Observational Cosmology with Large-scale Structure

Special Topics in Physics B XVI[1]

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Suggested readings

- S.Dodelson, "Modern cosmology" (Academic Press, 2003)
- H. Mo, F. van den Bosch, S. D. White, "Galaxy Formation and Evolution", (Cambridge Univ. Press, 2010)
- 松原隆彦, "現代宇宙論" (東京大学出版会, 2010)
- 松原隆彦, "宇宙論の物理 上・下" (東京大学出版会, 2014 年)
- 松原隆彦, "大規模構造の宇宙論" (共立出版, 2014 年)
- 須藤靖, "ものの大きさ" (東京大学出版会, 2006 年)

For more topical and focused reviews,

- F. Bernardeau, S. Colombi, E. Gaztañaga, R. Scoccimarro, "Large-scale structure of the Universe and cosmological perturbation theory", Physics Reports 367 (2002) 1-248 [2]
- A. Cooray and R. Sheth, "Halo models of large-scale structure", Physics Reports 372 (2002) 1-129 [10]
- J. Lesgourgues, S. Pastor, "Massive neutrinos and cosmology", Physics Reports 429 (2006) 307-379 [19]

Note –.

- I set $c = 1$ in most cases.
- I assume general relativity as the underlying theory of gravitation for structure formation and cosmic expansion. I also assume that you have learned general relativity (to some extent).

Chapter 1

Friedmann-Robertson-Walker cosmology

Friedmann-Lemeître-RobertsonWalker metric

$$ds^2 = -dt^2 + \{a(t)\}^2 \, d\vec{\ell}^2 \tag{1.1}$$

with the spatial metric given by

$$d\vec{\ell}^2 = \frac{dr^2}{1 - K\,r^2} + r^2(\theta^2 + \sin^2\theta\, d\phi^2) \tag{1.2}$$

$$= \begin{cases} d\chi^2 + \chi^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) & ; \quad (K = 0) \\ d\chi^2 + \left(\dfrac{\sinh\sqrt{-K}\chi}{\sqrt{-K}}\right)^2 (d\theta^2 + \sin^2\theta\, d\phi^2) & ; \quad (K \neq 0) \end{cases}$$

$$\equiv d\chi^2 + \{r(\chi)\}^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1.3}$$

where K is the spatial curvature, and χ is the comoving radial distance defined by

$$\chi \equiv \int \frac{dt}{a(t)} = \int \frac{dr}{\sqrt{1 - K\,r^2}} \tag{1.4}$$

Energy-momentum tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho,\, P,\, P,\, P), \tag{1.5}$$

The main components for energy density in the Universe are radiation, matter, and dark energy:

$$\rho = \rho_{\mathrm{r}} + \rho_{\mathrm{m}} + \rho_{\mathrm{DE}} \tag{1.6}$$

with the equation of state (EOS):

$$P_{\rm r} = \frac{1}{3}\,\rho_{\rm r}, \quad P_{\rm m} = 0, \quad P_{\rm DE} = w\,\rho_{\rm DE} \tag{1.7}$$

The EOS parameter of dark energy, $w\,(< 0)$, is assumed to be -1 in ΛCDM model (cosmological constant), but it may deviate from -1. Further, it may possibly depend on time, and is conveniently characterized in the literature by

$$w(a) = w_0 + w_a(1-a). \tag{1.8}$$

Einstein equation/Friedmann equation

$$G^\mu_\nu = 8\pi\,G\,T^\mu_\nu\,; \quad G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\,\delta^\mu_\nu \quad \Longrightarrow \quad \begin{cases} 3\left(\dfrac{\dot a}{a}\right)^2 = 8\pi\,G\,\rho - \dfrac{K}{a^2}, \\[2ex] 3\,\dfrac{\ddot a}{a} = -4\pi\,G\,(\rho + 3P). \end{cases} \tag{1.9}$$

The first equation is especially called Friedmann equation. One can check that these two equations are compatible with the following equation derived from the conservation law ($T^\mu_{\nu\,;\mu} = 0$):

$$\dot\rho + 3\,\frac{\dot a}{a}\,(\rho + P) = 0. \tag{1.10}$$

Cosmological parameters

$$\text{Hubble parameter}: \quad H \equiv \frac{\dot a}{a}, \tag{1.11}$$

$$\text{Density parameter}: \quad \Omega_* \equiv \frac{8\pi\,G}{3\,H^2}\,\rho_*, \qquad (* = {\rm r},\ {\rm m},\ {\rm DE}) \tag{1.12}$$

$$\text{Curvature parameter}: \quad \Omega_K \equiv -\frac{K}{3\,H^2} \tag{1.13}$$

Note that these are time-dependent quantities. The parameters given at present time are specifically denoted by H_0, $\Omega_{*,0}$, and $\Omega_{0,K}$.

With the definitions above, the Friedmann equation [first line of Eq. (1.9)] is reduced to

$$\Omega_{\rm r}(a) + \Omega_{\rm m}(a) + \Omega_{\rm DE}(a) + \Omega_K(a) = 1. \tag{1.14}$$

In terms of the parameters at present time, the Friedmann equation with a help of conservation law and EOS of each energy component leads to (using the redshift defined by $1 + z = 1/a$):

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\rm r,0}\,(1+z)^4 + \Omega_{\rm m,0}\,(1+z)^3 + \Omega_{\rm DE,0}\,\exp\left[3\int dz'\frac{1+w(z')}{1+z'}\right] + \Omega_{K,0}\,(1+z)^2. \tag{1.15}$$

To be more precise, the matter and radiation components are broken up into baryon (b), cold dark matter (c), neutrinos (ν), and photons (γ). While the baryon and cold dark matter are non-relativistic and thus their energy density evolves as $\rho_{\rm b,c} \propto a^{-3}$, the photons are relativistic and evolves as $\rho_\gamma \propto a^{-4}$. On the other hand, because of the small mass, the treatment of neutrinos needs a bit care. The neutrinos are initially relativistic, but become non-relativistic around $z \sim 200$. Taking a proper account of these facts, a refined version of Eq. (1.15) is given by [18]

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\gamma,0}\left\{1 + 0.227\,N_{\rm eff}\,f\left(\frac{m_\nu}{T_{\nu,0}(1+z)}\right)\right\}(1+z)^4 + (\Omega_{\rm b,0}+\Omega_{\rm c,0})\,(1+z)^3 + \Omega_{\rm DE,0}\,\exp\left[3\int dz'\frac{1+w(z')}{1+z'}\right] + \Omega_{K,0}\,(1+z)^2, \tag{1.16}$$

where $N_{\rm eff}$ is the effective number of neutrino species ($N_{\rm eff} = 3.046$ is the standard value), m_ν is the neutrino mass (assuming the equal mass for each species), and the present-day neutrino temperature, $T_{\nu,0} = (4/11)^{1/3}T_{\gamma,0} = 1.945$K. The function f is given by

$$f(y) \equiv \frac{120}{7\pi^4}\int_0^\infty dx\frac{x^2\sqrt{x^2+y^2}}{e^x+1} \simeq \{1+(0.3173y)^{1.83}\}^{1/1.83}. \tag{1.17}$$

Table 1.1: Cosmological parameters of ΛCDM models derived from Planck 2015 results [31]

$\Omega_{\rm m,0}$	0.315
$\Omega_{\rm b,0}$	0.049
$\Omega_{\rm c,0}$	0.265
$\Omega_{\nu,0}$	——*
$\Omega_{\rm DE,0}$	0.685†
h‡	0.673

* In ΛCDM model, neutrinos are supposed to be massless, but in Planck 2015, they assumed $m_\nu = 0.06$eV ($\Omega_{\nu,0}h^2 \approx \sum m_\nu/93.04\,{\rm eV} \approx 0.0006$), with the standard value of $N_{\rm eff} = 3.046$.

† ΛCDM assumes a spatially flat universe ($\Omega_{K,0} = 0$), which gives $\Omega_{\rm DE,0} = 1 - \Omega_{\rm m,0}$.

‡ h is dimensionless Hubble parameter defined by $H_0 = 100\,h\,{\rm km\,s^{-1}\,Mpc^{-1}}$.

Cosmological distances

- **Luminosity distance**: determined through the observation of apparent flux of the *standard candle*, for which the absolute luminosity of the distant object is a priori known:

$$d_{\rm L}(z) \equiv \sqrt{\frac{\rm Luminosity}{4\pi\,{\rm Flux}}} = (1+z)\,r(\chi(z)) \tag{1.18}$$

- **Angular-diameter distance**: estimated from measurement of apparent angular size of the *standard ruler*, for which the proper (physical) size is a priori known:

$$d_{\rm A}(z) \equiv \frac{\text{Proper size}}{\text{Angular size}} = \frac{1}{(1+z)}\, r(\chi(z)) \tag{1.19}$$

At $z \ll 1$, one can expand

$$d_{\rm L}(z) = (1+z)^2\, d_{\rm A}(z) \simeq \frac{z}{H_0}\left[1 + \frac{1}{2}(1-q_0)z + \cdots\right], \qquad q_0 \equiv -\left.\frac{a\ddot{a}}{\dot{a}^2}\right|_{t_0} = \left.\frac{d\ln H(z)}{dz}\right|_{z=0} - 1. \tag{1.20}$$

Chapter 2

Linear theory of structure formation

2.1 Basic equations

Metric (flat)

$$ds^2 = -(1+2\Psi)dt^2 + \{a(t)\}^2 (1+2\Phi)\,\delta_{ij}\,dx^i\,dx^j. \tag{2.1}$$

Perturbed quantities

$$\begin{aligned}
&\text{Photon} &:& \quad f_\gamma(p,x) = \left[\exp\left\{\tfrac{p}{T(1+\Theta)}\right\} - 1\right]^{-1} \\
&\text{CDM} &:& \quad \delta(\boldsymbol{x}), \quad \vec{v}(\boldsymbol{x}) \\
&\text{Baryon} &:& \quad \delta_{\rm b}(\boldsymbol{x}), \quad \vec{v}_{\rm b}(\boldsymbol{x}) \\
&\text{Neutrino} &:& \quad f_\nu = \left[\exp\left\{\tfrac{E}{T_\nu(1+\mathcal{N})}\right\} + 1\right]^{-1}
\end{aligned} \tag{2.2}$$

Fourier expansion

$$\delta(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\delta(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2.3}$$

$$\vec{v}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{i\,\boldsymbol{k}}{k}\,v(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} \tag{2.4}$$

Note that in the absence of vector/tensor metric fluctuations, the velocity field only possesses the gradient mode (i.e., irrotational flow).

