Weak lensing as a probe of large-scale structure

I. LENS EQUATIONS

Cosmic shear $\cdots$ Distorsion of distant-galaxy images via weak gravitational lensing by large-scale structure

$\longrightarrow$ its statistical correlation is sensitive to

- evolution of matter fluctuations
- cosmic expansion (through the weight function of geometric distances)

Brightness theorem

$$I_{\rm obs}(\vec{\theta}) = I_{\rm true}(\vec{\theta}_{\rm s}) \tag{1}$$

$I_{\rm obs}$: observed surface brightness of background galaxy
$I_{\rm true}$: surface brightness of background galaxy at its source redshift

The relation between $\vec{\theta}$ and $\vec{\theta}_{\rm s}$ is given by the lens equation. Below, we will derive the lens equation assuming flat cosmology (i.e., $K = 0$).

Photon geodesics

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \quad (i = 1 \sim 3) \tag{2}$$

x^i : (comoving) position of photon, $(x_1, x_2, x_3) = (\chi\,\vec{\theta},\ \chi)$, with $\chi(z) \equiv \int_0^z \frac{c\,dz'}{H(z')}$.
λ : affine parameter

Newton gauge : $\quad ds^2 = -\{1+2\,\Psi(\vec{x})\}\,dt^2 + a^2(t)\,\{1+2\,\Phi(\vec{x})\,\}\,\delta_{ij}\,dx^i dx^j$

We rewrite the geodesic equation (2) in terms of the derivative with respect to χ. To do this, we use

$$\frac{d\chi}{d\lambda} = \frac{d\chi}{dt}\frac{dt}{d\lambda} = -\frac{1}{a}p^0 \simeq -\frac{p}{a}(1-\Psi)\,; \qquad p^2 \equiv g_{ij}p^i p^j \tag{3}$$

$$\left(\because \quad g_{\mu\nu}p^\mu p^\nu = 0 \quad\Longrightarrow\quad -(1+2\Psi)\,(p^0)^2 + g_{ij}\,p^i p^j = 0\right)$$

Then, the transverse component of each term in Eq. (2) (i.e., $i = 1,\,2$) becomes

$$\begin{aligned}\frac{d^2x^i}{d\lambda^2} &= \frac{d^2}{d\lambda^2}(\chi\,\theta^i)\\ &\simeq \frac{p}{a}\frac{d}{d\chi}\left[\frac{p}{a}\frac{d}{d\chi}(\chi\,\theta^i)\right]\\ &= p^2\frac{d}{d\chi}\left[\frac{1}{a^2}\frac{d}{d\chi}(\chi\,\theta^i)\right] \qquad (\because \quad pa = \text{const.})\end{aligned} \tag{4}$$

$$\begin{aligned}\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} &= \frac{p^2}{a^2}(1-\Psi)^2\,\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\chi}\frac{dx^\beta}{d\chi}\\ &\simeq \frac{p^2}{a^2}\left[(\Psi-\Phi)_{,i} - 2a\,H\frac{d}{d\chi}(\chi\,\theta^i)\right],\end{aligned} \tag{5}$$

where we used the fact that $\Gamma^i_{00} = \Psi_{,i}/a^2$, $\Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}(H+\dot{\Phi})$, and $\Gamma^i_{jk} = \Gamma^i_{j0} = \delta_{ij}\Phi_{,k} + \delta_{ik}\Phi_{,j} - \delta_{jk}\Phi_{,i}$. Summing up the above two contributions, the geodesic equation can be rewritten with

$$\frac{d}{d\chi}\left[\frac{1}{a^2}\frac{d}{d\chi}(\chi\,\theta^i)\right] + \frac{1}{a^2}\left\{(\Psi-\Phi)_{,i} - 2a\,H\frac{d}{d\chi}(\chi\,\theta^i)\right\} = 0 \tag{6}$$

This is further reduced to a simplified equation:

$$\frac{d^2}{d\chi^2}(\chi\,\theta^i) = (\Phi-\Psi)_{,i} \tag{7}$$

Solving Eq. (7) with the boundary condition of $\theta^i = \theta^i_o$ at $\chi = 0$ and $\theta^i = \theta^i_s$ at $\chi = \chi_s(>0)$:

Lens equation

$$\begin{aligned}\theta^i_s &= \theta^i_o + \frac{1}{\chi_s}\int_0^{\chi_s} d\chi_1 \int_0^{\chi_1} d\chi_2\,\Big\{\Phi(\vec{x}(\chi_2)) - \Psi(\vec{x}(\chi_2))\Big\}_{,i}\\ &= \theta^i_o + \frac{1}{\chi_s}\int_0^{\chi_s} d\chi_2 \int_{\chi_2}^{\chi_s} d\chi_1\,\Big\{\Phi(\vec{x}(\chi_2)) - \Psi(\vec{x}(\chi_2))\Big\}_{,i}\\ &= \theta^i_o + \int_0^{\chi_s} d\chi'\,\frac{\chi_s-\chi'}{\chi_s}\Big\{\Phi(\vec{x}(\chi')) - \Psi(\vec{x}(\chi'))\Big\}_{,i}\end{aligned} \tag{8}$$

Note that subscript $(_{,i})$ implies $\frac{d}{dx^i} = \frac{1}{\chi}\frac{d}{d\theta^i}$.

II. CONVERGENCE AND SHEAR

The lens equation (8) describes how the image of background galaxy is deformed accodring to the gravitational potential of foreground large-scale structure. To see this more explicitly, we define the **deformation matrix**:

$$A_{ij} \equiv \frac{\partial \theta_s^i}{\partial \theta_o^j} = \delta_{ij} + \int_0^{\chi_s} d\chi \frac{(\chi_s - \chi)\chi}{\chi_s} \left(\Phi - \Psi\right)_{,ij} \tag{9}$$

Here we used $\frac{d}{d\theta_o^i} = \chi \frac{d}{dx^i}$ $\quad(\because \;\; x^i = \chi\, \theta_o^i)$. The above deformation matrix is rewritten in the following form:

$$A_{ij} \;\; = \;\; \delta_{ij} - \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \tag{10}$$

with κ and γ_i beging defined by

$$\text{convergence :} \qquad \kappa(\vec{\theta}) = -\frac{1}{2}\int_0^{\chi_s} d\chi \, \frac{(\chi_s - \chi)\chi}{\chi_s} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\Phi - \Psi) \tag{11}$$

$$\text{shear :} \quad \begin{cases} \gamma_1(\vec{\theta}) = -\dfrac{1}{2}\displaystyle\int_0^{\chi_s} d\chi \, \frac{(\chi_s - \chi)\chi}{\chi_s} \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)(\Phi - \Psi) \\[2ex] \gamma_2(\vec{\theta}) = \;\; -\displaystyle\int_0^{\chi_s} d\chi \, \frac{(\chi_s - \chi)\chi}{\chi_s} \;\; \frac{\partial^2}{\partial x_1 \partial x_2}(\Phi - \Psi) \end{cases} \tag{12}$$

Note that weak lensing implies κ, $|\gamma| \ll 1$.

For further simplification (but still practically useful treatment), we may write $(\Phi - \Psi) = -2\,\Psi$ in the absence of anisotropic stress. We then note that

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] \Psi = \left[\nabla^2 - \frac{\partial^2}{\partial x_3^2}\right] \Psi = \left[\nabla^2 - \frac{1}{\chi^2}\frac{\partial}{\partial \chi}\left(\chi^2 \frac{\partial}{\partial \chi}\right)\right] \Psi. \tag{13}$$

The second term is nothing but the time derivative of the potential. Since the potential is nearly constant in time, the second term is rather smaller than the first term, and we may ignore it. Thus, through the Poisson equation, Eq. (11) can be rewritten with

$$\begin{aligned} \kappa(\vec{\theta}) &= \int_0^{\chi_s} d\chi \, \frac{(\chi_s - \chi)\chi}{\chi_s} \, \nabla^2 \Psi(\vec{x};\chi) \\ &= \frac{3}{2}\,\Omega_{\rm m} H_0^2 \int_0^{\chi_s} d\chi \, \frac{(\chi_s - \chi)\chi}{\chi_s} \, \frac{\delta(\vec{x};\chi)}{a(\chi)}. \end{aligned} \tag{14}$$

Based on Eq. (14), a couple of generalizations to be noted is:

- Non-flat space:

  Replacing the comoving radial distance χ in the kernel of integration with $f_K(\chi)$:

$$\longrightarrow \quad \kappa(\vec{\theta}) = \frac{3}{2}\,\Omega_{\rm m}H_0^2 \int_0^{\chi_s} d\chi\, \frac{f_K(\chi_s-\chi)\, f_K(\chi)}{f_K(\chi_s)}\, \frac{\delta(\vec{x};\chi)}{a(\chi)}. \tag{15}$$

- Continuous source distrituion:

  Eq. (14) is only applied to the case with single-source plane at $\chi = \chi_s$, but with a broad source distribution of $w_g(\chi)$, the expression is generalized to

$$\begin{aligned}\longrightarrow \quad \kappa(\vec{\theta}) &= \frac{3}{2}\,\Omega_{\rm m}H_0^2 \int_0^{\infty} d\chi_s\, w_g(\chi_s) \int_0^{\chi_s} d\chi\, \frac{f_K(\chi_s-\chi)\, f_K(\chi)}{f_K(\chi_s)}\, \frac{\delta(\vec{x};\chi)}{a(\chi)} \\ &= \int_0^{\infty} d\chi\, \frac{g(\chi)}{a(\chi)}\, \delta(\vec{x};\chi) \end{aligned} \tag{16}$$

  with the function $g(\chi)$ given by

$$g(\chi) = \frac{3}{2}\,\Omega_{\rm m}H_0^2 \int_{\chi}^{\infty} d\chi_s\, \frac{f_K(\chi_s-\chi)\, f_K(\chi)}{f_K(\chi_s)}\, w_g(\chi_s) \tag{17}$$

<u>κ and γ in harmonic space</u>

The relation between convergence and shear fields [Eq. (11) and (12)] may become transparent when we go to harmonic space. In flat-sky limit, the harmonic expansion is simply reduced to the Fourier expansion:

$$\widetilde{\kappa}(\vec{\ell}) = \int d^2\vec{\theta}\; e^{i\,\vec{\ell}\cdot\vec{\theta}}\, \kappa(\vec{\theta}) \tag{18}$$

Then, we have

$$\widetilde{\gamma}_1(\vec{\ell}) = \frac{\ell_1^2-\ell_2^2}{\ell^2}\, \widetilde{\kappa}(\vec{\ell}), \tag{19}$$

$$\widetilde{\gamma}_2(\vec{\ell}) = 2\frac{\ell_1\ell_2}{\ell^2}\, \widetilde{\kappa}(\vec{\ell}). \tag{20}$$

Note that $\ell^2 = \ell_1^2 + \ell_2^2$. Defining $(\cos\phi_\ell,\ \sin\phi_\ell) = (\ell_1/\ell,\ \ell_2/\ell)$, the above expressions are written with

$$\widetilde{\gamma}_1(\vec{\ell}) = \cos(2\phi_\ell)\, \widetilde{\kappa}(\vec{\ell}), \tag{21}$$

$$\widetilde{\gamma}_2(\vec{\ell}) = \sin(2\phi_\ell)\, \widetilde{\kappa}(\vec{\ell}). \tag{22}$$

This relation implies that the shear field $\gamma = \gamma_1 + i\gamma_2$ has a spin-2 nature. We thus realize that the following decomposition is very useful to uniquely pick up the physically non-vanishing lensing effect:

E-/B-mode decomposition

$$\begin{pmatrix} \gamma_{\rm E}(\vec{\ell}) \\ \gamma_{\rm B}(\vec{\ell}) \end{pmatrix} \equiv \begin{pmatrix} \cos(2\phi_\ell) & \sin(2\phi_\ell) \\ -\sin(2\phi_\ell) & \cos(2\phi_\ell) \end{pmatrix} \begin{pmatrix} \widetilde{\gamma}_1(\vec{\ell}) \\ \widetilde{\gamma}_2(\vec{\ell}) \end{pmatrix} \tag{23}$$

$$\Longleftrightarrow \quad \gamma_{\rm E}(\vec{\ell}) \; + \; i \, \gamma_{\rm B}(\vec{\ell}) \; = \; e^{-i\,2\phi_\ell} \left[\widetilde{\gamma}_1(\vec{\ell}) \; + \; i \, \widetilde{\gamma}_2(\vec{\ell}) \right] \tag{24}$$

With this decomposition, we have

$$\gamma_{\rm E}(\vec{\ell}) = \widetilde{\kappa}(\vec{\ell}), \qquad \gamma_{\rm B}(\vec{\ell}) = 0. \tag{25}$$