暗黒物質宇宙における構造形成

Structure formation in the dark matter dominated universe

物理学基礎論特別講義 5 [1]

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Suggested readings

- P.J.E. Peebles, "The Large-scale structure of the Universe", (Princeton Univ. Press, 1980)
- S.Dodelson, "Modern cosmology" (Academic Press, 2003)
- H. Mo, F. van den Bosch, S. D. White, "Galaxy Formation and Evolution", (Cambridge Univ. Press, 2010)
- 松原隆彦, "現代宇宙論" (東京大学出版会, 2010)
- 松原隆彦, "宇宙論の物理 上・下" (東京大学出版会, 2014 年)
- 松原隆彦, "大規模構造の宇宙論" (共立出版, 2014 年)
- 須藤靖, "ものの大きさ" (東京大学出版会, 2006 年)

For more topical and focused reviews,

- F. Bernardeau, S. Colombi, E. Gaztañaga, R. Scoccimarro, "Large-scale structure of the Universe and cosmological perturbation theory", Physics Reports 367 (2002) 1-248 [5]
- A. Cooray and R. Sheth, "Halo models of large-scale structure", Physics Reports 372 (2002) 1-129 [20]
- J. Lesgourgues, S. Pastor, "Massive neutrinos and cosmology", Physics Reports 429 (2006) 307-379 [41]

Note –.

- I set $c = 1$ in most cases.
- I assume general relativity as the underlying theory of gravitation for structure formation and cosmic expansion. I also assume that you have learned general relativity (to some extent).

Contents

Chapter 1

Friedmann-Robertson-Walker cosmology

Friedmann-Lemeître-Robertson – Walker metric

$$ds^2 = -dt^2 + \{a(t)\}^2 \, d\vec{\ell}^2 \tag{1.1}$$

with the spatial metric given by

$$d\vec{\ell}^2 = \frac{dr^2}{1 - K\,r^2} + r^2(\theta^2 + \sin^2\theta \, d\phi^2) \tag{1.2}$$

$$= \begin{cases} d\chi^2 + \chi^2 \,(d\theta^2 + \sin^2\theta\, d\phi^2) & ; \quad (K = 0) \\ d\chi^2 + \left(\dfrac{\sinh\sqrt{-K}\chi}{\sqrt{-K}}\right)^2 (d\theta^2 + \sin^2\theta\, d\phi^2) & ; \quad (K \neq 0) \end{cases}$$

$$\equiv d\chi^2 + \{r(\chi)\}^2 \,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1.3}$$

where K is the spatial curvature, and χ is the comoving radial distance defined by

$$\chi \equiv \int \frac{dt}{a(t)} = \int \frac{dr}{\sqrt{1 - K\,r^2}} \tag{1.4}$$

Energy-momentum tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho,\, P,\, P,\, P), \tag{1.5}$$

The main components for energy density in the Universe are radiation, matter, and dark energy:

$$\rho = \rho_{\rm r} + \rho_{\rm m} + \rho_{\rm DE} \tag{1.6}$$

with the equation of state (EOS):

$$P_{\rm r} = \frac{1}{3}\,\rho_{\rm r}, \quad P_{\rm m} = 0, \quad P_{\rm DE} = w\,\rho_{\rm DE} \tag{1.7}$$

The EOS parameter of dark energy, $w\,(<0)$, is assumed to be -1 in ΛCDM model (cosmological constant), but it may deviate from -1. Further, it may possibly depend on time, and is conveniently characterized in the literature by

$$w(a) = w_0 + w_a(1-a). \tag{1.8}$$

Einstein equation/Friedmann equation

$$G^\mu_\nu = 8\pi\,G\,T^\mu_\nu\,; \quad G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\,\delta^\mu_\nu \quad \Longrightarrow \quad \begin{cases} 3\left(\dfrac{\dot a}{a}\right)^2 = 8\pi\,G\,\rho - \dfrac{K}{a^2}, \\[2ex] 3\,\dfrac{\ddot a}{a} = -4\pi\,G\,(\rho+3P). \end{cases} \tag{1.9}$$

The first equation is especially called Friedmann equation. One can check that these two equations are compatible with the following equation derived from the conservation law ($T^\mu_{\nu\,;\mu}=0$):

$$\dot\rho + 3\,\frac{\dot a}{a}\,(\rho+P) = 0. \tag{1.10}$$

Cosmological parameters

$$\text{Hubble parameter}: \quad H \equiv \frac{\dot a}{a}, \tag{1.11}$$

$$\text{Density parameter}: \quad \Omega_* \equiv \frac{8\pi\,G}{3\,H^2}\,\rho_*, \qquad (* = \text{r},\ \text{m},\ \text{DE}) \tag{1.12}$$

$$\text{Curvature parameter}: \quad \Omega_K \equiv -\frac{K}{3\,H^2} \tag{1.13}$$

Note that these are time-dependent quantities. The parameters given at present time are specifically denoted by H_0, $\Omega_{*,0}$, and $\Omega_{0,K}$.

With the definitions above, the Friedmann equation [first line of Eq. (1.9)] is reduced to

$$\Omega_{\rm r}(a) + \Omega_{\rm m}(a) + \Omega_{\rm DE}(a) + \Omega_K(a) = 1. \tag{1.14}$$

In terms of the parameters at present time, the Friedmann equation with a help of conservation law and EOS of each energy component leads to (using the redshift defined by $1+z=1/a$):

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\rm r,0}\,(1+z)^4 + \Omega_{\rm m,0}\,(1+z)^3 + \Omega_{\rm DE,0}\,\exp\left[3\int dz'\frac{1+w(z')}{1+z'}\right] + \Omega_{K,0}\,(1+z)^2. \tag{1.15}$$

To be more precise, the matter and radiation components are broken up into baryon (b), cold dark matter (c), neutrinos (ν), and photons (γ). While the baryon and cold dark matter are non-relativistic and thus their energy density evolves as $\rho_{\rm b,c} \propto a^{-3}$, the photons are relativistic and evolves as $\rho_\gamma \propto a^{-4}$. On the other hand, because of the small mass, the treatment of neutrinos needs a bit care. The neutrinos are initially relativistic, but become non-relativistic around $z \sim 200$. Taking a proper account of these facts, a refined version of Eq. (1.15) is given by [39]

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\gamma,0}\left\{1 + 0.227\,N_{\rm eff}\,f\left(\frac{m_\nu}{T_{\nu,0}(1+z)}\right)\right\}(1+z)^4 + (\Omega_{\rm b,0}+\Omega_{\rm c,0})\,(1+z)^3 + \Omega_{\rm DE,0}\,\exp\left[3\int dz'\frac{1+w(z')}{1+z'}\right] + \Omega_{K,0}\,(1+z)^2, \tag{1.16}$$

where $N_{\rm eff}$ is the effective number of neutrino species ($N_{\rm eff} = 3.046$ is the standard value), m_ν is the neutrino mass (assuming the equal mass for each species), and the present-day neutrino temperature, $T_{\nu,0} = (4/11)^{1/3} T_{\gamma,0} = 1.945$K. The function f is given by

$$f(y) \equiv \frac{120}{7\pi^4}\int_0^\infty dx \frac{x^2\sqrt{x^2+y^2}}{e^x+1} \simeq \{1 + (0.3173y)^{1.83}\}^{1/1.83}. \tag{1.17}$$

Table 1.1: Cosmological parameters of ΛCDM models derived from Planck 2015 results [57]

$\Omega_{\rm m,0}$	0.315
$\Omega_{\rm b,0}$	0.049
$\Omega_{\rm c,0}$	0.265
$\Omega_{\nu,0}$	——*
$\Omega_{\rm DE,0}$	0.685†
h‡	0.673

* In ΛCDM model, neutrinos are supposed to be massless, but in Planck 2015, they assumed $m_\nu = 0.06$eV ($\Omega_{\nu,0}h^2 \approx \sum m_\nu/93.04\,{\rm eV} \approx 0.0006$), with the standard value of $N_{\rm eff} = 3.046$.

† ΛCDM assumes a spatially flat universe ($\Omega_{K,0} = 0$), which gives $\Omega_{\rm DE,0} = 1 - \Omega_{\rm m,0}$.

‡ h is dimensionless Hubble parameter defined by $H_0 = 100\,h\,{\rm km\,s^{-1}\,Mpc^{-1}}$.

Cosmological distances

- **Luminosity distance**: determined through the observation of apparent flux of the *standard candle*, for which the absolute luminosity of the distant object is a priori known:

$$d_{\rm L}(z) \equiv \sqrt{\frac{\rm Luminosity}{4\pi\,{\rm Flux}}} = (1+z)\,r(\chi(z)) \tag{1.18}$$

- **Angular-diameter distance**: estimated from measurement of apparent angular size of the *standard ruler*, for which the proper (physical) size is a priori known:

$$d_{\rm A}(z) \equiv \frac{\text{Proper size}}{\text{Angular size}} = \frac{1}{(1+z)}\, r(\chi(z)) \tag{1.19}$$

The two distances given above generically holds the relation, $d_{\rm L}(z) = (1+z)^2\, d_{\rm A}(z)$, called Etherington's distance-duality relation.

At $z \ll 1$, one can expand

$$d_{\rm L}(z) = (1+z)^2\, d_{\rm A}(z) \simeq \frac{z}{H_0}\left[1 + \frac{1}{2}(1-q_0)z + \cdots\right], \qquad q_0 \equiv -\left.\frac{a\ddot{a}}{\dot{a}^2}\right|_{t_0} = \left.\frac{d\ln H(z)}{dz}\right|_{z=0} - 1. \tag{1.20}$$

Chapter 2

Linear theory of structure formation

2.1 Basic equations

Metric (flat)

$$ds^2 = -(1+2\Psi)dt^2 + \{a(t)\}^2\,(1+2\Phi)\,\delta_{ij}\,dx^i\,dx^j. \tag{2.1}$$

Perturbed quantities

$$\begin{array}{lcl} \text{Photon} & : & f_\gamma(p,x) = \left[\exp\left\{\frac{p}{T(1+\Theta)}\right\} - 1\right]^{-1} \\ \text{CDM} & : & \delta(\boldsymbol{x}), \quad \vec{v}(\boldsymbol{x}) \\ \text{Baryon} & : & \delta_{\rm b}(\boldsymbol{x}), \quad \vec{v}_{\rm b}(\boldsymbol{x}) \\ \text{Neutrino} & : & f_\nu = \left[\exp\left\{\frac{E}{T_\nu(1+\mathcal{N})}\right\} + 1\right]^{-1} \end{array} \tag{2.2}$$

Fourier expansion

$$\delta(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\delta(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2.3}$$

$$\vec{v}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{i\,\boldsymbol{k}}{k}\,v(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} \tag{2.4}$$

Note that in the absence of vector/tensor metric fluctuations, the velocity field only possesses the gradient mode (i.e., irrotational flow).

Evolution equations

$$\left(\frac{k}{a}\right)^2 \Phi + 3H(\dot{\Phi} - H\Psi) = 4\pi G \sum_i \rho_i \, \delta_i, \tag{2.5}$$

$$\left(\frac{k}{a}\right)^2 (\Phi + \Psi) = -8\pi G \Pi \; ; \qquad \Pi \equiv 4(\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2) \tag{2.6}$$

$$\dot{\Theta} + i\frac{k\mu}{a}(\Theta + \Psi) + \dot{\Phi} = n_e \, \sigma_{\rm T} \left[\Theta_0 - \Theta + i\mu \, v_{\rm b} - \frac{\mathcal{P}_2(\mu)}{2} \Theta_2\right], \tag{2.7}$$

$$\dot{\delta} - \frac{k}{a} v + 3\dot{\Phi} = 0, \tag{2.8}$$

$$\dot{v} + H \, v + \frac{k}{a} \Psi = 0, \tag{2.9}$$

$$\dot{\delta}_{\rm b} - \frac{k}{a} v_{\rm b} + 3\dot{\Phi} = 0, \tag{2.10}$$

$$\dot{v}_{\rm b} + H \, v_{\rm b} + \frac{k}{a} \Psi = -\frac{n_e \, \sigma_{\rm T}}{R} \left(3\Theta_1 + v_{\rm b}\right) \; ; \qquad R \equiv \frac{3\rho_{\rm b}}{4\rho_\gamma}, \tag{2.11}$$

$$\dot{\mathcal{N}} + i\frac{k\mu}{a} \left(\frac{p}{E}\mathcal{N} + \frac{E}{p} \Psi\right) + \dot{\Phi} = 0; \qquad E^2 = m_\nu^2 + p^2 \tag{2.12}$$

Note

- Polarization dependence of the Thomson scattering is ignored.