Evolution equations

$$\left(\frac{k}{a}\right)^2 \Phi + 3H(\dot{\Phi} - H\Psi) = 4\pi G \sum_i \rho_i \delta_i, \tag{2.5}$$

$$\left(\frac{k}{a}\right)^2 (\Phi + \Psi) = -8\pi G \Pi \; ; \qquad \Pi \equiv 4(\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2) \tag{2.6}$$

$$\dot{\Theta} + i\frac{k\mu}{a}(\Theta + \Psi) + \dot{\Phi} = n_e \sigma_{\mathrm{T}} \left[\Theta_0 - \Theta + i\mu v_{\mathrm{b}} - \frac{\mathcal{P}_2(\mu)}{2} \Theta_2\right], \tag{2.7}$$

$$\dot{\delta} - \frac{k}{a} v + 3\dot{\Phi} = 0, \tag{2.8}$$

$$\dot{v} + H v + \frac{k}{a} \Psi = 0, \tag{2.9}$$

$$\dot{\delta}_{\mathrm{b}} - \frac{k}{a} v_{\mathrm{b}} + 3\dot{\Phi} = 0, \tag{2.10}$$

$$\dot{v}_{\mathrm{b}} + H v_{\mathrm{b}} + \frac{k}{a} \Psi = -\frac{n_e \sigma_{\mathrm{T}}}{R} \left(3\Theta_1 + v_{\mathrm{b}}\right) \; ; \qquad R \equiv \frac{3\rho_{\mathrm{b}}}{4\rho_\gamma}, \tag{2.11}$$

$$\dot{\mathcal{N}} + i\frac{k\mu}{a} \left(\frac{p}{E}\mathcal{N} + \frac{E}{p} \Psi\right) + \dot{\Phi} = 0; \qquad E^2 = m_\nu^2 + p^2 \tag{2.12}$$

Note

- Polarization dependence of the Thomson scattering is ignored.

- Photon and neutrino fluctuations, Θ and $\mathcal{N}$, additionally have directional dependence of the momentum. Further, in the presence of non-zero mass, the neutrino fluctuation also depends on the momentum, i.e., $\mathcal{N}(\boldsymbol{k}, \mu, p)$, and $\Theta(\boldsymbol{k}, \mu)$ with $\mu \equiv \boldsymbol{k} \cdot /\boldsymbol{p}/(kp)$. It is thus convenient to characterize them by introducing the multipole expansion:

$$\Theta(\boldsymbol{k}, \mu) = \sum_\ell (-i)^\ell (2\ell + 1) \Theta_\ell(k) \mathcal{P}_\ell(\mu), \tag{2.13}$$

$$\mathcal{N}(\boldsymbol{k}, p, \mu) = \sum_\ell (-i)^\ell (2\ell + 1) \mathcal{N}_\ell(k, p) \mathcal{P}_\ell(\mu). \tag{2.14}$$

Boltzmann hierarchy

Applying the multipole expansion in Eqs. (2.13) and (2.14), with a help of formulas in Appendix A.2, Eqs.(2.7) and (2.12) respectively lead to a infinite set of hierarchy equations:

$$\dot{\Theta}_0 + \frac{k}{a}\Theta_1 + \dot{\Phi} = 0, \tag{2.15}$$

$$\dot{\Theta}_1 + \frac{k}{3a}(2\Theta_2 - \Theta_0 - \Psi) = n_e\sigma_{\rm T}\left(-\Theta_1 - \frac{v_{\rm b}}{3}\right), \tag{2.16}$$

$$\dot{\Theta}_2 + \frac{k}{5a}(3\Theta_3 - 2\Theta_2) = n_e\sigma_{\rm T}\left(-\Theta_2 + \frac{1}{10}\Theta_2\right), \tag{2.17}$$

$$\dot{\Theta}_\ell + \frac{k}{(2\ell+1)a}\{(\ell+1)\Theta_\ell - \ell\Theta_{\ell-1}\} = -n_e\sigma_{\rm T}\,\Theta_\ell, \quad (\ell \geq 3). \tag{2.18}$$

$$\dot{\mathcal{N}}_0 + \frac{k}{a}\frac{p}{\sqrt{p^2+m_\nu^2}}\mathcal{N}_1 + \dot{\Phi} = 0, \tag{2.19}$$

$$\dot{\mathcal{N}}_1 + \frac{k}{3a}\left\{\frac{p}{\sqrt{p^2+m_\nu^2}}(2\mathcal{N}_2 - \mathcal{N}_0) - \frac{\sqrt{p^2+m_\nu^2}}{p}\,\Psi\right\} = 0, \tag{2.20}$$

$$\dot{\mathcal{N}}_2 + \frac{k}{5a}\,\frac{p}{\sqrt{p^2+m_\nu^2}}\,(3\mathcal{N}_3 - 2\mathcal{N}_2) = 0, \tag{2.21}$$

$$\dot{\mathcal{N}}_\ell + \frac{k}{(2\ell+1)a}\,\frac{p}{\sqrt{p^2+m_\nu^2}}\,\{(\ell+1)\mathcal{N}_\ell - \ell\mathcal{N}_{\ell-1}\} = 0, \quad (\ell \geq 3). \tag{2.22}$$

2.2 Initial conditions

Adiabatic perturbations at the radiation-dominated epoch:

$$\Theta_0 = \mathcal{N}_0 = \frac{1}{2}\Phi_{\rm p}, \tag{2.23}$$

$$\delta = \delta_{\rm b} = \frac{3}{2}\Phi_{\rm p}, \tag{2.24}$$

$$v = v_{\rm b} = \frac{k}{2\,a\,H}\Phi_{\rm p}, \tag{2.25}$$

$$\Theta_1 = \mathcal{N}_1 = -\frac{k}{6\,a\,H}\Phi_{\rm p}. \tag{2.26}$$

$$\Theta_\ell = \mathcal{N}_\ell = 0, \qquad (\ell \geq 2) \tag{2.27}$$

Here, $\Phi_{\rm p}$ is the primordial curvature fluctuation, which is thought to be quantum-mechanically generated during the inflation. These initial conditions are given at super-horizon scales ($k \ll aH$).

Note –. To be precise, the neutrino quadrupole moment $\mathcal{N}_2$ is not negligibly small, and it leads to $\Phi + \Psi = (2/5)\,R_\nu\Psi$. This slightly alters the initial conditions given above.

2.3 Solutions: from radiation- to matter-dominated epoch

Below, assuming the massless neutrinos ($m_\nu = 0$) for simplicity, we derive the (approximate) solution for matter fluctuations.

Super-horizon evolution ($k \ll aH$)

$$\frac{d^2\Phi}{dy^2} + \frac{21y^2+54y+32}{2y(y+1)(3y+4)}\frac{d\Phi}{dy} + \frac{\Phi}{y(y+1)(3y+4)} = 0, \quad \left(y \equiv \frac{a}{a_{\rm eq}}\right) \tag{2.28}$$

The solution satisfying $\Phi \to \Phi_{\rm p}$ at $y \to 0$ becomes

$$\Phi(y) = \frac{\Phi_{\rm p}}{10}\frac{1}{y^3}\left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right] \tag{2.29}$$

$$\xrightarrow{y\gg1} \frac{9}{10}\,\Phi_{\rm p} \tag{2.30}$$

Sub-horizon evolution ($k \gg aH$)

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi\,G\,\rho_{\rm c}\,\delta = 0$$

$$\Longrightarrow \quad \frac{d^2\delta}{dy^2} + \frac{3y+2}{2y(y+1)}\frac{d\delta}{dy} - \frac{3}{2}\frac{1}{y(y+1)}\,\delta = 0, \quad \text{(Meszaros equation)} \tag{2.31}$$

$$\delta = c_1\,D_1(y) + c_2\,D_2(y), \quad \begin{cases} D_1(y) = \dfrac{2}{3} + y \\ D_2(y) = D_1(y)\ln\left[\dfrac{\sqrt{y+1}+1}{\sqrt{y+1}-1}\right] - 2\sqrt{y+1} \end{cases} \tag{2.32}$$

Horizon crossing at RD epoch

Using the conformal time defined by $a\,d\eta = dt$, we obtain

$$\delta'' + \frac{a'}{a}\delta' = -3\Phi'' + k^2\Phi - 3\frac{a'}{a}\Phi' \equiv S(\eta), \tag{2.33}$$

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{k^2}{3}\Phi = 0, \tag{2.34}$$

where the prime denotes the derivative with respect to η.

The solution of Eq. (2.33) is written as

$$\delta = d_1\,\ln a + d_2\, + \int_0^\eta d\eta'\,\{\ln a(\eta') - \ln a(\eta)\}\left(\frac{d\ln a(\eta')}{d\eta'}\right)^{-1} S(\eta'). \tag{2.35}$$

From the adiabatic initial condition given in Eq. (2.24), we have $d_1 = 0$ and $d_2 = (3/2)\Phi_{\rm p}$. For more explicit expression, we need to know the behavior of Φ from Eq. (2.34). The solution satisfying $\Phi \to \Phi_{\rm p}$ at $k \ll aH$ becomes

$$\Phi = \Phi_{\rm p} \left(3 \frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right) \tag{2.36}$$

$$\xrightarrow{k\eta \gg 1} \Phi_{\rm p} \left(-9 \frac{\cos(k\eta/\sqrt{3})}{(k\eta)^2} \right) \tag{2.37}$$

Substituting Eq. (2.37) into Eq. (2.35), the solution relevant at $a_H \ll a \ll a_{\rm eq}$ is approximately given by

$$\delta \simeq A\, \Phi_{\rm p} \ln \left(B \frac{a}{a_{\rm eq}} \right) \tag{2.38}$$

with $A \sim 9$ and $B \sim 0.6$.

Matching the solutions

From Eqs. (2.38) and (2.32), we have

$$\begin{aligned} &\text{Horizon crossing at RD epoch [Eq. (2.38)]} : \quad \delta \simeq A\, \Phi_{\rm p} \ln \left(B \frac{a}{a_{\rm eq}} \right), \\ &\text{Sub-horizon at MD/RD epoch [Eq. (2.32)]} : \quad \delta = c_1\, D_1 \left(\frac{a}{a_{\rm eq}} \right) + c_2\, D_2 \left(\frac{a}{a_{\rm eq}} \right) \end{aligned} \tag{2.39}$$

At $y_{\rm m} = a_{\rm m}/a_{\rm eq}$ satisfying the condition $y_{\rm H} \ll y_{\rm m} \ll 1$, matching the above two solutions give the explicit expression for the coefficients of the growing mode, c_1:

$$\begin{aligned} c_1 &= \frac{\dfrac{dD_2}{dy}|_{y_{\rm m}} \ln(B\dfrac{y_{\rm m}}{y_{\rm H}}) - D_2(y_{\rm m}) \dfrac{1}{y_{\rm m}}}{D_1(y_{\rm m}) \dfrac{dD_2}{dy}|_{y_{\rm m}} - D_2(y_{\rm m}) \dfrac{dD_1}{dy}|_{y_{\rm m}}}\, A\, \Phi_{\rm p} \\ &\xrightarrow{y_{\rm m} \ll 1} -\frac{9}{4} \left[-\frac{2}{3} \ln \left(B \frac{y_{\rm m}}{y_{\rm H}} \right) - \frac{2}{3} \ln \left(\frac{4}{y_{\rm m}} + 2 \right) \right] = \frac{3}{2}\, A\, \Phi_{\rm p}\, \ln \left(\frac{4B\, e^{-3}}{y_{\rm h}} \right). \end{aligned} \tag{2.40}$$

Thus, the sub-horizon solution of (CDM) density fluctuation at RD/MD epoch becomes

$$\delta \simeq \frac{3}{2}\, A\, \Phi_{\rm p}\, \ln \left(4\sqrt{2}\, B\, e^{-3} \frac{k}{k_{\rm eq}} \right) D_1(a), \qquad (k \gg k_{\rm eq}), \tag{2.41}$$

with $A \sim 9$ and $B \sim 0.6$.

