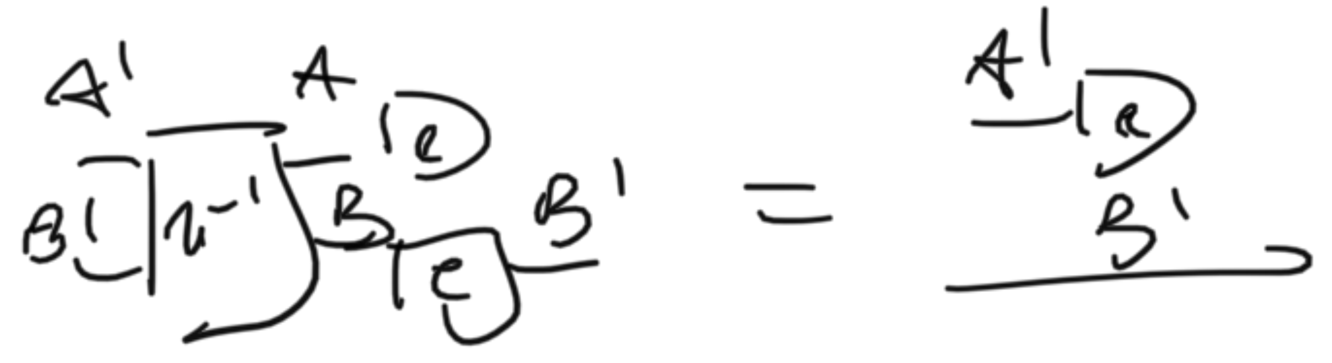
PP, arXiv:2012.15213

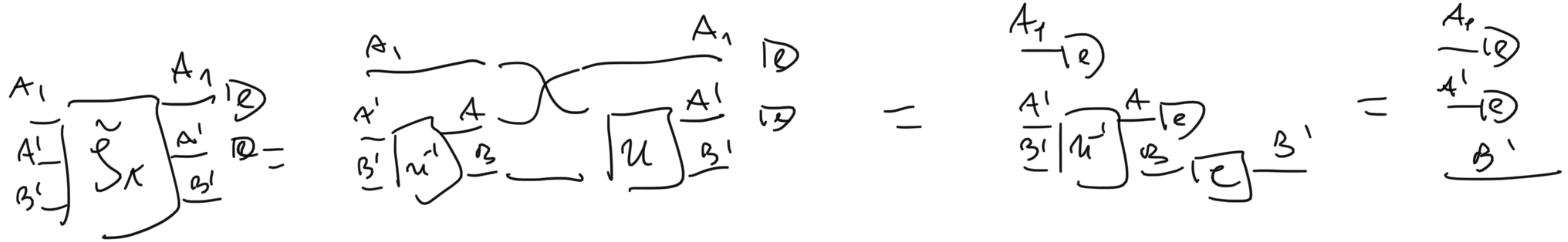
Proof

Depending on time

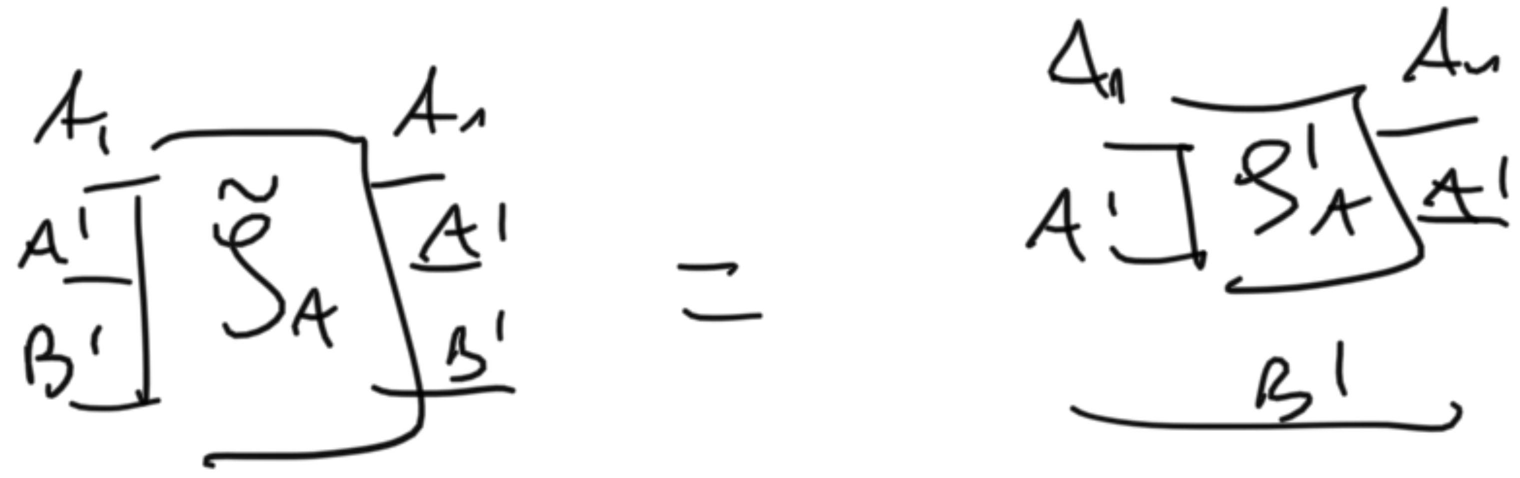


$(A \not\rightsquigarrow B) \Rightarrow$





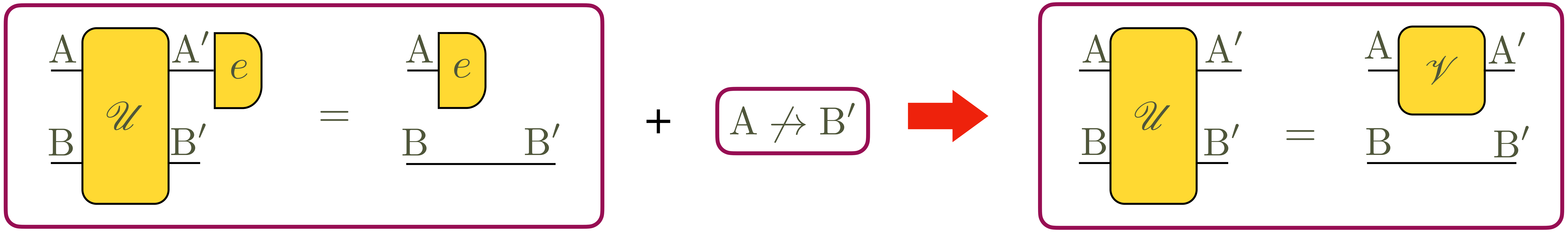
(no int. w.o. dist) $\Rightarrow$