- Photon and neutrino fluctuations, Θ and $\mathcal{N}$, additionally have directional dependence of the momentum. Further, in the presence of non-zero mass, the neutrino fluctuation also depends on the momentum, i.e., $\mathcal{N}(\boldsymbol{k}, \mu, p)$, and $\Theta(\boldsymbol{k}, \mu)$ with $\mu \equiv \boldsymbol{k} \cdot /\boldsymbol{p}/(kp)$. It is thus convenient to characterize them by introducing the multipole expansion:

$$\Theta(\boldsymbol{k}, \mu) = \sum_\ell (-i)^\ell \, (2\ell + 1) \, \Theta_\ell(k) \, \mathcal{P}_\ell(\mu), \tag{2.13}$$

$$\mathcal{N}(\boldsymbol{k}, p, \mu) = \sum_\ell (-i)^\ell \, (2\ell + 1) \, \mathcal{N}_\ell(k, p) \, \mathcal{P}_\ell(\mu). \tag{2.14}$$

Boltzmann hierarchy

Applying the multipole expansion in Eqs. (2.13) and (2.14), with a help of formulas in Appendix A.2, Eqs.(2.7) and (2.12) respectively lead to a infinite set of hierarchy equations:

$$\dot{\Theta}_0 + \frac{k}{a}\Theta_1 + \dot{\Phi} = 0, \tag{2.15}$$

$$\dot{\Theta}_1 + \frac{k}{3a}(2\Theta_2 - \Theta_0 - \Psi) = n_e\sigma_{\rm T}\left(-\Theta_1 - \frac{v_{\rm b}}{3}\right), \tag{2.16}$$

$$\dot{\Theta}_2 + \frac{k}{5a}(3\Theta_3 - 2\Theta_2) = n_e\sigma_{\rm T}\left(-\Theta_2 + \frac{1}{10}\Theta_2\right), \tag{2.17}$$

$$\dot{\Theta}_\ell + \frac{k}{(2\ell+1)a}\{(\ell+1)\Theta_\ell - \ell\Theta_{\ell-1}\} = -n_e\sigma_{\rm T}\,\Theta_\ell, \quad (\ell \geq 3). \tag{2.18}$$

$$\dot{\mathcal{N}}_0 + \frac{k}{a}\frac{p}{\sqrt{p^2+m_\nu^2}}\mathcal{N}_1 + \dot{\Phi} = 0, \tag{2.19}$$

$$\dot{\mathcal{N}}_1 + \frac{k}{3a}\left\{\frac{p}{\sqrt{p^2+m_\nu^2}}(2\mathcal{N}_2 - \mathcal{N}_0) - \frac{\sqrt{p^2+m_\nu^2}}{p}\,\Psi\right\} = 0, \tag{2.20}$$

$$\dot{\mathcal{N}}_2 + \frac{k}{5a}\,\frac{p}{\sqrt{p^2+m_\nu^2}}\,(3\mathcal{N}_3 - 2\mathcal{N}_2) = 0, \tag{2.21}$$

$$\dot{\mathcal{N}}_\ell + \frac{k}{(2\ell+1)a}\,\frac{p}{\sqrt{p^2+m_\nu^2}}\,\{(\ell+1)\mathcal{N}_\ell - \ell\mathcal{N}_{\ell-1}\} = 0, \quad (\ell \geq 3). \tag{2.22}$$

2.2 Initial conditions

Adiabatic perturbations at the radiation-dominated epoch:

$$\Theta_0 = \mathcal{N}_0 = \frac{1}{2}\Phi_{\rm p}, \tag{2.23}$$

$$\delta = \delta_{\rm b} = \frac{3}{2}\Phi_{\rm p}, \tag{2.24}$$

$$v = v_{\rm b} = \frac{k}{2\,a\,H}\Phi_{\rm p}, \tag{2.25}$$

$$\Theta_1 = \mathcal{N}_1 = -\frac{k}{6\,a\,H}\Phi_{\rm p}. \tag{2.26}$$

$$\Theta_\ell = \mathcal{N}_\ell = 0, \qquad (\ell \geq 2) \tag{2.27}$$

Here, $\Phi_{\rm p}$ is the primordial curvature fluctuation, which is thought to be quantum-mechanically generated during the inflation. These initial conditions are given at super-horizon scales ($k \ll aH$).

Note –. To be precise, the neutrino quadrupole moment $\mathcal{N}_2$ is not negligibly small, and it leads to $\Phi + \Psi = (2/5)\,R_\nu\Psi$. This slightly alters the initial conditions given above.

2.3 Solutions: from radiation- to matter-dominated epoch

Below, assuming the massless neutrinos ($m_\nu = 0$) for simplicity, we derive the (approximate) solution for matter fluctuations.

Super-horizon evolution ($k \ll aH$)

$$\frac{d^2\Phi}{dy^2} + \frac{21y^2+54y+32}{2y(y+1)(3y+4)}\frac{d\Phi}{dy} + \frac{\Phi}{y(y+1)(3y+4)} = 0, \quad \left(y \equiv \frac{a}{a_{\rm eq}}\right) \tag{2.28}$$

The solution satisfying $\Phi \to \Phi_{\rm p}$ at $y \to 0$ becomes

$$\Phi(y) = \frac{\Phi_{\rm p}}{10}\frac{1}{y^3}\left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right] \tag{2.29}$$

$$\xrightarrow{y \gg 1} \frac{9}{10}\Phi_{\rm p} \tag{2.30}$$

Sub-horizon evolution ($k \gg aH$)

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_{\rm c}\,\delta = 0$$

$$\Longrightarrow \quad \frac{d^2\delta}{dy^2} + \frac{3y+2}{2y(y+1)}\frac{d\delta}{dy} - \frac{3}{2}\frac{1}{y(y+1)}\delta = 0, \quad \text{(Meszaros equation)} \tag{2.31}$$

$$\delta = c_1 D_1(y) + c_2 D_2(y), \quad \begin{cases} D_1(y) = \frac{2}{3} + y \\ D_2(y) = D_1(y)\ln\left[\frac{\sqrt{y+1}+1}{\sqrt{y+1}-1}\right] - 2\sqrt{y+1} \end{cases} \tag{2.32}$$

Horizon crossing at RD epoch

Using the conformal time defined by $a\,d\eta = dt$, we obtain

$$\delta'' + \frac{a'}{a}\delta' = -3\Phi'' + k^2\Phi - 3\frac{a'}{a}\Phi' \equiv S(\eta), \tag{2.33}$$

$$\Phi'' + \frac{4}{\eta}\Phi' + \frac{k^2}{3}\Phi = 0, \tag{2.34}$$

where the prime denotes the derivative with respect to η.

The solution of Eq. (2.33) is written as

$$\delta = d_1\,\ln a + d_2 + \int_0^\eta d\eta'\,\{\ln a(\eta') - \ln a(\eta)\}\left(\frac{d\ln a(\eta')}{d\eta'}\right)^{-1} S(\eta'). \tag{2.35}$$

From the adiabatic initial condition given in Eq. (2.24), we have $d_1 = 0$ and $d_2 = (3/2)\Phi_{\rm p}$. For more explicit expression, we need to know the behavior of Φ from Eq. (2.34). The solution satisfying $\Phi \to \Phi_{\rm p}$ at $k \ll aH$ becomes

$$\Phi = \Phi_{\rm p}\left(3\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\,\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right) \tag{2.36}$$

$$\xrightarrow{k\eta\gg 1} \Phi_{\rm p}\left(-9\frac{\cos(k\eta/\sqrt{3})}{(k\eta)^2}\right) \tag{2.37}$$

Substituting Eq. (2.37) into Eq. (2.35), the solution relevant at $a_H \ll a \ll a_{\rm eq}$ is approximately given by

$$\delta \simeq A\,\Phi_{\rm p}\ln\left(B\frac{a}{a_{\rm eq}}\right) \tag{2.38}$$

with $A \sim 9$ and $B \sim 0.6$.

Matching the solutions

From Eqs. (2.38) and (2.32), we have

$$\begin{aligned}
&\text{Horizon crossing at RD epoch [Eq. (2.38)]}: \quad \delta \simeq A\,\Phi_{\rm p}\ln\left(B\frac{a}{a_{\rm eq}}\right),\\
&\text{Sub-horizon at MD/RD epoch [Eq. (2.32)]}: \quad \delta = c_1\,D_1\left(\frac{a}{a_{\rm eq}}\right) + c_2\,D_2\left(\frac{a}{a_{\rm eq}}\right)
\end{aligned} \tag{2.39}$$

At $y_{\rm m} = a_{\rm m}/a_{\rm eq}$ satisfying the condition $y_{\rm H} \ll y_{\rm m} \ll 1$, matching the above two solutions give the explicit expression for the coefficients of the growing mode, c_1:

$$\begin{aligned}
c_1 &= \frac{\frac{dD_2}{dy}|_{y_{\rm m}}\ln(B\frac{y_{\rm m}}{y_{\rm H}}) - D_2(y_{\rm m})\frac{1}{y_{\rm m}}}{D_1(y_{\rm m})\frac{dD_2}{dy}|_{y_{\rm m}} - D_2(y_{\rm m})\frac{dD_1}{dy}|_{y_{\rm m}}}\,A\,\Phi_{\rm p}\\
&\xrightarrow{y_{\rm m}\ll 1} -\frac{9}{4}\left[-\frac{2}{3}\ln\left(B\frac{y_{\rm m}}{y_{\rm H}}\right) - \frac{2}{3}\ln\left(\frac{4}{y_{\rm m}}+2\right)\right] = \frac{3}{2}\,A\,\Phi_{\rm p}\,\ln\left(\frac{4B\,e^{-3}}{y_{\rm h}}\right).
\end{aligned} \tag{2.40}$$

Thus, the sub-horizon solution of (CDM) density fluctuation at RD/MD epoch becomes

$$\delta \simeq \frac{3}{2}\,A\,\Phi_{\rm p}\,\ln\left(4\sqrt{2}\,B\,e^{-3}\frac{k}{k_{\rm eq}}\right)\,D_1(a), \qquad (k \gg k_{\rm eq}), \tag{2.41}$$

with $A \sim 9$ and $B \sim 0.6$.