Late-time sub-horizon evolution at MD/DE epoch

At the time after the radiation-matter equality time $a \gg a_{\rm eq}$, the radiation component becomes negligible, the universe is described by Einstein-de Sitter model. At later time, however, the deviation from Einstein-de Sitter model becomes significant, and it affects growth factor D_1. The late-time evolution for D_1 is described by

$$\ddot{D}_1 + 2H\,D_1 - 4\pi\,G\,\rho\,D_1 = 0\,; \quad \rho = \rho_{\rm m} + \rho_{\rm DE}. \tag{2.42}$$

The growing-mode solution is characterized by

$$D_1(a) \propto a\,g(a). \tag{2.43}$$

The deviation from the Einstein-de Sitter universe is also characterized by the linear growth-rate, defined by

$$f(a) \equiv \frac{d\ln D_1(a)}{d\ln a}. \tag{2.44}$$

For the Universe with cosmological constant, the functions $g(a)$ and $f(a)$ are approximately described by [8]

$$g(a) \simeq \frac{5}{2}\Omega_{\rm m}(a)\left[\Omega_{\rm m}^{4/7}(a) - \Omega_\Lambda(a) + \left\{1 + \frac{\Omega_{\rm m}(a)}{2}\right\}\left\{1 + \frac{\Omega_\Lambda(a)}{70}\right\}\right]^{-1}, \tag{2.45}$$

$$f(a) \simeq \Omega_{\rm m}^{4/7}(a) + \frac{\Omega_\Lambda(a)}{70}\left\{1 + \frac{\Omega_{\rm m}(a)}{2}\right\}. \tag{2.46}$$

In a flat Universe filled with matter and dark energy with constant EOS parameter (w), the exact solution is known, and the functions $g(a)$ and $f(a)$ are expressed in terms of the hyper-geometric function (e.g., [29]):

$$g(a) = \ {}_2F_1\left(-\frac{1}{3w}, \frac{w-1}{w}, 1-\frac{5}{6w}; -q(a)\right), \tag{2.47}$$

$$f(a) = 1 - \frac{3(w-1)}{6w-5}\,\frac{{}_2F_1\left[\frac{3w-1}{2w}, \frac{3w-1}{3w}, \frac{12w-5}{6w}, -q(a)\right]}{{}_2F_1\left[-\frac{1}{3w}, \frac{w-1}{2w}, \frac{6w-5}{6w}, -q(a)\right]} \tag{2.48}$$

with $q(a) \equiv \{(1-\Omega_{\rm m,0})/\Omega_{\rm m,0}\}\,a^{-3w}$.

2.4 Transfer function

While the wavelength of the observable fluctuations is basically shorter than the horizon size, these modes have experienced the super-horizon evolution. Since the evolution of fluctuations can change depending on when the mode crosses (or re-enters) the horizon scale, it is convenient to introduce the transfer function defined by

$$T(k;\,t_{\rm m}) \equiv \frac{\Phi(k;\,t_{\rm m})}{\Phi(k\to 0;\,t_{\rm m})}, \tag{2.49}$$

where the time $t_{\rm m}$ is chosen at the matter-dominated era, close to the Einstein-de Sitter Universe. Here, the $\Phi(k)$ in the our interest is the sub-horizon mode. From Eq. (2.30), we have $\Phi(k \to 0; t_{\rm m}) = (9/10)\,\Phi_{\rm p}$.

Using (2.49), the matter fluctuation at sub-horizon scales may be expressed as (at $t > t_{\rm m}$)

$$\begin{aligned} \delta_{\rm m}(\boldsymbol{k};\, a) &= \frac{3}{5}\,\frac{k^2}{\Omega_{\rm m,0}H_0^2}\,\Phi_{\rm p}(\boldsymbol{k})\,T(k)\;D_1(a) \\ &\equiv \delta_0(\boldsymbol{k})\,D_1(a). \end{aligned} \tag{2.50}$$

Asymptotic behavior of $T(k)$

From Eq. (2.41) and (2.49), we obtain[1]

$$T(k) \simeq \begin{cases} \dfrac{5}{2}\,A\left(\dfrac{k}{k_{\rm eq}}\right)^{-2}\ln\left(4\sqrt{2}\,B\,e^{-3}\,\dfrac{k}{k_{\rm eq}}\right) \simeq 12\left(\dfrac{k}{k_{\rm eq}}\right)^{-2}\ln\left(\dfrac{k}{8k_{\rm eq}}\right), & k \gg k_{\rm eq} \\ 1, & k \ll k_{\rm eq}. \end{cases} \tag{2.51}$$

Thus, the important characteristic scale is

$$k_{\rm eq} \equiv a_{\rm eq}\,H_{\rm eq} = \sqrt{\frac{2}{\Omega_{\rm r,0}H_0^2}}\,\frac{\Omega_{\rm m,0}H_0^2}{c} = 0.0095\left(\frac{\Omega_{\rm m,0}h^2}{0.13}\right)\;{\rm Mpc}^{-1}. \tag{2.52}$$

BBKS fitting formula

A simple but accurate formula for transfer function is given by [1]

$$T(k) = \frac{\ln[1+2.34q]}{2.34q}\left\{1+3.39q+(16.2)^2+(5.47)^3+(6.71q)^4\right\}^{-1/4}\;;\; q \equiv \frac{k}{\Gamma\,h\,{\rm Mpc}^{-1}} \tag{2.53}$$

with the shape parameter $\Gamma = \Omega_{\rm m,0}h$. This is the transfer function for CDM fluctuation, but simply replacing the shape parameter with $\Gamma = \Omega_{\rm m,0}h\;\exp[-\Omega_{\rm b,0} - (2h)^{1/2}\Omega_{\rm b}/\Omega_{\rm m,0}]$, it can represent in a good accuracy the transfer function for matter fluctuations [34].

Note

- An improved fitting formula for transfer function including the baryon acoustic oscillations is given by Ref. [12].

[1] Strictly, the asymptotic form given here is not the transfer function for matter fluctuation, but that for the CDM fluctuation.

- For the transfer function including massive neutrinos, a fitting formula relevant at small scales is presented in Ref. [15].
- For more accurate transfer function, use CMB Boltzmann code. Now, the available public codes are `camb`[2] and `class`[3].

2.5 Baryon acoustic oscillations

Acoustic oscillations

$$\frac{d^2\Theta_0}{d\eta^2}+\frac{R}{1+R}\mathcal{H}\frac{d\Theta_0}{d\eta}+\frac{k^2}{3(1+R)}\Theta_0=-\frac{k^2}{3}\Psi-\frac{d^2\Phi}{d\eta^2}-\frac{R}{1+R}\mathcal{H}\frac{d\Phi}{d\eta} \tag{2.54}$$

$$c_s\equiv\sqrt{\frac{1}{3(1+R)}}\;;\;R=\frac{3\rho_{\rm b}}{4\rho_\gamma}. \tag{2.55}$$

$$\Theta_0\;\propto\;\exp(i\,k\,r_s)\,;\quad r_s\equiv\int_0^\eta d\eta'\,c_s(\eta')\;\cdots\;\text{sound horizon scale} \tag{2.56}$$

$$\begin{aligned} r_s(\eta)&=\frac{2}{3\,k_{\rm eq}}\sqrt{\frac{6}{R_{\rm eq}}}\ln\left(\frac{\sqrt{1+R(\eta)}+\sqrt{R(\eta)+R_{\rm eq}}}{1+\sqrt{R_{\rm eq}}}\right)\\ &\approx 147\left(\frac{\Omega_{\rm m,0}h^2}{0.13}\right)^{-0.25}\left(\frac{\Omega_{\rm b,0}h^2}{0.024}\right)^{-0.008}\quad\text{at }\eta=\eta_{\rm rec}\end{aligned} \tag{2.57}$$

Relation to baryon fluctuations

$$v_{\rm b}\simeq-3\Theta_1\overset{k\gg aH}{\simeq}\frac{3}{k}\frac{d\Theta_0}{d\eta},\quad\frac{d\delta_{\rm b}}{d\eta}\simeq k\,v_{\rm b}\simeq3\frac{d\Theta_0}{d\eta}\quad\longrightarrow\quad\delta_{\rm b}\simeq3\Theta_0\;\propto\;\exp(ik\,r_s) \tag{2.58}$$

2.6 Baryon catch-up

Just after the time of decoupling, the baryon fluctuations is negligible, and the potential is basically determined by the CDM fluctuations. Thus, the evolution of baryon fluctuations is approximately described by

$$\ddot{\delta}_{\rm b}+2\,H\dot{\delta}_{\rm b}\simeq4\pi\,G\rho_{\rm c}\,\delta. \tag{2.59}$$

Using the fact that $\rho_{\rm c}\propto a^{-3}$, and $\delta\propto a$ at MD era, the above equation is reduced to

$$y^{1/2}\frac{d}{dy}\left(y^{3/2}\frac{d\delta_{\rm b}}{dy}\right)=\frac{3}{2}\delta\;;\quad y\equiv\frac{a}{a_{\rm dec}}. \tag{2.60}$$

[2] http://camb.info
[3] http://class-code.net

The solution which fulfills the condition, $\delta_{\rm b}=0$ at $y=1$, becomes

$$\delta_{\rm b}=\left(1-\frac{3}{y}+\frac{2}{y^{3/2}}\right)\,\delta. \tag{2.61}$$

This implies that the baryon fluctuations soon catch up the CDM fluctuations. Note importantly that the acoustic signature (BAO) imprinted on the baryon fluctuations still remains even after the baryon catch-up.