$\Rightarrow A \not\rightarrow B'$

Interaction without disturbance

- What about a theory featuring interactions without disturbance?



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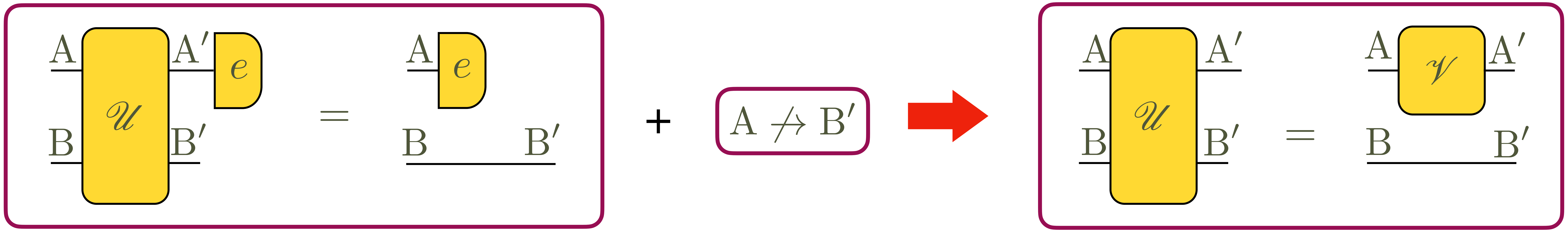
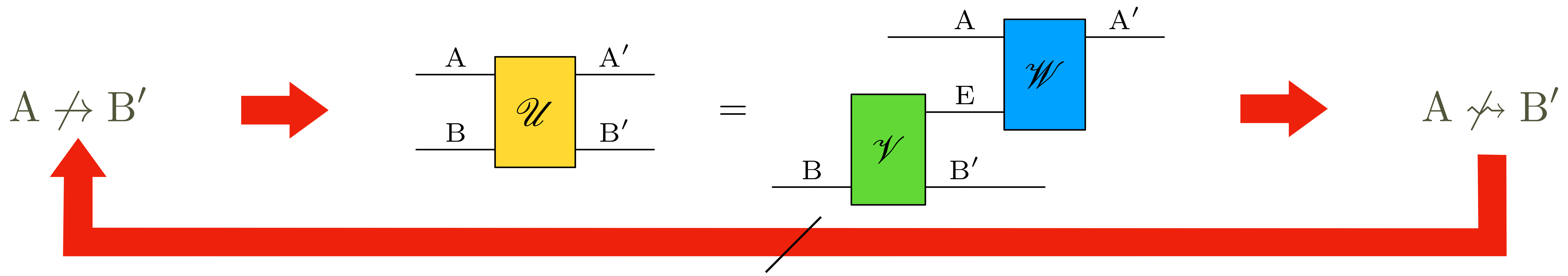