Late-time sub-horizon evolution at MD/DE epoch

At the time after the radiation-matter equality time $a \gg a_{\rm eq}$, the radiation component becomes negligible, the universe is described by Einstein-de Sitter model. At later time, however, the deviation from Einstein-de Sitter model becomes significant, and it affects growth factor D_1. The late-time evolution for D_1 is described by

$$\ddot{D}_1 + 2H\,D_1 - 4\pi\,G\,\rho\,D_1 = 0\,; \qquad \rho = \rho_{\rm m} + \rho_{\rm DE}. \tag{2.42}$$

The growing-mode solution is characterized by

$$D_1(a) \propto a\,g(a). \tag{2.43}$$

The deviation from the Einstein-de Sitter universe is also characterized by the linear growth-rate, defined by

$$f(a) \equiv \frac{d\ln D_1(a)}{d\ln a}. \tag{2.44}$$

For the Universe with cosmological constant, the functions $g(a)$ and $f(a)$ are approximately described by [15]

$$g(a) \simeq \frac{5}{2}\Omega_{\rm m}(a)\left[\Omega_{\rm m}^{4/7}(a) - \Omega_\Lambda(a) + \left\{1 + \frac{\Omega_{\rm m}(a)}{2}\right\}\left\{1 + \frac{\Omega_\Lambda(a)}{70}\right\}\right]^{-1}, \tag{2.45}$$

$$f(a) \simeq \Omega_{\rm m}^{4/7}(a) + \frac{\Omega_\Lambda(a)}{70}\left\{1 + \frac{\Omega_{\rm m}(a)}{2}\right\}. \tag{2.46}$$

In a flat Universe filled with matter and dark energy with constant EOS parameter (w), the exact solution is known, and the functions $g(a)$ and $f(a)$ are expressed in terms of the hyper-geometric function (e.g., [54]):

$$g(a) = {}_2F_1\left(-\frac{1}{3w}, \frac{w-1}{w}, 1 - \frac{5}{6w}; -q(a)\right), \tag{2.47}$$

$$f(a) = 1 - \frac{3(w-1)}{6w-5}\,\frac{{}_2F_1\left[\frac{3w-1}{2w}, \frac{3w-1}{3w}, \frac{12w-5}{6w}, -q(a)\right]}{{}_2F_1\left[-\frac{1}{3w}, \frac{w-1}{2w}, \frac{6w-5}{6w}, -q(a)\right]} \tag{2.48}$$

with $q(a) \equiv \{(1-\Omega_{\rm m,0})/\Omega_{\rm m,0}\}\,a^{-3w}$.

2.4 Transfer function

While the wavelength of the observable fluctuations is basically shorter than the horizon size, these modes have experienced the super-horizon evolution. Since the evolution of fluctuations can change depending on when the mode crosses (or re-enters) the horizon scale, it is convenient to introduce the transfer function defined by

$$T(k;\,t_{\rm m}) \equiv \frac{\Phi(k;\,t_{\rm m})}{\Phi(k\to 0;\,t_{\rm m})}, \tag{2.49}$$

where the time t_m is chosen at the matter-dominated era, close to the Einstein-de Sitter Universe. Here, the $\Phi(k)$ in the our interest is the sub-horizon mode. From Eq. (2.30), we have $\Phi(k \to 0;\, t_\mathrm{m}) = (9/10)\,\Phi_\mathrm{p}$.

Using (2.49), the matter fluctuation at sub-horizon scales may be expressed as (at $t > t_\mathrm{m}$)

$$\begin{aligned}\delta_\mathrm{m}(\boldsymbol{k};\, a) &= \frac{3}{5}\,\frac{k^2}{\Omega_{\mathrm{m},0} H_0^2}\,\Phi_\mathrm{p}(\boldsymbol{k})\,T(k)\;D_1(a)\\ &\equiv \delta_0(\boldsymbol{k})\,D_1(a).\end{aligned} \tag{2.50}$$

Asymptotic behavior of $T(k)$

From Eq. (2.41) and (2.49), we obtain[1]

$$T(k) \simeq \begin{cases} \dfrac{5}{2}\,A\left(\dfrac{k}{k_\mathrm{eq}}\right)^{-2}\ln\left(4\sqrt{2}\,B\,e^{-3}\,\dfrac{k}{k_\mathrm{eq}}\right) \simeq 12\left(\dfrac{k}{k_\mathrm{eq}}\right)^{-2}\ln\left(\dfrac{k}{8k_\mathrm{eq}}\right), & k \gg k_\mathrm{eq} \\ 1, & k \ll k_\mathrm{eq}. \end{cases} \tag{2.51}$$

Thus, the important characteristic scale is

$$k_\mathrm{eq} \equiv a_\mathrm{eq}\,H_\mathrm{eq} = \sqrt{\frac{2}{\Omega_{\mathrm{r},0} H_0^2}}\,\frac{\Omega_{\mathrm{m},0} H_0^2}{c} = 0.0095\left(\frac{\Omega_{\mathrm{m},0} h^2}{0.13}\right)\ \mathrm{Mpc}^{-1}. \tag{2.52}$$

BBKS fitting formula

A simple but accurate formula for transfer function is given by [1]

$$T(k) = \frac{\ln[1+2.34q]}{2.34q}\left\{1 + 3.39q + (16.2)^2 + (5.47)^3 + (6.71q)^4\right\}^{-1/4}\ ;\ q \equiv \frac{k}{\Gamma\,h\,\mathrm{Mpc}^{-1}} \tag{2.53}$$

with the shape parameter $\Gamma = \Omega_{\mathrm{m},0} h$. This is the transfer function for CDM fluctuation, but simply replacing the shape parameter with $\Gamma = \Omega_{\mathrm{m},0} h\ \exp[-\Omega_{\mathrm{b},0} - (2h)^{1/2}\Omega_\mathrm{b}/\Omega_{\mathrm{m},0}]$, it can represent in a good accuracy the transfer function for matter fluctuations [62].

Note

- An improved fitting formula for transfer function including the baryon acoustic oscillations is given by Ref. [23].

[1] Strictly, the asymptotic form given here is not the transfer function for matter fluctuation, but that for the CDM fluctuation.

- For the transfer function including massive neutrinos, a fitting formula relevant at small scales is presented in Ref. [31].
- For more accurate transfer function, use CMB Boltzmann code. Now, the available public codes are `camb`[2] and `class`[3].

Note finally the isocurvature perturbations. The asymptotic form of the transfer function is similarly given like Eq. (2.51), but one notable difference in the isocurvature mode is that at $k \gg k_{\rm eq}$, no logarithmic scale-dependence appears, and the transfer function simply scales as $T(k) \propto (k/k_{\rm eq})^{-2}$. This comes from the fact that there is no logarithmic growth before the onset of Mészaros effect [38].

2.5 Baryon acoustic oscillations

Acoustic oscillations

$$\frac{d^2\Theta_0}{d\eta^2} + \frac{R}{1+R}\mathcal{H}\frac{d\Theta_0}{d\eta} + \frac{k^2}{3(1+R)}\Theta_0 = -\frac{k^2}{3}\Psi - \frac{d^2\Phi}{d\eta^2} - \frac{R}{1+R}\mathcal{H}\frac{d\Phi}{d\eta} \tag{2.54}$$

$$c_s \equiv \sqrt{\frac{1}{3(1+R)}}\ ;\ R = \frac{3\rho_{\rm b}}{4\rho_\gamma}. \tag{2.55}$$

$$\Theta_0 \ \propto\ \exp(i\,k\,r_s)\,; \quad r_s \equiv \int_0^\eta d\eta'\,c_s(\eta') \ \cdots\ \text{sound horizon scale} \tag{2.56}$$

$$\begin{aligned} r_s(\eta) &= \frac{2}{3\,k_{\rm eq}}\sqrt{\frac{6}{R_{\rm eq}}}\,\ln\left(\frac{\sqrt{1+R(\eta)}+\sqrt{R(\eta)+R_{\rm eq}}}{1+\sqrt{R_{\rm eq}}}\right) \\ &\approx 147\left(\frac{\Omega_{\rm m,0}h^2}{0.13}\right)^{-0.25}\left(\frac{\Omega_{\rm b,0}h^2}{0.024}\right)^{-0.008} \quad \text{at } \eta=\eta_{\rm rec} \end{aligned} \tag{2.57}$$

Relation to baryon fluctuations

$$v_{\rm b} \simeq -3\Theta_1 \overset{k\gg aH}{\simeq} \frac{3}{k}\frac{d\Theta_0}{d\eta}, \quad \frac{d\delta_{\rm b}}{d\eta} \simeq k\,v_{\rm b} \simeq 3\frac{d\Theta_0}{d\eta} \quad \longrightarrow \quad \delta_{\rm b} \simeq 3\Theta_0 \ \propto\ \exp(ik\,r_s) \tag{2.58}$$

[2] http://camb.info
[3] http://class-code.net

2.6 Baryon catch-up

Just after the time of decoupling, the baryon fluctuations is negligible, and the potential is basically determined by the CDM fluctuations. Thus, the evolution of baryon fluctuations is approximately described by

$$\ddot{\delta}_{\rm b} + 2\,H\dot{\delta}_{\rm b} \simeq 4\pi\,G\rho_{\rm c}\,\delta. \tag{2.59}$$

Using the fact that $\rho_{\rm c} \propto a^{-3}$, and $\delta \propto a$ at MD era, the above equation is reduced to

$$y^{1/2}\frac{d}{dy}\left(y^{3/2}\frac{d\delta_{\rm b}}{dy}\right) = \frac{3}{2}\,\delta\,; \quad y \equiv \frac{a}{a_{\rm dec}}. \tag{2.60}$$

The solution which fulfills the condition, $\delta_{\rm b} = 0$ at $y = 1$, becomes

$$\delta_{\rm b} = \left(1 - \frac{3}{y} + \frac{2}{y^{3/2}}\right)\,\delta. \tag{2.61}$$

This implies that the baryon fluctuations soon catch up the CDM fluctuations. Note importantly that the acoustic signature (BAO) imprinted on the baryon fluctuations still remains even after the baryon catch-up.

2.7 Effect of non-zero mass of neutrinos

Even when the neutrinos become non-relativistic, they have a large velocity dispersion due to their small non-zero masses:

$$\sigma_\nu^2 = \frac{\displaystyle\int d^3q\,\left(\frac{q}{m_\nu}\right)^2\,f_\nu(q)}{\displaystyle\int d^3q\,f_\nu(q)} = \frac{15\zeta(5)}{\zeta(3)}\left(\frac{4}{11}\right)^{2/3}\frac{T_{\gamma,0}^2(1+z)^2}{m_\nu^2}$$
$$\simeq \left(6.03\times10^{-4}\,c\right)^2\left(\frac{1\,{\rm eV}}{m_\nu}\right)^2(1+z)^2. \tag{2.62}$$

This leads to the characteristic scale below.

Free-streaming scale, $k_{\rm FS}$[4]

$$k_{\rm FS} \equiv \sqrt{\frac{3}{2}\frac{a\,H}{c_s^2}} \simeq \sqrt{\frac{3}{2}\frac{a\,H}{\sigma_\nu}} = \frac{0.677}{(1+z)^2}\frac{m_\nu}{1\,{\rm eV}}\sqrt{\Omega_{\rm m,0}(1+z)^3+\Omega_\Lambda}\;h\,{\rm Mpc}^{-1}. \tag{2.63}$$

At the scales below the free-streaming scale, $k \ll k_{\rm FS}$, the neutrino fluctuations do not grow, and hence the fluctuations of the total matter is suppressed relative to those in the

[4]As remarked in Ref. [60], the sound velocity, $c_s = (\delta p/\delta\rho)^{1/2}$, slightly differs from σ_ν, and in the non-relativistic limit, it gives $c_s \simeq (\sqrt{5}/3)\sigma_\nu$.

massless neutrino case. The suppression of linear matter power spectrum is approximately characterized as

$$\frac{P(k)|_{f_\nu\neq 0}}{P(k)|_{f_\nu=0}} \simeq 1-8\,f_\nu\ ; \quad f_\nu \equiv \frac{\Omega_{\nu,0}}{\Omega_{\mathrm{m},0}} \simeq 0.075\left(\frac{0.1426}{\Omega_{\mathrm{m},0}h^2}\right)\left(\frac{\sum m_\nu}{1\,\mathrm{eV}}\right) \tag{2.64}$$

at $z=0$. A more refined (but partly empirical) formula is given by [see Eq. (141) of Ref. [41]]:

$$\frac{P(k)|_{f_\nu\neq 0}}{P(k)|_{f_\nu=0}} \simeq (1-f_\nu)^3\left(\frac{D_1(a)}{a_{\mathrm{nr}}}\right)^{-(6/5)f_\nu} = (1-f_\nu)^3\left\{1.9\times 10^5\frac{\Omega_{\nu,0}h^2}{N_{\mathrm{eff}}}\frac{D_1(a)}{a}\right\}^{-(6/5)f_\nu}. \tag{2.65}$$