2.7 Effect of non-zero mass of neutrinos

Even when the neutrinos become non-relativistic, they have a large velocity dispersion due to their small non-zero masses:

$$\sigma_\nu^2=\frac{\displaystyle\int d^3q\left(\frac{q}{m_\nu}\right)^2 f_\nu(q)}{\displaystyle\int d^3q\,f_\nu(q)}=\frac{15\zeta(5)}{\zeta(3)}\left(\frac{4}{11}\right)^{2/3}\frac{T_{\gamma,0}^2(1+z)^2}{m_\nu^2}$$
$$\simeq\left(6.03\times10^{-4}\,c\right)^2\left(\frac{1\,\mathrm{eV}}{m_\nu}\right)^2(1+z)^2. \tag{2.62}$$

This leads to the characteristic scale below.

Free-streaming scale, $k_{\rm FS}$[4]

$$k_{\rm FS}\equiv\sqrt{\frac{3}{2}}\frac{a\,H}{c_s^2}\simeq\sqrt{\frac{3}{2}}\frac{a\,H}{\sigma_\nu}=\frac{0.677}{(1+z)^2}\frac{m_\nu}{1\,\mathrm{eV}}\sqrt{\Omega_{\rm m,0}(1+z)^3+\Omega_\Lambda}\;h\,\mathrm{Mpc}^{-1}. \tag{2.63}$$

At the scales below the free-streaming scale, $k\ll k_{\rm FS}$, the neutrino fluctuations do not grow, and hence the fluctuations of the total matter is suppressed relative to those in the massless neutrino case. The suppression of linear matter power spectrum is approximately characterized as

$$\frac{P(k)|_{f_\nu\neq0}}{P(k)|_{f_\nu=0}}\;\simeq\;1-8\,f_\nu\;;\quad f_\nu\equiv\frac{\Omega_{\nu,0}}{\Omega_{\rm m,0}}\simeq0.075\left(\frac{0.1426}{\Omega_{\rm m,0}h^2}\right)\left(\frac{\sum m_\nu}{1\,\mathrm{eV}}\right) \tag{2.64}$$

at $z=0$. A more refined (but partly empirical) formula is given by [see Eq. (141) of Ref. [19]]:

$$\frac{P(k)|_{f_\nu\neq0}}{P(k)|_{f_\nu=0}}\;\simeq\;(1-f_\nu)^3\left(\frac{D_1(a)}{a_{\rm nr}}\right)^{-(6/5)f_\nu}=(1-f_\nu)^3\left\{1.9\times10^5\frac{\Omega_{\nu,0}h^2}{N_{\rm eff}}\frac{D_1(a)}{a}\right\}^{-(6/5)f_\nu}. \tag{2.65}$$

[4]As remarked in Ref. [33], the sound velocity, $c_s=(\delta p/\delta\rho)^{1/2}$, slightly differs from σ_ν, and in the non-relativistic limit, it gives $c_s\simeq(\sqrt{5}/3)\sigma_\nu$.

Chapter 3

Observational effects

3.1 Redshift-space distortions

Redshift space

$$1 + z_{\rm obs} \simeq (1+z)(1+v_{\parallel}) \;\longrightarrow\; \boldsymbol{s} = \boldsymbol{x} + \frac{1+z}{H(z)}\, v_{\parallel}\, \hat{x}. \tag{3.1}$$

For distance galaxies, the observer's line-of-sight to the galaxy-clustering region is approximately fixed so that one can introduce a particular direction, $\hat{\boldsymbol{z}}$, and write $v_{\parallel} = (\boldsymbol{v}\cdot\hat{z})$. We then have

$$\boldsymbol{s} = \boldsymbol{x} + \frac{1+z}{H(z)}\, (\boldsymbol{v}\cdot\hat{z})\, \hat{z}. \tag{3.2}$$

Galaxy density field in redshift space:

$$\{1+\delta^{(\rm S)}(\boldsymbol{s})\} d^3\boldsymbol{s} = \{1+\delta_{\rm g}(\boldsymbol{x})\} d^3\boldsymbol{x}$$
$$\longrightarrow \quad \delta^{(\rm S)}(\boldsymbol{s}) = \{1+\delta_{\rm g}(\boldsymbol{x})\} \left|\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{x}}\right|^{-1} - 1. \tag{3.3}$$

In Fourier space,

$$\begin{aligned} \delta^{(\rm S)}(\boldsymbol{k}) &= \int d^3\boldsymbol{s}\, \delta^{(\rm S)}(\boldsymbol{s})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{s}} \\ &= \int d^3\boldsymbol{x} \left[\delta_{\rm g}(\boldsymbol{x}) - \frac{1+z}{H(z)}\frac{\partial v_z(\boldsymbol{x})}{\partial z}\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{x} - ik\mu_k(1+z)/H(z)\, v_z(\boldsymbol{x})} \end{aligned} \tag{3.4}$$

with $\mu_k \equiv (\boldsymbol{k}\cdot\hat{z})/|\boldsymbol{k}|$.

Linear perturbation (Kaiser formula)

Linearizing RHS of Eq. (3.4) yields

$$\delta^{(\rm S)}(\boldsymbol{k}) \simeq \int d^3\boldsymbol{x} \left[\delta_{\rm g}(\boldsymbol{x}) - \frac{1+z}{H(z)}\frac{\partial v_z}{\partial z}\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \delta_{\rm g}(\boldsymbol{k}) + \frac{1+z}{H(z)}\, k\, \mu_k^2\, v(\boldsymbol{k}). \tag{3.5}$$

Using the linearized continuity equation $\dot{\delta}_{\rm m}-(k/a)v=0$ [see e.g., Eq. (2.8) on sub-horizon scales] and assuming the linear galaxy bias $\delta_{\rm g}=b\,\delta_{\rm m}$, we obtain

$$\delta^{(\mathrm{S})}(\boldsymbol{k})=(b\,+\,f\,\mu_k^2)\delta_{\rm m}(\boldsymbol{k}), \tag{3.6}$$

where f is the linear growth rate defined by Eq. (2.44). Then, the (linear) redshift-space power spectrum becomes

$$P^{(\mathrm{S})}(\boldsymbol{k})=(b+f\,\mu_k^2)^2\,P_{\rm m}(k)=\sum_\ell P_\ell^{(\mathrm{S})}(k)\,\mathcal{P}_\ell(\mu_k)\ ;\quad \begin{cases} P_0^{(\mathrm{S})}(k)=\left(b^2+\dfrac{2}{3}fb+\dfrac{1}{5}f^2\right)\,P_{\rm m}(k) \\ P_2^{(\mathrm{S})}(k)=\left(\dfrac{4}{3}fb+\dfrac{4}{7}f^2\right)\,P_{\rm m}(k) \\ P_4^{(\mathrm{S})}(k)=\dfrac{8}{35}f^2\,P_{\rm m}(k) \end{cases} \tag{3.7}$$

The corresponding redshift-space correlation function is

$$\begin{aligned}\xi^{(\mathrm{S})}(\boldsymbol{s})&=\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,P^{(\mathrm{S})}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{s}}\\ &=\sum_{\ell=0,2,4}\xi_\ell^{(\mathrm{S})}(s)\mathcal{P}_\ell(\mu_{\rm s})\ ;\quad \xi_\ell^{(\mathrm{S})}(s)=(-i)^\ell\int\frac{dkk^2}{2\pi^2}\,j_\ell(k\,s)\,P_\ell^{(\mathrm{S})}(k),\end{aligned} \tag{3.8}$$

where the directional cosine $\mu_{\rm s}$ is defined by $\mu_{\rm s}=\boldsymbol{s}\cdot\hat{z}/|\boldsymbol{s}|$.

3.2 Geometric distortions (Alcock-Paczynski effect)

On top of the redshift-space distortions, there appears another anisotropies induced by the geometric distortions (Alcock-Paczynski effect). This distortion arises if the background expansion of the real universe differs from the fiducial cosmology used to convert the observed galaxy positions (i.e., redshift and angular positions) to the comoving radial and transverse distances.

Correlation function

Denoting the transverse and radial separations of a galaxy pair in the *true* cosmology by $s'_\perp$ and $s'_\parallel$, their relation to the observed separation in the fiducial cosmological model, $s_\perp^{\rm obs}$ and $s_\parallel^{\rm obs}$, is given by

$$s'_\perp=\frac{d_A}{d_{A,\rm fid}}s_\perp^{\rm obs},\qquad s'_\parallel=\frac{H^{-1}}{H_{\rm fid}^{-1}}s_\parallel^{\rm obs}. \tag{3.9}$$

Here, the quantities with subscript $_{\rm fid}$ represent those in the fiducial cosmology. The above relation indicates that the measured correlation function in the fiducial cosmological model, $\xi^{(\mathrm{S})}_{\mathrm{obs}}$ is related to the actual one in the true cosmology, $\xi^{(\mathrm{S})}$, through

$$\xi^{(\mathrm{S})}_{\mathrm{obs}}(s^{\mathrm{obs}},\,\mu_{\mathrm{s}}^{\mathrm{obs}}) = \xi^{(\mathrm{S})}(s',\,\mu_{\mathrm{s}}')\ ; \qquad \begin{cases} s' \equiv \sqrt{(s'_\perp)^2+(s'_\parallel)^2} = s^{\mathrm{obs}}\,\beta(\mu_{\mathrm{s}}^{\mathrm{obs}}) \\[2ex] \mu_{\mathrm{s}}' \equiv s'_\parallel/s' = \dfrac{H_{\mathrm{fid}}}{H}\dfrac{\mu_{\mathrm{s}}^{\mathrm{obs}}}{\beta(\mu_{\mathrm{s}}^{\mathrm{obs}})} \end{cases} \tag{3.10}$$

with

$$\beta(\mu_{\mathrm{s}}^{\mathrm{obs}}) = \sqrt{\left(\frac{d_A}{d_{A,\mathrm{fid}}}\right)^2 + \left\{\left(\frac{H_{\mathrm{fid}}}{H}\right)^2 - \left(\frac{d_A}{d_{A,\mathrm{fid}}}\right)^2\right\}(\mu_{\mathrm{s}}^{\mathrm{obs}})^2}. \tag{3.11}$$

Power spectrum

Similarly, one has the Fourier counterpart of Eq. (3.9):

$$k'_\perp = \frac{d_{A,\mathrm{fid}}}{d_A}k_\perp^{\mathrm{obs}}, \qquad k'_\parallel = \frac{H_{\mathrm{fid}}^{-1}}{H^{-1}}k_\parallel^{\mathrm{obs}}. \tag{3.12}$$

Then, from Eq. (3.10), we obtain

$$P^{(\mathrm{S})}_{\mathrm{obs}}(k^{\mathrm{obs}},\,\mu_k^{\mathrm{obs}}) = \frac{H}{H_{\mathrm{fid}}}\left(\frac{d_{A,\mathrm{fid}}}{d_A}\right)^2 P^{(\mathrm{S})}(k',\,\mu_k')\ ; \qquad \begin{cases} k' = k^{\mathrm{obs}}\,\alpha(\mu_k^{\mathrm{obs}}) \\[2ex] \mu_k' = \dfrac{H}{H_{\mathrm{fid}}}\dfrac{\mu_k^{\mathrm{obs}}}{\alpha(\mu_k^{\mathrm{obs}})} \end{cases} \tag{3.13}$$

with

$$\alpha(\mu_k^{\mathrm{obs}}) = \sqrt{\left(\frac{d_{A,\mathrm{fid}}}{d_A}\right)^2 + \left\{\left(\frac{H}{H_{\mathrm{fid}}}\right)^2 - \left(\frac{d_{A,\mathrm{fid}}}{d_A}\right)^2\right\}(\mu_k^{\mathrm{obs}})^2}. \tag{3.14}$$

The expressions given in Eqs. (3.10) and (3.13) imply that in the presence of mismatch between the fiducial and true cosmologies, the higher-multipole moments of the power spectrum/correlation function ($\ell > 4$) naturally arises even if the linear formula of redshift-space distortions holds [Eqs. (3.7) and (3.8)].