- Thus, if the special interaction without disturbance is reversible, one has



PP, arXiv:2012.15213

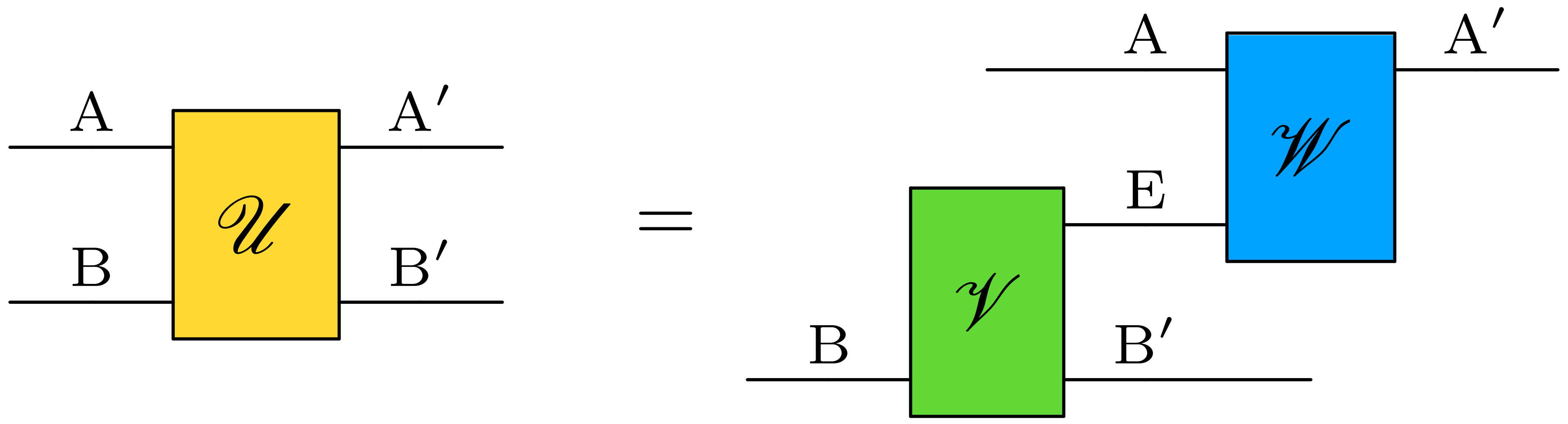
Open question

The quest for counterexamples

Open question

The quest for counterexamples

$\mathrm{A} \not\to \mathrm{B}'$ $\Rightarrow$

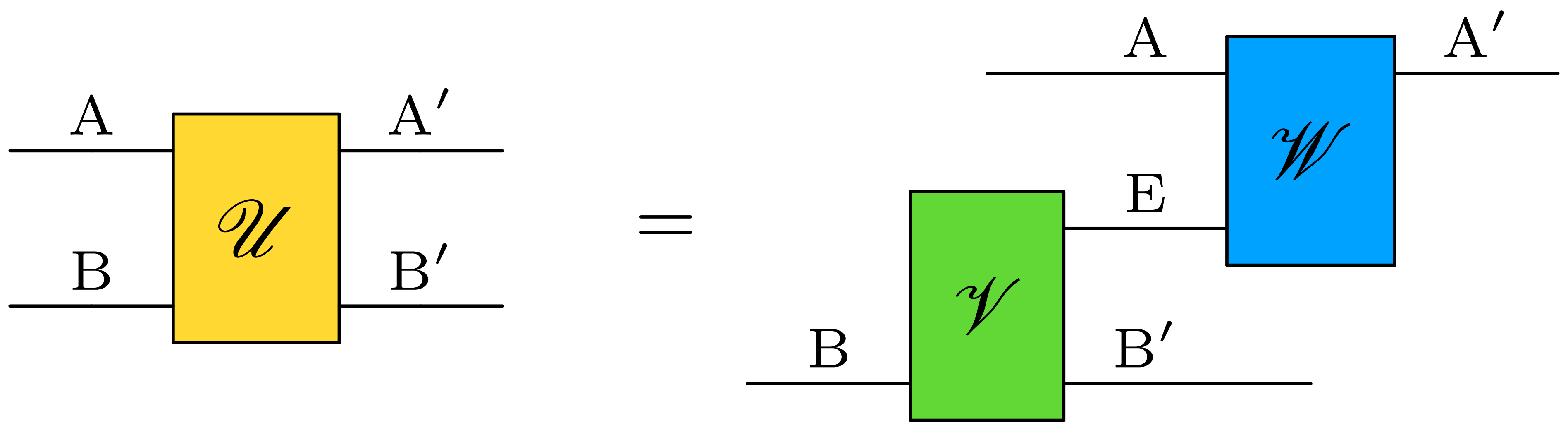


$\Rightarrow$ $\mathrm{A} \not\rightsquigarrow \mathrm{B}'$

Open question

The quest for counterexamples

$A \not\to B' \quad \Rightarrow \quad \not\Leftarrow$



$\Rightarrow \quad A \not\rightsquigarrow B'$

Open question

The quest for counterexamples