Chapter 3

Observational effects

3.1 Redshift-space distortions

Redshift space

$$1 + z_{\rm obs} \simeq (1+z)(1+v_\parallel) \;\longrightarrow\; \boldsymbol{s} = \boldsymbol{x} + \frac{1+z}{H(z)}\, v_\parallel\, \hat{x}. \tag{3.1}$$

For distance galaxies, the observer's line-of-sight to the galaxy-clustering region is approximately fixed so that one can introduce a particular direction, $\hat{\boldsymbol{z}}$, and write $v_\parallel = (\boldsymbol{v}\cdot\hat{z})$. We then have

$$\boldsymbol{s} = \boldsymbol{x} + \frac{1+z}{H(z)}\,(\boldsymbol{v}\cdot\hat{z})\,\hat{z}. \tag{3.2}$$

Galaxy density field in redshift space:

$$\begin{aligned} \{1+\delta^{(\rm S)}(\boldsymbol{s})\}d^3\boldsymbol{s} &= \{1+\delta_{\rm g}(\boldsymbol{x})\}d^3\boldsymbol{x} \\ &\longrightarrow\;\; \delta^{(\rm S)}(\boldsymbol{s}) = \{1+\delta_{\rm g}(\boldsymbol{x})\}\left|\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{x}}\right|^{-1} - 1. \end{aligned} \tag{3.3}$$

In Fourier space,

$$\begin{aligned} \delta^{(\rm S)}(\boldsymbol{k}) &= \int d^3\boldsymbol{s}\,\delta^{(\rm S)}(\boldsymbol{s})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{s}} \\ &= \int d^3\boldsymbol{x}\left[\delta_{\rm g}(\boldsymbol{x}) - \frac{1+z}{H(z)}\frac{\partial v_z(\boldsymbol{x})}{\partial z}\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{x} - ik\mu_k(1+z)/H(z)\,v_z(\boldsymbol{x})} \end{aligned} \tag{3.4}$$

with $\mu_k \equiv (\boldsymbol{k}\cdot\hat{z})/|\boldsymbol{k}|$.

Linear perturbation (Kaiser formula)

Linearizing RHS of Eq. (3.4) yields

$$\delta^{(\rm S)}(\boldsymbol{k}) \simeq \int d^3\boldsymbol{x}\left[\delta_{\rm g}(\boldsymbol{x}) - \frac{1+z}{H(z)}\frac{\partial v_z}{\partial z}\right] e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \delta_{\rm g}(\boldsymbol{k}) + \frac{1+z}{H(z)}\,k\,\mu_k^2\,v(\boldsymbol{k}). \tag{3.5}$$

Using the linearized continuity equation $\dot{\delta}_{\rm m}-(k/a)v=0$ [see e.g., Eq. (2.8) on sub-horizon scales] and assuming the linear galaxy bias $\delta_{\rm g}=b\,\delta_{\rm m}$, we obtain

$$\delta^{\rm (S)}(\boldsymbol{k})=(b\,+\,f\,\mu_k^2)\delta_{\rm m}(\boldsymbol{k}), \tag{3.6}$$

where f is the linear growth rate defined by Eq. (2.44). Then, the (linear) redshift-space power spectrum becomes

$$P^{\rm (S)}(\boldsymbol{k})=(b+f\,\mu_k^2)^2\,P_{\rm m}(k)=\sum_\ell P_\ell^{\rm (S)}(k)\,\mathcal{P}_\ell(\mu_k)\ ;\quad \begin{cases} P_0^{\rm (S)}(k)=\left(b^2+\dfrac{2}{3}fb+\dfrac{1}{5}f^2\right)\,P_{\rm m}(k) \\ P_2^{\rm (S)}(k)=\left(\dfrac{4}{3}fb+\dfrac{4}{7}f^2\right)\,P_{\rm m}(k) \\ P_4^{\rm (S)}(k)=\dfrac{8}{35}f^2\,P_{\rm m}(k) \end{cases} \tag{3.7}$$

The corresponding redshift-space correlation function is

$$\begin{aligned}\xi^{\rm (S)}(\boldsymbol{s})&=\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,P^{\rm (S)}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{s}}\\ &=\sum_{\ell=0,2,4}\xi_\ell^{\rm (S)}(s)\mathcal{P}_\ell(\mu_{\rm s})\ ;\quad \xi_\ell^{\rm (S)}(s)=(-i)^\ell\int\frac{dkk^2}{2\pi^2}\,j_\ell(k\,s)\,P_\ell^{\rm (S)}(k),\end{aligned} \tag{3.8}$$

where the directional cosine $\mu_{\rm s}$ is defined by $\mu_{\rm s}=\boldsymbol{s}\cdot\hat{z}/|\boldsymbol{s}|$.

The linear theory description given above gives a clear indication that measuring the anisotropies in the correlation function or power spectrum enables us to determine the linear growth rate induced by gravity, f. In fact, what we can estimate from the observed galaxies is a combination of the parameters, $f\,\sigma_8$ and $b\,\sigma_8$, where σ_8 is the normalization amplitude of the power spectrum, $P_{\rm m}$[1]. While this effect, combined with Alcock-Paczynski effect (see below), offers an interesting opportunity to test or constrain gravity on cosmological scales, applicability of the linear theory prediction is known to be restricted to a certain narrow range, and the development of the treatment beyond linear theory is crucial for a precision test of gravity.

3.2 Geometric distortions (Alcock-Paczynski effect)

On top of the redshift-space distortions, there appears another anisotropies induced by the geometric distortions (Alcock-Paczynski effect). This distortion arises if the background

[1]

$$sigma_8^2\equiv\int\frac{dk^2k}{2\pi^2}\,W_{\rm th}^2(k\,R_8)\,P_{\rm m}(k),\quad (R_8=8\,h^{-1}\,{\rm Mpc}) \tag{3.9}$$

with the function $W(x)$ is the Fourier transform of the top-hat filter function given by $W(x)=(3/x^3)\{\sin x-x\cos x\}$.

expansion of the real universe differs from the fiducial cosmology used to convert the observed galaxy positions (i.e., redshift and angular positions) to the comoving radial and transverse distances.

Correlation function

Denoting the transverse and radial separations of a galaxy pair in the *true* cosmology by $s'_\perp$ and $s'_\parallel$, their relation to the observed separation in the fiducial cosmological model, $s_\perp^{\rm obs}$ and $s_\parallel^{\rm obs}$, is given by

$$s'_\perp = \frac{d_A}{d_{A,\rm fid}} s_\perp^{\rm obs}, \qquad s'_\parallel = \frac{H^{-1}}{H_{\rm fid}^{-1}} s_\parallel^{\rm obs}. \tag{3.10}$$

Here, the quantities with subscript $_{\rm fid}$ represent those in the fiducial cosmology. The above relation indicates that the measured correlation function in the fiducial cosmological model, $\xi^{\rm (S)}_{\rm obs}$ is related to the actual one in the true cosmology, $\xi^{\rm (S)}$, through

$$\xi^{\rm (S)}_{\rm obs}(s^{\rm obs},\,\mu_{\rm s}^{\rm obs}) = \xi^{\rm (S)}(s',\,\mu'_{\rm s})\ ; \quad \begin{cases} s' \equiv \sqrt{(s'_\perp)^2+(s'_\parallel)^2} = s^{\rm obs}\,\beta(\mu_{\rm s}^{\rm obs}) \\ \\ \mu'_{\rm s} \equiv s'_\parallel/s' = \dfrac{H_{\rm fid}}{H}\dfrac{\mu_{\rm s}^{\rm obs}}{\beta(\mu_{\rm s}^{\rm obs})} \end{cases} \tag{3.11}$$

with

$$\beta(\mu_{\rm s}^{\rm obs}) = \sqrt{\left(\frac{d_A}{d_{A,\rm fid}}\right)^2 + \left\{\left(\frac{H_{\rm fid}}{H}\right)^2 - \left(\frac{d_A}{d_{A,\rm fid}}\right)^2\right\}(\mu_{\rm s}^{\rm obs})^2}. \tag{3.12}$$

Power spectrum

Similarly, one has the Fourier counterpart of Eq. (3.10):

$$k'_\perp = \frac{d_{A,\rm fid}}{d_A} k_\perp^{\rm obs}, \qquad k'_\parallel = \frac{H_{\rm fid}^{-1}}{H^{-1}} k_\parallel^{\rm obs}. \tag{3.13}$$

Then, from Eq. (3.11), we obtain

$$P^{\rm (S)}_{\rm obs}(k^{\rm obs},\,\mu_k^{\rm obs}) = \frac{H}{H_{\rm fid}}\left(\frac{d_{A,\rm fid}}{d_A}\right)^2 P^{\rm (S)}(k',\,\mu'_k)\ ; \quad \begin{cases} k' = k^{\rm obs}\,\alpha(\mu_k^{\rm obs}) \\ \\ \mu'_k = \dfrac{H}{H_{\rm fid}}\dfrac{\mu_k^{\rm obs}}{\alpha(\mu_k^{\rm obs})} \end{cases} \tag{3.14}$$

with

$$\alpha(\mu_k^{\rm obs}) = \sqrt{\left(\frac{d_{A,\rm fid}}{d_A}\right)^2 + \left\{\left(\frac{H}{H_{\rm fid}}\right)^2 - \left(\frac{d_{A,\rm fid}}{d_A}\right)^2\right\}(\mu_k^{\rm obs})^2}. \tag{3.15}$$

The expressions given in Eqs. (3.11) and (3.14) imply that in the presence of mismatch between the fiducial and true cosmologies, the higher-multipole moments of the power spectrum/correlation function ($\ell > 4$) naturally arises even if the linear formula of redshift-space distortions holds [Eqs. (3.7) and (3.8)].

3.3 Weak lensing effect

The gravitational lensing, the deflection of light path by the massive objects, is now used as a tool to probe structure formation and cosmology. Among several types of lensing effects, weak lensing is relevant phenomena, and is imprinted in the large-scale structure observations. The standard way to detect the weak lensing effect is to observe the galaxy image through the imaging/photometric surveys, and to statistically estimate the distortion of image by lensing effect. Lensing effect also affects the measurement of projected galaxy map through the magnification. Below, we present a basic description of weak lensing, and see how the lensing distortions are induced by the matter fluctuations or large-scale structure.

Lens equations

Cosmic shear $\cdots$ Distortion of distant-galaxy images via weak gravitational lensing by large-scale structure

$\longrightarrow$ its statistical correlation is sensitive to

- evolution of matter fluctuations
- cosmic expansion (through the weight function of geometric distances)

Brightness theorem

$$I_{\rm obs}(\vec{\theta}) = I_{\rm true}(\vec{\theta}_{\rm s}) \tag{3.16}$$

$I_{\rm obs}$: observed surface brightness of background galaxy
$I_{\rm true}$: surface brightness of background galaxy at its source redshift

The relation between $\vec{\theta}$ and $\vec{\theta}_{\rm s}$ is given by the lens equation. Below, we will derive the lens equation assuming flat cosmology (i.e., $K = 0$).