Chapter 4

Analytic approaches to nonlinear structure formation

4.1 Spherical collapse model

A simple nonlinear model for gravitational collapse which tells you characteristic properties of gravitationally bound objects (i.e., dark matter halos).

Consider a homogeneous (uniform) density of sphere with radius R and mass M. The motion of the shell at R is described by

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2}. \tag{4.1}$$

The solution which becomes collapsed at finite time is parametrically expressed as

$$R = \frac{GM}{2|E|}(1-\cos\theta), \quad t = \frac{GM}{(2|E|)^{3/2}}(\theta - \sin\theta). \tag{4.2}$$

Here, E is the total energy of the shell ($E = \dot{R}^2/2 - GM/R$). Note that at $t_{\rm ta} = t(\theta = \pi)$, the radius of the sphere becomes maximum, and one has $R = R_{\rm max} = GR/|E|$. On the other hand, the radius becomes zero at $t_{\rm coll} = t(\theta = 2\pi) = 2t_{\rm ta}$.

Then the density of the sphere is estimated to give

$$\rho \equiv \frac{M}{(4\pi/3)R^3} = \frac{6}{\pi}\frac{|E|^3}{G^3M^2}(1-\cos\theta)^{-3}. \tag{4.3}$$

Taking the ratio to the background density of the universe $\rho_{\rm m} = 3H^2/(8\pi G)$, the density contrast becomes

$$\delta \equiv \frac{\rho}{\rho_{\rm m}} - 1 = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} - 1, \tag{4.4}$$

Here, for simplicity, we assumed the Einstein-de Sitter universe [i.e., $\rho_{\rm m}^{\rm EdS} = 1/(6\pi G t^2)$].

Critical density contrast

$$\delta_{\rm crit} \equiv \delta_{\rm lin}(t_{\rm coll}) = \frac{3}{20}(12\pi)^{2/3} \simeq 1.68647. \tag{4.5}$$

Virial overdensity

$$\Delta_{\rm vir} \equiv \frac{\rho_{\rm vir}}{\rho_{\rm m}^{\rm EdS}(t_{\rm coll})} = \frac{8\rho(t_{\rm ta})}{\rho_{\rm m}^{\rm EdS}(t_{\rm ta})/4} = 18\pi^2 \simeq 177.6 \tag{4.6}$$

Extension to ΛCDM cosmology [27, 7, 26]

$$\delta_{\rm crit} = 1.686\{\Omega_{\rm m}(t_{\rm coll})\}^{0.055} \tag{4.7}$$

$$\Delta_{\rm vir} = \frac{18\pi^2 + 82y - 39y^2}{\Omega_{\rm m}(t_{\rm coll})} \;; \quad y \equiv \Omega_{\rm m}(t_{\rm coll}) - 1. \tag{4.8}$$

4.2 Zel'dovich approximation

This section is based on Ref. [22].

Zel'dovich approximation is the first-order Lagrangian perturbation theory, and describes the quasi-linear evolution of matter fluctuations [40]. In contrast to the spherical collapse model, Zel'dovich approximation tells us (qualitatively) how the asphericity of the structure develops according to its initial configuration. In particular, it is used to generate the initial conditions for cosmological N-body simulations.

As mentioned above, the Zel'dovich approximation is the Lagrangian-based treatment by following the trajectories of particles. Motion of each mass particle is described by

$$L = \frac{1}{2}ma^2\dot{\boldsymbol{x}}^2 - m\Psi(\boldsymbol{x}), \tag{4.9}$$

which gives

$$\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}} + \frac{1}{a^2}\nabla_x\Psi(\boldsymbol{x}) = 0. \tag{4.10}$$

The quantity of interest here is the displacement field $\boldsymbol{\psi}(\boldsymbol{q})$ which maps the initial particle positions $\boldsymbol{q}$ into the final Eulerian particle positions $\boldsymbol{x}$,

$$\boldsymbol{x}(\boldsymbol{q}, t) = \boldsymbol{q} + \boldsymbol{\psi}(\boldsymbol{q}, t). \tag{4.11}$$

With this definition, taking the divergence of Eq. (4.10) gives

$$\nabla_x\left(\ddot{\boldsymbol{\psi}} + 2H\dot{\boldsymbol{\psi}}\right) = -4\pi\, G\rho_{\rm m}\,\delta_{\rm m}(\boldsymbol{x}) \tag{4.12}$$

In the above, the sources of nonlinearity are

$$1+\delta_{\rm m}(\boldsymbol{x})=\left|\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}}\right|^{-1}\equiv\frac{1}{J}, \tag{4.13}$$

$$\frac{\partial}{\partial\boldsymbol{x}_i}=\left(\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}}\right)^{-1}_{ij}\frac{\partial}{\partial\boldsymbol{q}_j}\equiv(J^{-1})_{ij}\frac{\partial}{\partial\boldsymbol{q}_j}. \tag{4.14}$$

Regarding $\boldsymbol{\psi}$ as a perturbed quantity, the leading-order evaluation leads to

$$J=\frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}J_{ip}J_{jq}J_{kr}\simeq 1+\nabla_q\cdot\boldsymbol{\psi}, \tag{4.15}$$

$$(J^{-1})_{ij}=\frac{1}{2J}\epsilon_{jkp}\epsilon_{iqr}J_{kq}J_{pr}\simeq\delta_{ij}+\mathcal{O}(\boldsymbol{\psi}). \tag{4.16}$$

Eq. (4.12) is then rewritten at leading order with

$$\begin{aligned}(J^{-1})_{ij}\frac{\partial}{\partial\boldsymbol{q}_j}\Big(\ddot{\boldsymbol{\psi}}+2H\dot{\boldsymbol{\psi}}\Big)&=-4\pi\,G\rho_{\rm m}\Big(\frac{1}{J}-1\Big)\\ \Longrightarrow\quad(\nabla_q\cdot\boldsymbol{\psi})^{\cdot\cdot}+2H(\nabla_q\cdot\boldsymbol{\psi})^{\cdot}&-4\pi\,G\rho_{\rm m}(\nabla_q\cdot\boldsymbol{\psi})\simeq 0.\end{aligned} \tag{4.17}$$

Eq. (4.17) is nothing but the evolution equation for linear density field. Since $\delta_{\rm m}\simeq-\nabla_q\cdot\boldsymbol{\psi}$ at $t\to 0$, we may write the displacement field as

$$\boldsymbol{\psi}(\boldsymbol{q};a)=-\,D_1(a)\,\nabla_q\varphi(\boldsymbol{q}),\qquad\nabla_q^2\varphi(\boldsymbol{q})=\delta_0(\boldsymbol{q}). \tag{4.18}$$

Here, D_1 is the linear growth factor, and δ_0 is the initial density field.

A crucial point may be that the density field is not assumed to be small. Thus, it is often said that the solution may be applied to the quasi-linear regime. Plugging Eq. (4.18) into Eq. (4.13), we obtain

$$1+\delta_{\rm m}(\boldsymbol{x})\simeq\frac{1}{(1-D_1\,\lambda_1)(1-D_1\,\lambda_2)(1-D_1\,\lambda_3)}. \tag{4.19}$$

where λ_i is the eigenvalue of the vector $\varphi_{,i}$. This illustrates how the non-sphericity of the structure develops according to the initial condition. In particular, the above equation implies that in the Gaussian initial condition, most of the nonlinear structure is aspherical.

4.3 Perturbation theory

Collisionless Boltzmann equation (Vlasov-Poisson system)

$$\left[\frac{\partial}{\partial t}+\frac{\boldsymbol{p}}{ma^2}\frac{\partial}{\partial\boldsymbol{x}}-m\frac{\partial\Psi}{\partial\boldsymbol{x}}\frac{\partial}{\partial\boldsymbol{p}}\right]f(\boldsymbol{x},\boldsymbol{p})=0, \tag{4.20}$$

supplemented with the Poisson equation:

$$\nabla^2\Psi(\boldsymbol{x})=4\pi\,G\,a^2\left[\frac{m}{a^3}\int d^3\boldsymbol{p}\,f(\boldsymbol{x},\boldsymbol{p})-\rho_{\rm m}\right]. \tag{4.21}$$

Here, m is the mass of CDM (+baryon) particle.

Single-stream approximation

$$\boxed{\text{Ansatz}}\quad f(\boldsymbol{x},\boldsymbol{p})=\overline{n}\,a^3\,\{1+\delta_{\rm m}(\boldsymbol{x})\}\,\delta_{\rm D}\big[\boldsymbol{p}-m\,a\,\boldsymbol{v}(\boldsymbol{x})\big]. \tag{4.22}$$

With this ansatz, taking the zeroth and first velocity moments of Eq. (4.20) yields

$$\frac{\partial\delta_{\rm m}}{\partial t}+\frac{1}{a}\nabla\left[(1+\delta)\boldsymbol{v}\right]=0, \tag{4.23}$$

$$\frac{\partial\boldsymbol{v}}{\partial t}+\frac{1}{a}(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}=-\frac{1}{a}\frac{\partial\Psi}{\partial\boldsymbol{x}}, \tag{4.24}$$

$$\frac{1}{a^2}\,\nabla^2\Psi=4\pi\,G\,\rho_{\rm m}\,\delta_{\rm m}. \tag{4.25}$$

In what follows, we omit the subscript of $\delta_{\rm m}$, and simply denote the mass density field by δ.