Photon geodesics

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \quad (i = 1 \sim 3) \tag{3.17}$$

x^i : (comoving) position of photon, $(x_1, x_2, x_3) = (\chi\,\vec{\theta},\ \chi)$, with $\chi(z) \equiv \int_0^z \frac{c\,dz'}{H(z')}$.
λ : affine parameter

Newton gauge : $ds^2 = -\{1 + 2\,\Psi(\vec{x})\}\,dt^2 + a^2(t)\,\{1 + 2\,\Phi(\vec{x})\}\,\delta_{ij}\,dx^i dx^j$

We rewrite the geodesic equation (3.17) in terms of the derivative with respect to χ. To do this, we use

$$\frac{d\chi}{d\lambda} = \frac{d\chi}{dt}\frac{dt}{d\lambda} = -\frac{1}{a}p^0 \simeq -\frac{p}{a}(1 - \Psi)\ ; \qquad p^2 \equiv g_{ij}p^i p^j \tag{3.18}$$
$$\left(\because \quad g_{\mu\nu}p^\mu p^\nu = 0 \quad \Longrightarrow \quad -(1 + 2\Psi)\,(p^0)^2 + g_{ij}\,p^i p^j = 0\right)$$

Then, the transverse component of each term in Eq. (3.17) (i.e., $i = 1,\,2$) becomes

$$\begin{aligned}
\frac{d^2x^i}{d\lambda^2} &= \frac{d^2}{d\lambda^2}\,(\chi\,\theta^i) \\
&\simeq \frac{p}{a}\,\frac{d}{d\chi}\left[\frac{p}{a}\,\frac{d}{d\chi}\,(\chi\,\theta^i)\right] \\
&= p^2\,\frac{d}{d\chi}\left[\frac{1}{a^2}\,\frac{d}{d\chi}\,(\chi\,\theta^i)\right] \qquad (\because \quad pa = \text{const.}) \qquad (3.19)\\
\Gamma^i_{\alpha\beta}\,\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} &= \frac{p^2}{a^2}\,(1 - \Psi)^2\,\Gamma^i_{\alpha\beta}\,\frac{dx^\alpha}{d\chi}\frac{dx^\beta}{d\chi} \\
&\simeq \frac{p^2}{a^2}\,\left[(\Psi - \Phi)_{,i} - 2a\,H\,\frac{d}{d\chi}\,(\chi\,\theta^i)\right], \qquad (3.20)
\end{aligned}$$

where we used the fact that $\Gamma^i_{00} = \Psi_{,i}/a^2$, $\Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}(H + \dot{\Phi})$, and $\Gamma^i_{jk} = \Gamma^i_{j0} = \delta_{ij}\Phi_{,k} + \delta_{ik}\Phi_{,j} - \delta_{jk}\Phi_{,i}$. Summing up the above two contributions, the geodesic equation can be rewritten with

$$\frac{d}{d\chi}\left[\frac{1}{a^2}\,\frac{d}{d\chi}\,(\chi\,\theta^i)\right] + \frac{1}{a^2}\left\{(\Psi - \Phi)_{,i} - 2a\,H\,\frac{d}{d\chi}\,(\chi\,\theta^i)\right\} = 0 \tag{3.21}$$

This is further reduced to a simplified equation:

$$\frac{d^2}{d\chi^2}\,(\chi\,\theta^i) = (\Phi - \Psi)_{,i} \tag{3.22}$$

Solving Eq. (3.22) with the boundary condition of $\theta^i = \theta^i_o$ at $\chi = 0$ and $\theta^i = \theta^i_s$ at $\chi = \chi_s(> 0)$:

Lens equation

$$\begin{aligned}\theta_s^i &= \theta_o^i + \frac{1}{\chi_s}\int_0^{\chi_s} d\chi_1 \int_0^{\chi_1} d\chi_2 \,\Big\{\Phi(\vec{x}(\chi_2)) - \Psi(\vec{x}(\chi_2))\Big\}_{,i} \\ &= \theta_o^i + \frac{1}{\chi_s}\int_0^{\chi_s} d\chi_2 \int_{\chi_2}^{\chi_s} d\chi_1 \,\Big\{\Phi(\vec{x}(\chi_2)) - \Psi(\vec{x}(\chi_2))\Big\}_{,i} \\ &= \theta_o^i + \int_0^{\chi_s} d\chi' \,\frac{\chi_s - \chi'}{\chi_s}\Big\{\Phi(\vec{x}(\chi')) - \Psi(\vec{x}(\chi'))\Big\}_{,i} \end{aligned} \tag{3.23}$$

Note that subscript $(_{,i})$ implies $\frac{d}{dx^i} = \frac{1}{\chi}\frac{d}{d\theta^i}$.

Convergence and shear

Eq. (3.23) describes how the image of background galaxy is deformed according to the gravitational potential of foreground large-scale structure. To see this more explicitly, we define the **deformation matrix**:

$$A_{ij} \equiv \frac{\partial \theta_s^i}{\partial \theta_o^j} = \delta_{ij} + \int_0^{\chi_s} d\chi \frac{(\chi_s - \chi)\chi}{\chi_s}\big(\Phi - \Psi\big)_{,ij} \tag{3.24}$$

Here we used $\frac{d}{d\theta_o^i} = \chi\frac{d}{dx^i}$ $\quad(\because \; x^i = \chi\,\theta_o^i)$. The above deformation matrix is rewritten in the following form:

$$A_{ij} \;=\; \delta_{ij} - \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \tag{3.25}$$

with κ and γ_i being defined by

$$\text{convergence :} \qquad \kappa(\vec{\theta}) = -\frac{1}{2}\int_0^{\chi_s} d\chi\,\frac{(\chi_s - \chi)\chi}{\chi_s}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\Phi - \Psi) \tag{3.26}$$

$$\text{shear :} \qquad \begin{cases} \gamma_1(\vec{\theta}) = -\dfrac{1}{2}\displaystyle\int_0^{\chi_s} d\chi\,\frac{(\chi_s - \chi)\chi}{\chi_s}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)(\Phi - \Psi) \\ \gamma_2(\vec{\theta}) = \;-\displaystyle\int_0^{\chi_s} d\chi\,\frac{(\chi_s - \chi)\chi}{\chi_s}\;\frac{\partial^2}{\partial x_1 \partial x_2}(\Phi - \Psi) \end{cases} \tag{3.27}$$

Note that weak lensing implies κ, $|\gamma| \ll 1$.

For further simplification (but still practically useful treatment), we may write $(\Phi - \Psi) = -2\,\Psi$ in the absence of anisotropic stress. We then note that

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right]\Psi = \left[\nabla^2 - \frac{\partial^2}{\partial x_3^2}\right]\Psi = \left[\nabla^2 - \frac{1}{\chi^2}\frac{\partial}{\partial \chi}\left(\chi^2\frac{\partial}{\partial \chi}\right)\right]\Psi. \tag{3.28}$$

The second term is nothing but the time derivative of the potential. Since the potential is nearly constant in time, the second term is rather smaller than the first term, and we may ignore it. Thus, through the Poisson equation, Eq. (3.26) can be rewritten with

$$\begin{aligned}\kappa(\vec{\theta}) &= \int_0^{\chi_s} d\chi\,\frac{(\chi_s-\chi)\chi}{\chi_s}\,\nabla^2\Psi(\vec{x};\chi)\\ &= \frac{3}{2}\,\Omega_{\rm m}H_0^2\int_0^{\chi_s} d\chi\,\frac{(\chi_s-\chi)\chi}{\chi_s}\,\frac{\delta(\vec{x};\chi)}{a(\chi)}.\end{aligned}\tag{3.29}$$

Based on Eq. (3.29), a couple of generalizations to be noted is:

- Non-flat space:
  Replacing the comoving radial distance χ in the kernel of integration with $f_K(\chi)$:

$$\longrightarrow\quad \kappa(\vec{\theta}) = \frac{3}{2}\,\Omega_{\rm m}H_0^2\int_0^{\chi_s} d\chi\,\frac{f_K(\chi_s-\chi)\,f_K(\chi)}{f_K(\chi_s)}\,\frac{\delta(\vec{x};\chi)}{a(\chi)}.\tag{3.30}$$

- Continuous source distribution:
  Eq. (3.29) is only applied to the case with single-source plane at $\chi=\chi_s$, but with a broad source distribution of $w_g(\chi)$, the expression is generalized to

$$\begin{aligned}\longrightarrow\quad \kappa(\vec{\theta}) &= \frac{3}{2}\,\Omega_{\rm m}H_0^2\int_0^{\infty} d\chi_s\,w_g(\chi_s)\int_0^{\chi_s} d\chi\,\frac{f_K(\chi_s-\chi)\,f_K(\chi)}{f_K(\chi_s)}\,\frac{\delta(\vec{x};\chi)}{a(\chi)}\\ &= \int_0^{\infty} d\chi\,\frac{g(\chi)}{a(\chi)}\,\delta(\vec{x};\chi)\end{aligned}\tag{3.31}$$

  with the function $g(\chi)$ given by

$$g(\chi) = \frac{3}{2}\,\Omega_{\rm m}H_0^2\int_{\chi}^{\infty} d\chi_s\,\frac{f_K(\chi_s-\chi)\,f_K(\chi)}{f_K(\chi_s)}\,w_g(\chi_s)\tag{3.32}$$

κ and γ in harmonic space

The relation between convergence and shear fields [Eq. (3.26) and (3.27)] may become transparent when we go to harmonic space. In flat-sky limit, the harmonic expansion is simply reduced to the Fourier expansion:

$$\widetilde{\kappa}(\vec{\ell}) = \int d^2\vec{\theta}\;e^{i\,\vec{\ell}\cdot\vec{\theta}}\,\kappa(\vec{\theta})\tag{3.33}$$

Then, we have

$$\widetilde{\gamma}_1(\vec{\ell}) = \frac{\ell_1^2-\ell_2^2}{\ell^2}\,\widetilde{\kappa}(\vec{\ell}),\tag{3.34}$$

$$\widetilde{\gamma}_2(\vec{\ell}) = 2\frac{\ell_1\ell_2}{\ell^2}\,\widetilde{\kappa}(\vec{\ell}).\tag{3.35}$$

Note that $\ell^2=\ell_1^2+\ell_2^2$. Defining $(\cos\phi_\ell,\ \sin\phi_\ell)=(\ell_1/\ell,\ \ell_2/\ell)$, the above expressions are written with

$$\widetilde{\gamma}_1(\vec{\ell}) = \cos(2\phi_\ell)\,\widetilde{\kappa}(\vec{\ell}),\tag{3.36}$$

$$\widetilde{\gamma}_2(\vec{\ell}) = \sin(2\phi_\ell)\,\widetilde{\kappa}(\vec{\ell}).\tag{3.37}$$

This relation implies that the shear field $\gamma = \gamma_1 + i\gamma_2$ has a spin-2 nature. We thus realize that the following decomposition is very useful to uniquely pick up the physically non-vanishing lensing effect:

E-/B-mode decomposition

$$\begin{pmatrix} \gamma_{\rm E}(\vec{\ell}) \\ \gamma_{\rm B}(\vec{\ell}) \end{pmatrix} \equiv \begin{pmatrix} \cos(2\phi_\ell) & \sin(2\phi_\ell) \\ -\sin(2\phi_\ell) & \cos(2\phi_\ell) \end{pmatrix} \begin{pmatrix} \widetilde{\gamma}_1(\vec{\ell}) \\ \widetilde{\gamma}_2(\vec{\ell}) \end{pmatrix} \tag{3.38}$$

$$\Longleftrightarrow \quad \gamma_{\rm E}(\vec{\ell}) \; + \; i\,\gamma_{\rm B}(\vec{\ell}) \; = \; e^{-i\,2\phi_\ell} \left[\widetilde{\gamma}_1(\vec{\ell}) \; + \; i\,\widetilde{\gamma}_2(\vec{\ell}) \right] \tag{3.39}$$

With this decomposition, we have

$$\gamma_{\rm E}(\vec{\ell}) = \widetilde{\kappa}(\vec{\ell}), \qquad \gamma_{\rm B}(\vec{\ell}) = 0. \tag{3.40}$$

Chapter 4

Analytic approaches to nonlinear structure formation

4.1 Spherical collapse model

A simple nonlinear model for gravitational collapse which tells you characteristic properties of gravitationally bound objects (i.e., dark matter halos).

Consider a homogeneous (uniform) density of sphere with radius R and mass M. The motion of the shell at R is described by

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2}. \tag{4.1}$$

The solution which becomes collapsed at finite time is parametrically expressed as

$$R = \frac{GM}{2|E|}(1-\cos\theta), \quad t = \frac{GM}{(2|E|)^{3/2}}(\theta - \sin\theta). \tag{4.2}$$

Here, E is the total energy of the shell ($E = \dot{R}^2/2 - GM/R$). Note that at $t_{\rm ta} = t(\theta=\pi)$, the radius of the sphere becomes maximum, and one has $R = R_{\rm max} = GR/|E|$. On the other hand, the radius becomes zero at $t_{\rm coll} = t(\theta = 2\pi) = 2t_{\rm ta}$.

Then the density of the sphere is estimated to give

$$\rho \equiv \frac{M}{(4\pi/3)R^3} = \frac{6}{\pi}\frac{|E|^3}{G^3M^2}(1-\cos\theta)^{-3}. \tag{4.3}$$

Taking the ratio to the background density of the universe $\rho_{\rm m} = 3H^2/(8\pi G)$, the density contrast becomes

$$\delta \equiv \frac{\rho}{\rho_{\rm m}} - 1 = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} - 1, \tag{4.4}$$

Here, for simplicity, we assumed the Einstein-de Sitter universe [i.e., $\rho_{\rm m}^{\rm EdS} = 1/(6\pi G t^2)$]. It is to be noted that density contrast in Eq. (4.4) satisfies the following evolution equation:

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} - 4\pi G\,\rho_{\rm m}\,\delta = -4\pi G\,\rho_{\rm m}\,\delta^2 + \frac{4}{3(1+\delta)}\left(\frac{d\delta}{dt}\right)^2. \tag{4.5}$$

Critical density contrast

$$\delta_{\rm crit} \equiv \delta_{\rm lin}(t_{\rm coll}) = \frac{3}{20}(12\pi)^{2/3} \simeq 1.68647. \tag{4.6}$$

Virial overdensity

$$\Delta_{\rm vir} \equiv \frac{\rho_{\rm vir}}{\rho_{\rm m}^{\rm EdS}(t_{\rm coll})} = \frac{8\rho(t_{\rm ta})}{\rho_{\rm m}^{\rm EdS}(t_{\rm ta})/4} = 18\pi^2 \simeq 177.6 \tag{4.7}$$

Extension to ΛCDM cosmology [52, 11, 50]

$$\delta_{\rm crit} = 1.686\{\Omega_{\rm m}(t_{\rm coll})\}^{0.055} \tag{4.8}$$

$$\Delta_{\rm vir} = \frac{18\pi^2 + 82y - 39y^2}{\Omega_{\rm m}(t_{\rm coll})} \ ; \quad y \equiv \Omega_{\rm m}(t_{\rm coll}) - 1. \tag{4.9}$$

4.2 Zel'dovich approximation

This section is based on Ref. [46].

Zel'dovich approximation is the first-order Lagrangian perturbation theory, and describes the quasi-linear evolution of matter fluctuations [70]. In contrast to the spherical collapse model, Zel'dovich approximation tells us (qualitatively) how the asphericity of the structure develops according to its initial configuration. In particular, it is used to generate the initial conditions for cosmological N-body simulations.

As mentioned above, the Zel'dovich approximation is the Lagrangian-based treatment by following the trajectories of particles. Motion of each mass particle is described by

$$L = \frac{1}{2}ma^2\dot{\boldsymbol{x}}^2 - m\Psi(\boldsymbol{x}), \tag{4.10}$$

which gives

$$\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}} + \frac{1}{a^2}\nabla_x\Psi(\boldsymbol{x}) = 0. \tag{4.11}$$

The quantity of interest here is the displacement field $\boldsymbol{\psi}(\boldsymbol{q})$ which maps the initial particle positions $\boldsymbol{q}$ into the final Eulerian particle positions $\boldsymbol{x}$,

$$\boldsymbol{x}(\boldsymbol{q}, t) = \boldsymbol{q} + \boldsymbol{\psi}(\boldsymbol{q}, t). \tag{4.12}$$

With this definition, taking the divergence of Eq. (4.11) gives

$$\nabla_x\Big(\ddot{\boldsymbol{\psi}} + 2H\dot{\boldsymbol{\psi}}\Big) = -4\pi\, G\rho_{\rm m}\,\delta_{\rm m}(\boldsymbol{x}) \tag{4.13}$$

In the above, the sources of nonlinearity are

$$1+\delta_{\rm m}(\boldsymbol{x}) = \left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right|^{-1} \equiv \frac{1}{J}, \tag{4.14}$$

$$\frac{\partial}{\partial \boldsymbol{x}_i} = \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right)^{-1}_{ij}\frac{\partial}{\partial \boldsymbol{q}_j} \equiv (J^{-1})_{ij}\frac{\partial}{\partial \boldsymbol{q}_j}. \tag{4.15}$$

Regarding $\boldsymbol{\psi}$ as a perturbed quantity, the leading-order evaluation leads to

$$J = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}J_{ip}J_{jq}J_{kr} \simeq 1+\nabla_q\cdot\boldsymbol{\psi}, \tag{4.16}$$

$$(J^{-1})_{ij} = \frac{1}{2J}\epsilon_{jkp}\epsilon_{iqr}J_{kq}J_{pr} \simeq \delta_{ij}+\mathcal{O}(\boldsymbol{\psi}). \tag{4.17}$$

Eq. (4.13) is then rewritten at leading order with

$$(J^{-1})_{ij}\frac{\partial}{\partial \boldsymbol{q}_j}\Big(\ddot{\boldsymbol{\psi}} + 2H\dot{\boldsymbol{\psi}}\Big) = -4\pi\, G\rho_{\rm m}\Big(\frac{1}{J}-1\Big)$$
$$\Longrightarrow \quad (\nabla_q\cdot\boldsymbol{\psi})^{\cdot\cdot} + 2H(\nabla_q\cdot\boldsymbol{\psi})^{\cdot} - 4\pi\, G\rho_{\rm m}(\nabla_q\cdot\boldsymbol{\psi}) \simeq 0. \tag{4.18}$$

Eq. (4.18) is nothing but the evolution equation for linear density field. Since $\delta_{\rm m} \simeq -\nabla_q\cdot\boldsymbol{\psi}$ at $t\to 0$, we may write the displacement field as

$$\boldsymbol{\psi}(\boldsymbol{q};a) = -\,D_1(a)\,\nabla_q\varphi(\boldsymbol{q}), \qquad \nabla_q^2\varphi(\boldsymbol{q}) = \delta_0(\boldsymbol{q}). \tag{4.19}$$

Here, D_1 is the linear growth factor, and δ_0 is the initial density field.

A crucial point may be that the density field is not assumed to be small. Thus, it is often said that the solution may be applied to the quasi-linear regime. Plugging Eq. (4.19) into Eq. (4.14), we obtain

$$1+\delta_{\rm m}(\boldsymbol{x}) \simeq \frac{1}{(1-D_1\,\lambda_1)(1-D_1\,\lambda_2)(1-D_1\,\lambda_3)}. \tag{4.20}$$

where λ_i is the eigenvalue of the vector $\varphi_{,i}$. This illustrates how the non-sphericity of the structure develops according to the initial condition. In particular, the above equation implies that in the Gaussian initial condition, most of the nonlinear structure is aspherical.

Beyond Zel'dovich approximation

Zel'dovich approximation is regarded as the first-order approximation to the displacement field $\boldsymbol{\psi}$, and there is a systematic way of perturbative expansion to improve the displacement field. This is the Lagrangian perturbation theory (e.g., [12, 51, 13, 14, 30, 16]). Here, we give the basic equations for $\boldsymbol{\psi}$, with which we can systematically construct the perturbative solution. To do this, notice that the displacement field is the vector quantity whose dynamical degree of freedom is divided to two parts: longitudinal ($\psi_{k,k}$) and transverse ($\epsilon_{ijk}\psi_{j,k}$) parts. While Eq. (4.13) directly leads to the evolution equation for longitudinal mode, the equation for transverse mode is obtained by taking the rotation to Eq. (4.11) with respect to Eulerian coordinate, i.e., $\nabla \times (\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}}) = 0$. A set of basic equations then becomes [46]

$$\left(\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t} - 4\pi G\rho_{\rm m}\right)\psi_{k,k} = -\,\epsilon_{ijk}\epsilon_{ipq}\,\psi_{j,p}\left(\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t} - 2\pi G\rho_{\rm m}\right)\psi_{k,q} - \frac{1}{2}\,\epsilon_{ijk}\epsilon_{pqr}\psi_{i,p}\psi_{j,q}\left(\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t} - \frac{4\pi}{3}\rho_{\rm m}\right)\psi_{k,r}, \tag{4.21}$$

$$\left(\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t}\right)\epsilon_{ijk}\,\psi_{j,k} = -\,\epsilon_{ijk}\,\psi_{p,j}\left(\frac{\partial^2}{\partial t^2} + 2H\frac{\partial}{\partial t}\right)\psi_{p,k}, \tag{4.22}$$

where $\psi_{j,k} = \partial\psi_j/\partial q_k$. The right-hand-side of the above equations represent the non-linear source terms, which have to be evaluated by order-by-order calculation. Once we get the perturbative solutions for longitudinal and transverse modes (i.e., $\psi_{k,k}$ and $\epsilon_{ijk}\psi_{j,k}$), a final step is to explicitly construct the displacement field itself. This is not trivial at all, but can be systematically done in Fourier space (e.g., [46]).

4.3 (Eulerian) Perturbation theory

Collisionless Boltzmann equation (Vlasov-Poisson system)

$$\left[\frac{\partial}{\partial t} + \frac{\boldsymbol{p}}{ma^2}\frac{\partial}{\partial \boldsymbol{x}} - m\frac{\partial\Psi}{\partial \boldsymbol{x}}\frac{\partial}{\partial \boldsymbol{p}}\right] f(\boldsymbol{x},\boldsymbol{p}) = 0, \tag{4.23}$$

supplemented with the Poisson equation:

$$\nabla^2\Psi(\boldsymbol{x}) = 4\pi G\,a^2\left[\frac{m}{a^3}\int d^3\boldsymbol{p}\,f(\boldsymbol{x},\boldsymbol{p}) - \rho_{\rm m}\right]. \tag{4.24}$$

Here, m is the mass of CDM (+baryon) particle.

Single-stream approximation

$$\boxed{\text{Ansatz}}\quad f(\boldsymbol{x},\boldsymbol{p}) = \overline{n}\,a^3\,\{1+\delta_{\rm m}(\boldsymbol{x})\}\;\delta_{\rm D}\big[\boldsymbol{p} - m\,a\,\boldsymbol{v}(\boldsymbol{x})\big]. \tag{4.25}$$

With this ansatz, taking the zeroth and first velocity moments of Eq. (4.23) yields

$$\frac{\partial\delta_{\rm m}}{\partial t} + \frac{1}{a}\nabla\left[(1+\delta_{\rm m})\boldsymbol{v}\right] = 0, \tag{4.26}$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \frac{1}{a}(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\frac{1}{a}\frac{\partial\Psi}{\partial\boldsymbol{x}}, \tag{4.27}$$

$$\frac{1}{a^2}\nabla^2\Psi = 4\pi G\,\rho_{\rm m}\,\delta_{\rm m}. \tag{4.28}$$

Note that using the Lagrangian time derivative $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla$, Eqs. (4.26)-(4.28) can give the following evolution equation:

$$\frac{d^2\delta_{\rm m}}{dt^2} + 2H\frac{d\delta_{\rm m}}{dt} - \frac{4}{3(1+\delta_{\rm m})}\left(\frac{d\delta_{\rm m}}{dt}\right)^2 = H^2(1+\delta)\Big(\frac{3}{2}\Omega_{\rm m}(a)\,\delta_{\rm m} + \sigma^{ij}\sigma_{ij} - \omega^{ij}\omega_{ij}\Big),$$

which is compared to Eq. (4.5). Here, the quantities σ_{ij} and ω_{ij} being the shear and vorticity tensor defined by

$$\sigma_{ij} = \frac{1}{2\,a\,H}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\,(\nabla\cdot\boldsymbol{v})\,\delta_{ij}\Big),$$
$$\omega_{ij} = \frac{1}{2\,a\,H}\Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big).$$

For initial condition only with scalar perturbations, the vorticity tensor is not generated, and as long as we consider the system described by Eqs. (4.26)-(4.28), the vorticity is not generated even through the nonlinear evolution. This means that the velocity follows the potential flow, and dynamics of the above system is characterized by δ and velocity potential.