Evolution equations in Fourier space

Introducing the velocity-divergence field, $\theta\equiv\nabla\cdot\boldsymbol{v}/(aH)$, Eqs. (4.23)-(4.25) are rewritten in terms of the Fourier-space quantities[1]:

$$a\frac{d\delta(\boldsymbol{k})}{da}+\theta(\boldsymbol{k})=-\int\frac{d^3\boldsymbol{k}_1d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12})\,\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\,\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2), \tag{4.26}$$

$$a\frac{d\theta(\boldsymbol{k})}{da}+\left(2+\frac{\dot{H}}{H^2}\right)\theta(\boldsymbol{k})+\frac{3}{2}\Omega_{\rm m}(a)\,\delta(\boldsymbol{k})=-\int\frac{d^3\boldsymbol{k}_1d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12})\,\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\,\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) \tag{4.27}$$

with the functions α and β given by

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)=1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{|\boldsymbol{k}_1|^2},\qquad\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)=1+\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}{|\boldsymbol{k}_1|^2|\boldsymbol{k}_2|^2}. \tag{4.28}$$

Standard PT expansion

Eqs .(4.26) and (4.27) may be expanded as[2]

$$\delta(\boldsymbol{k})=\sum_n D_1^n\,\delta_n(\boldsymbol{k}),\qquad\theta(\boldsymbol{k})=-f\,\sum_n D_1^n\,\theta_n(\boldsymbol{k}). \tag{4.29}$$

[1]The vorticity component of the velocity field is ignored.

[2]Strictly speaking, the expansion in Eq. (4.29) is correct only in the Einstein-de Sitter cosmology, in which we have $D_1=a$. Nevertheless, this expansion is shown to give a very good approximation.

The perturbative solutions δ_n and θ_n are formally expressed as

$$\delta_n(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}_1 \cdots d^3\boldsymbol{k}_n}{(2\pi)^{3(n-1)}} \, \delta_D(\boldsymbol{k} - \boldsymbol{k}_{12\cdots n}) \, F_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) \, \delta_0(\boldsymbol{k}_1) \cdots \delta_0(\boldsymbol{k}_n), \tag{4.30}$$

$$\theta_n(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}_1 \cdots d^3\boldsymbol{k}_n}{(2\pi)^{3(n-1)}} \, \delta_D(\boldsymbol{k} - \boldsymbol{k}_{12\cdots n}) \, G_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) \, \delta_0(\boldsymbol{k}_1) \cdots \delta_0(\boldsymbol{k}_n) \tag{4.31}$$

with $\boldsymbol{k}_{12\cdots n} = \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_n$. The function δ_0 is the initial density field originated from primordial curvature perturbation, $\Phi_{\rm p}$ [see Eq. (2.50)]. The functions F_n and G_n are called perturbation theory (PT) kernels, whose functional forms are determined recursively.

Constructing PT kernels

First defining the kernel

$$\mathcal{F}_a^{(n)}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \begin{pmatrix} F_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) \\ G_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) \end{pmatrix}, \tag{4.32}$$

then the recursion relation of the PT kernel is obtained from Eqs. (4.26) and (4.27)[3]:

$$\mathcal{F}_a^{(n)}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) = \sum_{m=1}^{n-1} \sigma_{ab}^{(n)} \, \gamma_{bcd}(\boldsymbol{q}_1, \boldsymbol{q}_2) \, \mathcal{F}_c^{(m)}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_m) \mathcal{F}_d^{(n-m)}(\boldsymbol{k}_{m+1}, \cdots, \boldsymbol{k}_n), \tag{4.33}$$

with $\mathcal{F}^{(1)} = (1, 1)$. Here, $\boldsymbol{q}_1 = \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_m$ and $\boldsymbol{q}_1 = \boldsymbol{k}_{m+1} + \cdots + \boldsymbol{k}_n$. The $\sigma_{ab}^{(n)}$ and γ_{abc} are respectively given by (e.g., Ref. [2, 11, 28])

$$\sigma_{ab}^{(n)} = \frac{1}{(2n+3)(n-1)} \begin{pmatrix} 2n+1 & 2 \\ 3 & 2n \end{pmatrix}, \tag{4.34}$$

$$\gamma_{abc}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \begin{cases} \alpha(\boldsymbol{k}_2, \boldsymbol{k}_1)/2 & (a, b, c) = (1, 1, 2) \\ \alpha(\boldsymbol{k}_1, \boldsymbol{k}_2)/2 & (a, b, c) = (1, 2, 1) \\ \beta(\boldsymbol{k}_1, \boldsymbol{k}_2) & (a, b, c) = (2, 2, 2) \\ 0 & \text{otherwise} \end{cases} \tag{4.35}$$

Note that the PT kernels obtained from recursion relation are not yet symmetric for all possible permutations of the variables, e.g., $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2$. For later statistical calculations, they must be symmetrized:

$${}^s\mathcal{F}_a^{(3)}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n) \equiv \frac{1}{n!} \sum_{\text{permutations}} \mathcal{F}_a^{(n)}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n). \tag{4.36}$$

Examples

[3] Again, the recursion relation obtained here is exact only in the Einstein-de Sitter Universe.

Here, we give explicit expression for PT kernels up to third order:

$$
{}^sF_2(\boldsymbol{k}_1,\boldsymbol{k}_2)=\frac{5}{7}+\frac{1}{2}\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)+\frac{2}{7}\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}, \tag{4.37}
$$

$$
{}^sG_2(\boldsymbol{k}_1,\boldsymbol{k}_2)=\frac{3}{7}+\frac{1}{2}\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)+\frac{4}{7}\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}. \tag{4.38}
$$

$$
{}^sF_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\frac{1}{6}\Big[\frac{7}{9}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_3)}{k_3^2}\,{}^sF_2(\boldsymbol{k}_1,\boldsymbol{k}_2)+\Big\{\frac{7}{9}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_{12})}{k_{12}^2}+\frac{2}{9}\frac{k_{123}^2(\boldsymbol{k}_3\cdot\boldsymbol{k}_{12})}{k_3^2k_{12}^2}\Big\}\,{}^sG_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\Big]
$$
$$
+\,(\text{cyclic perm.})\ , \tag{4.39}
$$

$$
{}^sG_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\frac{1}{6}\Big[\frac{1}{3}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_3)}{k_3^2}\,{}^sF_2(\boldsymbol{k}_1,\boldsymbol{k}_2)+\Big\{\frac{1}{3}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_{12})}{k_{12}^2}+\frac{2}{3}\frac{k_{123}^2(\boldsymbol{k}_3\cdot\boldsymbol{k}_{12})}{k_3^2k_{12}^2}\Big\}\,{}^sG_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\Big]
$$
$$
+\,(\text{cyclic perm.})\ . \tag{4.40}
$$

Relation to Lagrangian PT

Instead of expanding δ and θ like Eq. (4.29), we may introduce the Lagrangian frame in which the mass distribution looks homogeneous ($\boldsymbol{q}$, the rest frame of mass element), and follow the motion of flow with the displacement field (vector), $\boldsymbol{\psi}$ [Eq. (4.11)]:

$$
\boldsymbol{x}(\boldsymbol{q},t)=\boldsymbol{q}+\boldsymbol{\psi}(\boldsymbol{q},t).
$$

Note $\boldsymbol{\psi}\to 0$ at $t\to 0$. The perturbative treatment of the displacement field is called Lagrangian PT. In Fourier space, we can expand

$$
\boldsymbol{\psi}(\boldsymbol{k};\,a)\equiv\int d^3\boldsymbol{q}\,\boldsymbol{\psi}(\boldsymbol{q})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}}=\sum_n D_1^n(a)\,\boldsymbol{\psi}_n(\boldsymbol{k})\ ;
$$
$$
\boldsymbol{\psi}_n(\boldsymbol{k})=i\int\frac{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}{(2\pi)^{3(n-1)}}\,\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{p}_{12\cdots n})\,\boldsymbol{L}_n(\boldsymbol{p}_1,\cdots,\boldsymbol{p}_n)\,\delta_0(\boldsymbol{p}_1)\cdots\delta_0(\boldsymbol{p}_n). \tag{4.41}
$$

The function $\boldsymbol{L}_n$ is called Lagrangian PT kernel. Note that the first-order Lagrangian PT is the Zel'dovich approximation.

The Lagrangian PT kernels are related to standard PT kernels as follows. In terms of the displacement field, the density field in Fourier space is described as (using the relation $d^3\boldsymbol{q}=\{1+\delta(\boldsymbol{x})\}d^3\boldsymbol{x}$):

$$
\begin{aligned}
\delta(\boldsymbol{k})&=\int d^3\boldsymbol{x}\,\delta(\boldsymbol{x})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\\
&=\int d^3\boldsymbol{q}\,e^{-i\boldsymbol{k}\cdot\{\boldsymbol{q}+\boldsymbol{\psi}(\boldsymbol{q})\}}-(2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}),\qquad\left(\because\ \ \delta(\boldsymbol{x})=\left|\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}}\right|-1\right)\\
&=\sum_n\left(\int\frac{d^3\boldsymbol{k}_1\cdots d^3\boldsymbol{k}_n}{(2\pi)^{3(n-1)}}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{p}_{12\cdots n})\,\frac{(-i)^n}{n!}\,\{\boldsymbol{k}\cdot\boldsymbol{\psi}(\boldsymbol{k}_1)\}\cdots\{\boldsymbol{k}\cdot\boldsymbol{\psi}(\boldsymbol{k}_n)\}\right)-(2\pi)^3\delta_{\rm D}(\boldsymbol{k})
\end{aligned} \tag{4.42}
$$

Substituting Eq. (4.41) into the above, the order-by-order comparison of the Fourier kernels with those of the standard PT kernel given in Eq. (4.30) leads to

$$\begin{aligned}
&F_1(\boldsymbol{k}) = 1 = \boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}),\\
&F_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_1,\boldsymbol{k}_2) + \frac{1}{2}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)\},\\
&F_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \boldsymbol{k}\cdot\boldsymbol{L}_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) + \frac{1}{3}\big[\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_2,\boldsymbol{k}_3)\} + (\text{cyclic perm.})\big]\\
&\qquad\qquad + \frac{1}{6}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_3)\}. \qquad (4.43)
\end{aligned}$$

Using these relations, one can reconstruct the *longitudinal* mode of the Lagrangian PT kernels recursively without solving evolution equation of $\boldsymbol{\psi}$. Notice that the reconstructed kernels in this way actually miss *transverse* modes, which appears non-vanishing at $n \geq 3$. A systematic calculation of the Lagrangian PT kernels including both longitudinal and transverse modes is given by Ref. [22].