In what follows, we omit the subscript of $\delta_{\rm m}$, and simply denoting the mass density field by δ, we shall solve the above fluid system perturbatively.

Evolution equations in Fourier space

Introducing the velocity-divergence field, $\theta \equiv \nabla\cdot\boldsymbol{v}/(aH)$, Eqs. (4.26)-(4.28) are rewritten in terms of the Fourier-space quantities[1]:

$$a\frac{d\delta(\boldsymbol{k})}{da} + \theta(\boldsymbol{k}) = -\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12})\,\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\,\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2), \tag{4.29}$$

$$a\frac{d\theta(\boldsymbol{k})}{da} + \left(2+\frac{\dot{H}}{H^2}\right)\theta(\boldsymbol{k}) + \frac{3}{2}\Omega_{\rm m}(a)\,\delta(\boldsymbol{k}) = -\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12})\,\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\,\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2) \tag{4.30}$$

with the functions α and β given by

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{|\boldsymbol{k}_1|^2}, \qquad \beta(\boldsymbol{k}_1,\boldsymbol{k}_2) = 1 + \frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}{|\boldsymbol{k}_1|^2|\boldsymbol{k}_2|^2}. \tag{4.31}$$

Standard PT expansion

Eqs .(4.29) and (4.30) may be expanded as[2]

[1] The vorticity component of the velocity field is ignored.

[2] Strictly speaking, the expansion in Eq. (4.32) is correct only in the Einstein-de Sitter cosmology, in which we have $D_1 = a$. Nevertheless, this expansion is shown to give a very good approximation.

$$\delta(\boldsymbol{k}) = \sum_n D_1^n\,\delta_n(\boldsymbol{k}), \qquad \theta(\boldsymbol{k}) = -f\,\sum_n D_1^n\,\theta_n(\boldsymbol{k}). \tag{4.32}$$

The perturbative solutions δ_n and θ_n are formally expressed as

$$\delta_n(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}_1\cdots d^3\boldsymbol{k}_n}{(2\pi)^{3(n-1)}}\,\delta_D(\boldsymbol{k}-\boldsymbol{k}_{12\cdots n})\,F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\,\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_n), \tag{4.33}$$

$$\theta_n(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}_1\cdots d^3\boldsymbol{k}_n}{(2\pi)^{3(n-1)}}\,\delta_D(\boldsymbol{k}-\boldsymbol{k}_{12\cdots n})\,G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\,\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_n) \tag{4.34}$$

with $\boldsymbol{k}_{12\cdots n} = \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_n$. The function δ_0 is the initial density field originated from primordial curvature perturbation, $\Phi_{\rm p}$ [see Eq. (2.50)]. The functions F_n and G_n are called perturbation theory (PT) kernels, whose functional forms are determined recursively.

Constructing PT kernels

First defining the kernel

$$\mathcal{F}_a^{(n)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \begin{pmatrix} F_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \\ G_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \end{pmatrix}, \tag{4.35}$$

then the recursion relation of the PT kernel is obtained from Eqs. (4.29) and (4.30)[3]:

$$\mathcal{F}_a^{(n)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \sum_{m=1}^{n-1}\sigma_{ab}^{(n)}\,\gamma_{bcd}(\boldsymbol{q}_1,\boldsymbol{q}_2)\,\mathcal{F}_c^{(m)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_m)\mathcal{F}_d^{(n-m)}(\boldsymbol{k}_{m+1},\cdots,\boldsymbol{k}_n), \tag{4.36}$$

with $\mathcal{F}^{(1)} = (1,1)$. Here, $\boldsymbol{q}_1 = \boldsymbol{k}_1 + \cdots + \boldsymbol{k}_m$ and $\boldsymbol{q}_1 = \boldsymbol{k}_{m+1} + \cdots + \boldsymbol{k}_n$. The $\sigma_{ab}^{(n)}$ and γ_{abc} are respectively given by (e.g., Ref. [5, 21, 53])

$$\sigma_{ab}^{(n)} = \frac{1}{(2n+3)(n-1)}\begin{pmatrix} 2n+1 & 2 \\ 3 & 2n \end{pmatrix}, \tag{4.37}$$

$$\gamma_{abc}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \begin{cases} \alpha(\boldsymbol{k}_2,\boldsymbol{k}_1)/2 & (a,b,c)=(1,1,2) \\ \alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)/2 & (a,b,c)=(1,2,1) \\ \beta(\boldsymbol{k}_1,\boldsymbol{k}_2) & (a,b,c)=(2,2,2) \\ 0 & \text{otherwise} \end{cases} \tag{4.38}$$

Note that the PT kernels obtained from recursion relation are not yet symmetric for all possible permutations of the variables, e.g., $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2$. For later statistical calculations, they must be symmetrized:

$${}^s\mathcal{F}_a^{(3)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \equiv \frac{1}{n!}\sum_{\text{permutations}}\mathcal{F}_a^{(n)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n). \tag{4.39}$$

[3] Again, the recursion relation obtained here is exact only in the Einstein-de Sitter Universe.

Examples

Below, we give explicit expression for PT kernels up to third order:

$$
{}^sF_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{5}{7} + \frac{1}{2}\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{2}{7}\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}, \tag{4.40}
$$

$$
{}^sG_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{3}{7} + \frac{1}{2}\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{4}{7}\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}. \tag{4.41}
$$

$$
\begin{aligned}
{}^sF_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) &= \frac{1}{6}\Big[\frac{7}{9}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_3)}{k_3^2}\,{}^sF_2(\boldsymbol{k}_1,\boldsymbol{k}_2) + \Big\{\frac{7}{9}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_{12})}{k_{12}^2} + \frac{2}{9}\frac{k_{123}^2(\boldsymbol{k}_3\cdot\boldsymbol{k}_{12})}{k_3^2k_{12}^2}\Big\}\,{}^sG_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\Big] \\
&\quad + (\text{cyclic perm.})\,, \qquad (4.42)\\
{}^sG_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) &= \frac{1}{6}\Big[\frac{1}{3}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_3)}{k_3^2}\,{}^sF_2(\boldsymbol{k}_1,\boldsymbol{k}_2) + \Big\{\frac{1}{3}\frac{(\boldsymbol{k}_{123}\cdot\boldsymbol{k}_{12})}{k_{12}^2} + \frac{2}{3}\frac{k_{123}^2(\boldsymbol{k}_3\cdot\boldsymbol{k}_{12})}{k_3^2k_{12}^2}\Big\}\,{}^sG_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\Big] \\
&\quad + (\text{cyclic perm.})\,. \qquad (4.43)
\end{aligned}
$$

Here, some of the important properties of the PT kernels are summarized [5]:

- If $\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\cdots$ goes to zero, but the individual $\boldsymbol{k}_i$ do not, we have

$$
{}^sF_n(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) \propto k^2. \tag{4.44}
$$

- If some of the arguments of ${}^sF_n$ get larger but the total sum $\boldsymbol{k}=\boldsymbol{q}_1+\cdots+\boldsymbol{q}_{n-2}$ stays fixed, the kernels vanishes in inverse square law. That is, for $p \gg q_i$, we have

$$
{}^sF_n(\boldsymbol{q}_1,\cdots,\boldsymbol{q}_{n-2},\boldsymbol{p},-\boldsymbol{p}) \propto \frac{k^2}{p^2}. \tag{4.45}
$$

Note that the above properties also hold for the kernel ${}^sG_n$.

Below, we assume that the constructed kernels are symmetrized, and simply denote ${}^sF_n$ and ${}^sG_n$ by F_n and G_n.

Relation to Lagrangian PT

Here, we comment on the relation between standard PT expansion and Lagrangian PT which we have briefly discussed. In Lagrangian PT, instead of expanding δ and θ like Eq. (4.32), we introduce the Lagrangian frame in which the mass distribution looks homogeneous ($\boldsymbol{q}$, the rest frame of mass element), and follow the motion of flow with the displacement field (vector), $\boldsymbol{\psi}$ [Eq. (4.12)]:

$$
\boldsymbol{x}(\boldsymbol{q},t) = \boldsymbol{q} + \boldsymbol{\psi}(\boldsymbol{q},t).
$$

Note $\boldsymbol{\psi} \to 0$ at $t \to 0$. Moving to Fourier space, we expand the displacement field as

$$\boldsymbol{\psi}(\boldsymbol{k};\,a) \equiv \int d^3\boldsymbol{q}\,\boldsymbol{\psi}(\boldsymbol{q})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{q}} = \sum_n D_1^n(a)\,\boldsymbol{\psi}_n(\boldsymbol{k})\ ;$$
$$\boldsymbol{\psi}_n(\boldsymbol{k}) = i\int \frac{d^3\boldsymbol{p}_1\cdots d^3\boldsymbol{p}_n}{(2\pi)^{3(n-1)}}\,\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{p}_{12\cdots n})\,\boldsymbol{L}_n(\boldsymbol{p}_1,\cdots,\boldsymbol{p}_n)\,\delta_0(\boldsymbol{p}_1)\cdots\delta_0(\boldsymbol{p}_n). \quad (4.46)$$

The function $\boldsymbol{L}_n$ is called Lagrangian PT kernel. Note that the first-order Lagrangian PT is nothing but the Zel'dovich approximation.

The Lagrangian PT kernels are related to standard PT kernels as follows. In terms of the displacement field, the density field in Fourier space is described as (using the relation $d^3\boldsymbol{q} = \{1+\delta(\boldsymbol{x})\}d^3\boldsymbol{x}$):

$$\begin{aligned}
\delta(\boldsymbol{k}) &= \int d^3\boldsymbol{x}\,\delta(\boldsymbol{x})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= \int d^3\boldsymbol{q}\,e^{-i\boldsymbol{k}\cdot\{\boldsymbol{q}+\boldsymbol{\psi}(\boldsymbol{q})\}} - (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}), \qquad \left(\because\ \ \delta(\boldsymbol{x}) = \left|\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}}\right| - 1\right) \\
&= \sum_n\left(\int \frac{d^3\boldsymbol{k}_1\cdots d^3\boldsymbol{k}_n}{(2\pi)^{3(n-1)}}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{p}_{12\cdots n})\,\frac{(-i)^n}{n!}\,\{\boldsymbol{k}\cdot\boldsymbol{\psi}(\boldsymbol{k}_1)\}\cdots\{\boldsymbol{k}\cdot\boldsymbol{\psi}(\boldsymbol{k}_n)\}\right) - (2\pi)^3\delta_{\rm D}(\boldsymbol{k})
\end{aligned} \quad (4.47)$$

Substituting Eq. (4.46) into the above, the order-by-order comparison of the Fourier kernels with those of the standard PT kernel given in Eq. (4.33) leads to

$$\begin{aligned}
&F_1(\boldsymbol{k}) = 1 = \boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}), \\
&F_2(\boldsymbol{k}_1,\boldsymbol{k}_2) = \boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_1,\boldsymbol{k}_2) + \frac{1}{2}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)\}, \\
&F_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \boldsymbol{k}\cdot\boldsymbol{L}_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) + \frac{1}{3}\big[\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_2(\boldsymbol{k}_2,\boldsymbol{k}_3)\} + (\text{cyclic perm.})\big] \\
&\qquad\qquad + \frac{1}{6}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_1)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_2)\}\{\boldsymbol{k}\cdot\boldsymbol{L}_1(\boldsymbol{k}_3)\}.
\end{aligned} \quad (4.48)$$

Using these relations, one can reconstruct the *longitudinal* mode of the Lagrangian PT kernels recursively without solving evolution equation of $\boldsymbol{\psi}$. Notice that the reconstructed kernels in this way actually miss *transverse* modes, which appears non-vanishing at $n \geq 3$. For a systematic calculation of both longitudinal and transverse modes, we need to go to Lagrangian space [see Eqs. (4.21) and (4.21), also Ref. [46]].