Gaussian initial condition

For explicit calculations of the statistical quantities based on PT, one needs to specify the statistical properties of the density field δ_0 as a seed of PT expansion [Eqs. (4.30), (4.31) and (4.41)]. Standard assumption/hypothesis may be the *Gaussian initial condition*. In this case, all the statistical information is encoded in the initial power spectrum $P_0(k)$, and any statistical quantity is constructed with P_0. We have

$$\langle\delta_0(\boldsymbol{k})\rangle = 0, \qquad (4.44)$$
$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\rangle = (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}_{12})P_0(k) \qquad (4.45)$$
$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_3)\rangle = 0, \qquad (4.46)$$
$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_3)\delta_0(\boldsymbol{k}_4)\rangle = (2\pi)^6\Big[\delta_{\rm D}(\boldsymbol{k}_{12})\delta_{\rm D}(\boldsymbol{k}_{34})P_0(k_1)P_0(k_2) + (\text{cyclic perm.})\Big], \qquad (4.47)$$
$$\vdots$$

In general, for positive integer n,

$$\langle\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_{2n+1})\rangle = 0, \qquad (4.48)$$
$$\langle\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_{2n})\rangle = \sum_{\text{all pair associations }p}\ \prod_{\text{pairs }(i,j)}\langle\delta_0(\boldsymbol{k}_i)\delta_0(\boldsymbol{k}_j)\rangle. \qquad (4.49)$$

These properties are known as Wick's theorem or Isserlis' theorem.

Statistical calculations

- **Power spectrum** : $\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle = (2\pi)^3\delta_{\rm D}(\boldsymbol{k}_{12})\,P(k_1)$

An explicit calculation of the ensemble average at next-to-leading order (called *one-loop*) leads to

$$\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle \simeq \langle\delta_1(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\rangle + \langle\delta_2(\boldsymbol{k}_1)\delta_2(\boldsymbol{k}_2)\rangle + \langle\delta_1(\boldsymbol{k}_1)\delta_3(\boldsymbol{k}_2)\rangle + \langle\delta_3(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\rangle + \cdots . \tag{4.50}$$

We thus obtain

$$P(k,a) \simeq \{D_1(a)\}^2\,P_0(k) + \{D_1(a)\}^4\,\big\{P_{22}(k) + P_{13}(k)\big\}. \tag{4.51}$$

The first term at RHS is nothing but the linear power spectrum. The parenthesis represents the contributions from the higher-order PT, given by

$$P_{22}(k) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\{F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\}^2\,P_0(|\boldsymbol{k}-\boldsymbol{p}|)P_0(p), \tag{4.52}$$

$$P_{22}(k) = 2\,P_0(k)\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\{F_3(\boldsymbol{k},\boldsymbol{p},-\boldsymbol{p})\}^2\,P_0(p). \tag{4.53}$$

Because of the different dependence on the linear growth factor, these nonlinear contributions give rise to the *scale-dependent growth* of power spectrum.

- **Bispectrum** : $\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\delta(\boldsymbol{k}_3)\rangle = (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}_{123})\,B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$

  At leading-order, we have

$$\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\delta(\boldsymbol{k}_3)\rangle \simeq \langle\delta_1(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\delta_2(\boldsymbol{k}_3)\rangle + \langle\delta_2(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\delta_1(\boldsymbol{k}_3)\rangle + \langle\delta_1(\boldsymbol{k}_1)\delta_2(\boldsymbol{k}_2)\delta_1(\boldsymbol{k}_3)\rangle + \cdots . \tag{4.54}$$

  Thus, the non-vanishing bispectrum arises:

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \simeq \{D_1(a)\}^4\,\Big\{2F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)P_0(k_1)P_0(k_2) + (\text{cyclic perm.})\Big\}. \tag{4.55}$$

  This implies that the nonlinear gravitational evolution generally produces non-Gaussianity, and other higher-order statistics also become non-vanishing. In other words, the statistical information initially encoded in the power spectrum is partly transferred to the higher-order statistics.

Resummed perturbation theory

In order to improve the performance of PT-based prediction, we need to include the higher-order PT corrections. A crucial remark is that the applicable range of the PT prediction largely depends on the PT scheme itself. Indeed, standard PT is known to have bad convergence properties, and produces ill-behaved higher-order corrections. The resummation or renormalization is one of the techniques to improve the convergence of PT expansion.

Among various methods proposed so far (e.g., [38, 30, 20, 11, 35]), we here present a specific prescription for resummed PT treatment, called *multi-point propagator expansion* or Γ-*expansion* [3, 21]

For the evolved (nonlinear) density field $\delta(\boldsymbol{k},a)$, one defines

$$\frac{1}{p!}\Big\langle\frac{\delta^p\delta(\boldsymbol{k},a)}{\delta\delta_0(\boldsymbol{k}_1)\cdots\delta\delta_0(\boldsymbol{k}_p)}\Big\rangle\equiv\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12\cdots p})\frac{1}{(2\pi)^{3(p-1)}}\,\Gamma^{(p)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p) \tag{4.56}$$

The function $\Gamma^{(p)}$ is called $(p{+}1)$-point propagator. This is the non-perturbative statistical quantity characterizing nonlinear mode-coupling. In terms of the standard PT expansion, it is expressed as

$$\Gamma^{(p)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p;\,a)=\{D_1(a)\}^p\,F_p(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)+\sum_{n=1}\{D_1(a)\}^{p+2n}\,\Gamma^{(p)}_{n\text{-loop}}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p) \tag{4.57}$$

with higher-order correction $\Gamma^{(p)}_{n\text{-loop}}$ given by

$$\Gamma^{(p)}_{n\text{-loop}}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)$$
$$=c_n^{(p)}\int\frac{d^3\boldsymbol{q}_1\cdots d^3\boldsymbol{q}_n}{(2\pi)^{3n}}F_{2n+p}(\boldsymbol{q}_1,-\boldsymbol{q}_1,\cdots,\boldsymbol{q}_n,-\boldsymbol{q}_n,\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)\,P_0(q_1)\cdots P_0(q_n). \tag{4.58}$$

with the symmetric factor $c_n^{(p)}=(2-1)!!\,{}_{2n+p}C_p$. Using these expressions, it is straightforward to show that the standard PT expansion of the power spectrum is re-organized in terms of the multi-point propagators as follows:

$$P(k,a)=\{\Gamma^{(1)}(k;a)\}^2\,P_0(k)$$
$$+\sum_{n=2}n!\int\frac{d^3\boldsymbol{q}_1\cdots d^3\boldsymbol{q}_n}{(2\pi)^{3(n-1)}}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{q}_{12\cdots n})\left\{\Gamma^{(n)}(\boldsymbol{q}_1,\cdots,\boldsymbol{q}_n;\,a)\right\}^2P_0(q_1)\cdots P_0(q_n). \tag{4.59}$$

Similarly, other statistical quantities such as bispectrum and trispectrum are systematically constructed with multi-point propagators [3, 4, 39].

A crucial point is how to accurately construct a *regularized* multi-point propagator that can describe their global shape, i.e., their whole k-dependence. A couple of important properties to be noted is

- High-k behaviors: In the limit $k\to\infty$, resummation of the standard PT expansion at all order is possible, and one gets [3, 5]

$$\Gamma^{(p)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p;\,a)\ \stackrel{k\to\infty}{\longrightarrow}\ \{D_1(a)\}^p\,F_p(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)\,e^{-k^2\sigma_d^2/2} \tag{4.60}$$

  with σ_d being the rms of the displacement field given by

$$\sigma_d^2=\{D_1(a)\}^2\int\frac{dq}{6\pi^2}P_0(q). \tag{4.61}$$

- Low-k behaviors: At low-k, a perturbative calculation with standard PT expansion may be applied. While this should be restricted to a certain low-k regime, each perturbative correction in Eq. (4.58) possesses the following asymptotic form:

$$\Gamma^{(p)}_{n\text{-loop}}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p;\,a)\ \stackrel{k\to\infty}{\longrightarrow}\ \frac{1}{n!}\left(-\frac{k^2\sigma_d^2}{2}\right)^n\{D_1(a)\}^p\,F_p(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p) \tag{4.62}$$

The above properties indicate that there exists well-defined matching scheme that smoothly interpolates between the low-k and high-k results of any multi-point propagator [4]. Construction of such a *regularized* propagator is given by

$$\Gamma^{(p)}_{\rm reg}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p;\,a)=\{D_1(a)\}^p$$
$$\times\Bigl[F_p(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)\Bigl\{1+\frac{k^2\sigma_d^2}{2}\Bigr\}+\{D_1(a)\}^2\,\Gamma^{(p)}_{\text{1-loop}}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)\Bigr]\exp\Bigl\{-\frac{k^2\sigma_d^2}{2}\Bigr\},\quad(4.63)$$

which consistently reproduces one-loop PT results at low-k. At high-k, it approaches the asymptotic behavior in Eq. (4.60). This construction is easily generalized to include the higher-order PT corrections at low-k.

Finally, comments to be noted (or advertisement) are

- Code to compute power spectrum based on this resummed PT scheme is publicly available [37][4].
- Based on this resummed scheme, an algorithm that allows accelerated power spectrum calculations is proposed and is implemented in the public code [37].

4.4 Halo model

Halo model provides a qualitative view of gravitational clustering of large-scale structure on both large and small scales, and it can be even applied for a quantitative study of matter/halo clustering. The underlying assumption of the halo model is that the spatial volume of the Universe is entirely filled with the self-gravitating bound objects called dark matter halos, and all the CDM particles (and baryons) constitute these halos.

To start with, consider the two-point correlation function as the Fourier counterpart of the power spectrum, $\xi(r)=\langle\delta(\boldsymbol{x})\delta(\boldsymbol{x}+\boldsymbol{r})\rangle$. The two-point correlation function measures the excess probability above the Poisson distribution of finding pair of objects (CDM particles) with separation r. We can write the contributions to ξ as two separate terms, one from particle pairs in the same halo, and the other from pairs that reside in two different halos. The dark matter halos exhibit a spectrum of masses that can be characterized by a distribution function $n_{\rm halo}(M)$ called the halo mass function, and the halo centers are spatially correlated. Taking these factors into consideration, we can write the two- point correlation function in terms of the halo density profile $\rho_{\rm halo}(r)$, halo mass function $n_{\rm halo}(M)$, and halo-halo correlation $\xi_{\rm hh}(r)$:

$$\xi(\boldsymbol{r})=\xi_{\text{1-halo}}(r)+\xi_{\text{2-halo}}(r)\,;\qquad(4.64)$$
$$\xi_{\text{1-halo}}(r)=\int dM\,n_{\rm halo}(M)\int d^3\boldsymbol{x}\frac{\rho_{\rm halo}(\boldsymbol{x};\,M)}{\rho_{\rm m}}\,\frac{\rho_{\rm halo}(\boldsymbol{x}+\boldsymbol{r};\,M)}{\rho_{\rm m}},\qquad(4.65)$$
$$\xi_{\text{2-halo}}(|\boldsymbol{x}-\boldsymbol{x}'|)=\int dM_1\,n_{\rm halo}(M_1)\int dM_2\,n_{\rm halo}(M_2)$$
$$\times\int d^3\boldsymbol{x}_1\frac{\rho_{\rm halo}(\boldsymbol{x}-\boldsymbol{x}_1;\,M_1)}{\rho_{\rm m}}\int d^3\boldsymbol{x}_1\,\frac{\rho_{\rm halo}(\boldsymbol{x}'-\boldsymbol{x}_2;\,M_2)}{\rho_{\rm m}}\,\xi_{\rm hh}(\boldsymbol{x}_1-\boldsymbol{x}_2;\,M_1,M_2).\qquad(4.66)$$

[4]http://ascl.net/1404.012

Then, the corresponding Fourier counter-part (power spectrum) becomes

$$P(k) = P_{\text{1-halo}}(k) + P_{\text{2-halo}}(k)\,; \tag{4.67}$$

$$P_{\text{1-halo}}(k) = \int dM\, n_{\text{halo}}(M) \left|\frac{\tilde{\rho}_{\text{halo}}(k;\, M)}{\rho_{\text{m}}}\right|^2, \tag{4.68}$$

$$P_{\text{2-halo}}(k) = \int dM_1\, n_{\text{halo}}(M_1) \int dM_2\, n_{\text{halo}}(M_2)\, \frac{\tilde{\rho}_{\text{halo}}(k;\, M_1)}{\rho_{\text{m}}} \frac{\tilde{\rho}_{\text{halo}}(k;\, M_2)}{\rho_{\text{m}}}\, P_{\text{hh}}(k;\, M_1, M_2). \tag{4.69}$$

where the quantities $\tilde{\rho}_{\text{halo}}$ and P_{hh} are the Fourier transform of the density profile and correlation function, respectively.

Given the explicit functional forms of n_{halo}, ρ_{halo}, and ξ_{hh}, the above equations describe the nonlinear matter clustering reasonably well. In particular, halo model description is powerful to predict the statistics at small scales, where the perturbation theory treatment cannot reach. For more detail, see Ref. [10] for applications and extension to the galaxy clustering (for recent interesting applications, see Ref. [14] to the redshift-space distortions, Refs. [36, 17] to the non-Gaussian covariance).

4.5 Galaxy/halo bias

So far, we have focused on the dark matter clustering on large scales. However, the fundamental observables of the large-scale structure by the galaxy redshift surveys are the galaxies. To be precise, what we can observe/measure is the number density fluctuations of galaxies, which differs from the mass density fluctuations we have so far considered. In this respect, the galaxy distribution may be a biased tracer of large-scale matter distribution. Connecting the dark matter clustering with galaxy clustering is thus important issues, and needs to be addressed[5].

$$\delta_{\text{gal}}(\boldsymbol{x}) = \frac{n_{\text{gal}}(\boldsymbol{x})}{\overline{n}_{\text{gal}}} - 1 \qquad \longleftrightarrow \qquad \delta(\boldsymbol{x}) = \frac{\rho_{\text{m}}(\boldsymbol{x})}{\rho_{\text{m}}} - 1. \tag{4.70}$$

<u>Prescriptions for galaxy bias</u>[6]

- Linear bias: the simplest prescription that has been first invented by Ref. [16]:

$$\delta_{\text{gal}}(\boldsymbol{x}) = b\, \delta_{\text{m}}(\boldsymbol{x}). \tag{4.71}$$

[5] Comprehensive review on galaxy bias by V. Desjacques, D. Jeong, and F. Schmidt is supposed to appear soon.

[6] Examples presented here are regarded as Eulerian local bias. Another class of local bias prescription in terms of the Lagrangian quantities is called Lagrangian local bias.

- Nonlinear bias : the second simplest prescription (e.g., [13])

$$\delta_{\rm gal}(\boldsymbol{x}) = \sum_n \frac{b_n}{n!}\Big[\{\delta_{\rm m}(\boldsymbol{x})\}^n - \langle\{\delta_{\rm m}(\boldsymbol{x})\}^n\rangle\Big]. \qquad (4.72)$$

A more general prescription of the bias may be non-local, nonlinear and stochastic bias. But, generic prescription of it looks intractable.

Toy models for bias

Representative models of galaxy biasing are halo bias and peak bias. These are classified as the Lagrangian bias model, and the density fields are defined in Lagrangian space:

$$\text{Halo bias:} \qquad 1+\delta_{\rm halo}(\boldsymbol{q};\,M) = \frac{\hat{n}_{\rm halo}(\boldsymbol{q};\,M)}{n_{\rm halo}(M)}, \qquad (4.73)$$

$$\text{Peak bias:} \qquad 1+\delta_{\rm peak}(\boldsymbol{q};\,\nu_{\rm c}) = \frac{\hat{n}_{\rm peak}(\boldsymbol{q};\,\nu_{\rm c})}{n_{\rm peak}(\nu_{\rm c})}, \qquad (4.74)$$

where $\hat{n}_{\rm halo}$ and $\hat{n}_{\rm peak}$ are the random fields, and $n_{\rm halo}$ and $n_{\rm peak}$ represent their mean, i.e., $\langle\hat{n}_{\rm halo}\rangle = n_{\rm halo}$ and $\langle\hat{n}_{\rm peak}\rangle = n_{\rm peak}$. Explicit expression for $\hat{n}_{\rm halo}$ is (see Ref. [23]):

$$\hat{n}_{\rm halo}(\boldsymbol{q};\,M) = -2\frac{\rho_{\rm m}}{M}\frac{\partial}{\partial M}\Theta\big[\delta(\boldsymbol{q}|M) - \delta_{\rm crit}\big] \qquad (4.75)$$

with Θ being the Heaviside step function. The quantity $\delta_{\rm crit}$ is the critical density determined by the spherical collapse model [see Eq. (4.5) or (4.7)], and $\delta(\boldsymbol{q}|M)$ is the linearly extrapolated density field smoothed over the radius $\{M/(4\pi\,\rho_{\rm m}/3)\}^{1/3}$ with top-hat filter. For the peak, it is defined in terms of the smoothed linear density field with Gaussian filter [23]:

$$\hat{n}_{\rm peak}(\boldsymbol{q};\,\nu_{\rm c}) = \frac{3^{3/2}}{R_*^3}\,\delta_{\rm D}(\nu-\nu_{\rm c})\delta_{\rm D}(\vec{\eta})\Theta(\lambda_3)|{\rm det}(\zeta_{ij})| \qquad (4.76)$$

with $R_* = \sqrt{3}\sigma_1/\sigma_2$, $\nu = \delta/\sigma$, $\eta_i = \nabla_i\delta/\sigma_1$, and $\zeta_{ij} = \nabla_i\nabla_j\delta/\sigma_2$. Here, σ_n is defined by $\sigma_n^2 = \langle(\nabla^n\delta)^2\rangle$.

The halo/peak density fields in Lagrangian space are given as nonlinear and scale-dependent functions of linear density field. Moreover, they are mapped into Eulerian space:

$$1+\delta_{\rm X}(\boldsymbol{x}) = \int d^3\boldsymbol{q}\,[1+\delta_X(\boldsymbol{q})]\,\delta_{\rm D}\big(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{\psi}(\boldsymbol{q})\big), \qquad ({\rm X}={\rm halo},\ {\rm peak}). \qquad (4.77)$$

This further induces non-locality of the bias through the gravitational evolution (see Ref [25] for some attempts).

Practical bias parameterization ?

For practical application to observed galaxy power spectrum or correlation function (specifically BOSS), the bias prescription recently used in the literature is [32, 24, 9]

$$\delta_{\rm gal}(\boldsymbol{x}) = b_1\,\delta(\boldsymbol{x}) + \frac{b_2}{2}\left[\delta(\boldsymbol{x})^2 - \langle\delta(\boldsymbol{x})^2\rangle\right] + \frac{1}{2}\,b_{s^2}\left[s(\boldsymbol{x})^2 - \langle s(\boldsymbol{x})^2\rangle\right] + \cdots \tag{4.78}$$

with the non-local field s defined by

$$s(\boldsymbol{x})^2 = s_{ij}(\boldsymbol{x})s^{ij}(\boldsymbol{x})\ ; \quad s_{ij}(\boldsymbol{x}) = \left(\nabla_i\nabla_j\,\nabla^{-2} - \frac{1}{3}\delta_{ij}\right)\delta(\boldsymbol{x}) \tag{4.79}$$

Note that when you go to third-order, another non-local correction arises from the coupling between the tidal fields of density and velocity, whose coefficient is called $b_{3\rm nl}$ [32]. At the end, on top of the cosmological parameters of our interest, there additionally appear 5 free parameters in the galaxy power spectrum, which have to be determined simultaneously from the measured power spectrum [6][7]:

$$\begin{aligned} P_{\rm gal}(k) = {} & b_1^2\,P_{\delta\delta}(k) + 2b_2b_1\,P_{b2,\delta}(k) + 2b_{s2}b_1\,P_{bs2,\delta}(k) + 2b_{3\rm nl}b_1\,\sigma_3^2(k)\,P_{\rm lin}(k) \\ & + b_2^2\,P_{b22}(k) + 2\,b_2b_{s2}\,P_{b2s2}(k) + b_{s2}^2\,P_{bs22}(k) + N, \end{aligned} \tag{4.80}$$

where $P_{\delta\delta}$ and $P_{\rm lin}$ are the nonlinear and linear matter power spectrum. The definitions and expressions for other power spectra are given in Ref. [32, 6].

[7] In redshift space, one more free parameter arises from the Finger-of-God damping factor.

Appendix A

Useful formulas

A.1 Fourier transformation

$$A(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, A(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.1}$$

$$A(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, A(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.2}$$

Dirac's delta-function:

$$\delta_{\mathrm{D}}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.3}$$

Relation between $\xi(r)$ and $P(k)$ (Wiener-Khintchine relation):

$$\xi(r) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, P(|\boldsymbol{k}|)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.4}$$

$$P(k) = \int d^3\boldsymbol{r}\, \xi(|\boldsymbol{r}|)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{A.5}$$

A.2 Legendre polynomials

$$(1-\mu^2)\frac{d^2\mathcal{P}_\ell(\mu)}{d\mu^2} - 2\mu\frac{d\mathcal{P}_\ell(\mu)}{d\mu} + \ell(\ell+1)\,\mathcal{P}_\ell(\mu) = 0 \tag{A.6}$$

$$\int_{-1}^{1} d\mu\, \mathcal{P}_\ell(\mu)\, \mathcal{P}_{\ell'}(\mu) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} \tag{A.7}$$

$$\ell\,\mathcal{P}_\ell(\mu) - (2\ell-1)\mu\,\mathcal{P}_{\ell-1}(\mu) + (\ell-1)\mathcal{P}_{\ell-2}(\mu) = 0 \tag{A.8}$$

$$(\mu^2-1)\frac{d\mathcal{P}_\ell(\mu)}{d\mu} = \ell\,\{\mu\,\mathcal{P}_\ell(\mu) - \mathcal{P}_{\ell-1}(\mu)\} = (\ell+1)\{\mathcal{P}_{\ell+1}(\mu) - \mu\mathcal{P}_{\ell-1}(\mu)\} \tag{A.9}$$

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