Gaussian initial condition

For explicit calculations of the statistical quantities based on PT, one needs to specify the statistical properties of the density field δ_0 as a seed of PT expansion [Eqs. (4.33), (4.34) and (4.46)]. Standard assumption/hypothesis may be the *Gaussian initial condition.* In this case, all the statistical information is encoded in the initial power spectrum

$P_0(k)$, and any statistical quantity is constructed with P_0. We have

$$\langle\delta_0(\boldsymbol{k})\rangle = 0, \tag{4.49}$$

$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\rangle = (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}_{12})P_0(k) \tag{4.50}$$

$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_3)\rangle = 0, \tag{4.51}$$

$$\langle\delta_0(\boldsymbol{k}_1)\delta_0(\boldsymbol{k}_2)\delta_0(\boldsymbol{k}_3)\delta_0(\boldsymbol{k}_4)\rangle = (2\pi)^6\Big[\delta_{\rm D}(\boldsymbol{k}_{12})\delta_{\rm D}(\boldsymbol{k}_{34})P_0(k_1)P_0(k_2) + (\text{cyclic perm.})\Big], \tag{4.52}$$

$$\vdots$$

In general, for positive integer n,

$$\langle\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_{2n+1})\rangle = 0, \tag{4.53}$$

$$\langle\delta_0(\boldsymbol{k}_1)\cdots\delta_0(\boldsymbol{k}_{2n})\rangle = \sum_{\text{all pair associations } p}\ \prod_{\text{pairs } (i,j)}\langle\delta_0(\boldsymbol{k}_i)\delta_0(\boldsymbol{k}_j)\rangle. \tag{4.54}$$

These properties are known as Wick's theorem or Isserlis' theorem.

Statistical calculations

- **Power spectrum** : $\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle = (2\pi)^3\delta_{\rm D}(\boldsymbol{k}_{12})\,P(k_1)$

  An explicit calculation of the ensemble average at next-to-leading order (called *one-loop*) leads to

$$\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle \simeq \langle\delta_1(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\rangle + \langle\delta_2(\boldsymbol{k}_1)\delta_2(\boldsymbol{k}_2)\rangle + \langle\delta_1(\boldsymbol{k}_1)\delta_3(\boldsymbol{k}_2)\rangle + \langle\delta_3(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\rangle + \cdots . \tag{4.55}$$

  We thus obtain

$$P(k,a) \simeq \{D_1(a)\}^2\,P_0(k) + \{D_1(a)\}^4\big\{P_{22}(k) + P_{13}(k)\big\}. \tag{4.56}$$

  The first term at RHS is nothing but the linear power spectrum. The parenthesis represents the contributions from the higher-order PT, given by

$$P_{22}(k) = 2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\{F_2(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})\}^2\,P_0(|\boldsymbol{k}-\boldsymbol{p}|)P_0(p), \tag{4.57}$$

$$P_{13}(k) = 6\,P_0(k)\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\{F_3(\boldsymbol{k},\boldsymbol{p},-\boldsymbol{p})\}^2\,P_0(p). \tag{4.58}$$

  Because of the different dependence on the linear growth factor, these nonlinear contributions give rise to the *scale-dependent growth* of power spectrum.

- **Bispectrum** : $\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\delta(\boldsymbol{k}_3)\rangle = (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}_{123})\,B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$

  At leading-order, we have

$$\begin{aligned}\langle\delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\delta(\boldsymbol{k}_3)\rangle \simeq\ & \langle\delta_1(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\delta_2(\boldsymbol{k}_3)\rangle + \langle\delta_2(\boldsymbol{k}_1)\delta_1(\boldsymbol{k}_2)\delta_1(\boldsymbol{k}_3)\rangle \\ &+ \langle\delta_1(\boldsymbol{k}_1)\delta_2(\boldsymbol{k}_2)\delta_1(\boldsymbol{k}_3)\rangle + \cdots .\end{aligned} \tag{4.59}$$

Thus, the non-vanishing bispectrum arises:

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\simeq\{D_1(a)\}^4\Big\{2F_2(\boldsymbol{k}_1,\boldsymbol{k}_2)P_0(k_1)P_0(k_2)+(\text{cyclic perm.})\Big\}. \tag{4.60}$$

This implies that the nonlinear gravitational evolution generally produces non-Gaussianity, and other higher-order statistics also become non-vanishing. In other words, the statistical information initially encoded in the power spectrum is partly transferred to the higher-order statistics.

Resummed perturbation theory

In order to improve the performance of PT-based prediction, we need to include the higher-order PT corrections. A crucial remark is that the applicable range of the PT prediction largely depends on the PT scheme itself. Indeed, standard PT is known to have bad convergence properties, and produces ill-behaved higher-order corrections. The resummation or renormalization is one of the techniques to improve the convergence of PT expansion.

Among various methods proposed so far (e.g., [68, 56, 44, 21, 63]), we here present a specific prescription for resummed PT treatment, called *multi-point propagator expansion* or Γ-*expansion* [6, 45]

For the evolved (nonlinear) density field $\delta(\boldsymbol{k},a)$, one defines

$$\frac{1}{p!}\Big\langle\frac{\delta^p\delta(\boldsymbol{k},a)}{\delta\delta_0(\boldsymbol{k}_1)\cdots\delta\delta_0(\boldsymbol{k}_p)}\Big\rangle\equiv\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12\cdots p})\frac{1}{(2\pi)^{3(p-1)}}\,\Gamma^{(p)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p) \tag{4.61}$$

The function $\Gamma^{(p)}$ is called $(p+1)$-point propagator. This is the non-perturbative statistical quantity characterizing nonlinear mode-coupling. In terms of the standard PT expansion, it is expressed as

$$\Gamma^{(p)}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p;\,a)=\{D_1(a)\}^p\,F_p(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)+\sum_{n=1}\{D_1(a)\}^{p+2n}\,\Gamma^{(p)}_{n\text{-loop}}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p) \tag{4.62}$$

with higher-order correction $\Gamma^{(p)}_{n\text{-loop}}$ given by

$$\begin{aligned}\Gamma^{(p)}_{n\text{-loop}}&(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)\\ &=c^{(p)}_n\int\frac{d^3\boldsymbol{q}_1\cdots d^3\boldsymbol{q}_n}{(2\pi)^{3n}}F_{2n+p}(\boldsymbol{q}_1,-\boldsymbol{q}_1,\cdots,\boldsymbol{q}_n,-\boldsymbol{q}_n,\boldsymbol{k}_1,\cdots,\boldsymbol{k}_p)\,P_0(q_1)\cdots P_0(q_n).\end{aligned} \tag{4.63}$$

with the symmetric factor $c^{(p)}_n=(2-1)!!\,{}_{2n+p}C_p$. Using these expressions, it is straightforward to show that the standard PT expansion of the power spectrum is re-organized in terms of the multi-point propagators as follows:

$$P(k,a) = \{\Gamma^{(1)}(k;a)\}^2 P_0(k) + \sum_{n=2} n! \int \frac{d^3\boldsymbol{q}_1 \cdots d^3\boldsymbol{q}_n}{(2\pi)^{3(n-1)}} \delta_{\rm D}(\boldsymbol{k} - \boldsymbol{q}_{12\cdots n}) \left\{\Gamma^{(n)}(\boldsymbol{q}_1, \cdots, \boldsymbol{q}_n;\, a)\right\}^2 P_0(q_1) \cdots P_0(q_n). \tag{4.64}$$

Similarly, other statistical quantities such as bispectrum and trispectrum are systematically constructed with multi-point propagators [6, 7, 69].

A crucial point is how to accurately construct a *regularized* multi-point propagator that can describe their global shape, i.e., their whole k-dependence. A couple of important properties to be noted is

- High-k behaviors: In the limit $k \to \infty$, resummation of the standard PT expansion at all order is possible, and one gets [6, 9]

$$\Gamma^{(p)}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p;\, a) \overset{k\to\infty}{\longrightarrow} \{D_1(a)\}^p F_p(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p)\, e^{-k^2\sigma_d^2/2} \tag{4.65}$$

  with σ_d being the rms of the displacement field given by

$$\sigma_d^2 = \{D_1(a)\}^2 \int \frac{dq}{6\pi^2} P_0(q). \tag{4.66}$$

- Low-k behaviors: At low-k, a perturbative calculation with standard PT expansion may be applied. While this should be restricted to a certain low-k regime, each perturbative correction in Eq. (4.63) possesses the following asymptotic form:

$$\Gamma^{(p)}_{n\text{-loop}}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p;\, a) \overset{k\to\infty}{\longrightarrow} \frac{1}{n!}\left(-\frac{k^2\sigma_d^2}{2}\right)^n \{D_1(a)\}^p F_p(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p) \tag{4.67}$$

The above properties indicate that there exists well-defined matching scheme that smoothly interpolates between the low-k and high-k results of any multi-point propagator [7]. Construction of such a *regularized* propagator is given by

$$\Gamma^{(p)}_{\rm reg}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p;\, a) = \{D_1(a)\}^p \times \left[F_p(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p)\left\{1 + \frac{k^2\sigma_d^2}{2}\right\} + \{D_1(a)\}^2\,\Gamma^{(p)}_{\text{1-loop}}(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_p)\right] \exp\left\{-\frac{k^2\sigma_d^2}{2}\right\}, \tag{4.68}$$

which consistently reproduces one-loop PT results at low-k. At high-k, it approaches the asymptotic behavior in Eq. (4.65). This construction is easily generalized to include the higher-order PT corrections at low-k.

Finally, comments to be noted (or advertisement) are

- Code to compute power spectrum based on this resummed PT scheme is publicly available [67][4].
- Based on this resummed scheme, an algorithm that allows accelerated power spectrum calculations is proposed and is implemented in the public code [67].

[4] http://ascl.net/1404.012

4.4 Halo model

Halo model provides a qualitative view of gravitational clustering of large-scale structure on both large and small scales, and it can be even applied for a quantitative study of matter/halo clustering. The underlying assumption of the halo model is that the spatial volume of the Universe is entirely filled with the self-gravitating bound objects called dark matter halos, and all the CDM particles (and baryons) constitute these halos.

To start with, consider the two-point correlation function as the Fourier counter-part of the power spectrum, $\xi(r) = \langle\delta(\boldsymbol{x})\delta(\boldsymbol{x}+\boldsymbol{r})\rangle$. The two-point correlation function measures the excess probability above the Poisson distribution of finding pair of objects (CDM particles) with separation r. We can write the contributions to ξ as two separate terms, one from particle pairs in the same halo, and the other from pairs that reside in two different halos. The dark matter halos exhibit a spectrum of masses that can be characterized by a distribution function $n_{\rm halo}(M)$ called the halo mass function, and the halo centers are spatially correlated. Taking these factors into consideration, we can write the two- point correlation function in terms of the halo density profile $\rho_{\rm halo}(r)$, halo mass function $n_{\rm halo}(M)$, and halo-halo correlation $\xi_{\rm hh}(r)$:

$$\xi(\boldsymbol{r}) = \xi_{\text{1-halo}}(r) + \xi_{\text{2-halo}}(r)\,; \tag{4.69}$$

$$\xi_{\text{1-halo}}(r) = \int dM\, n_{\rm halo}(M) \int d^3\boldsymbol{x} \frac{\rho_{\rm halo}(\boldsymbol{x};\, M)}{\rho_{\rm m}} \frac{\rho_{\rm halo}(\boldsymbol{x}+\boldsymbol{r};\, M)}{\rho_{\rm m}}, \tag{4.70}$$

$$\xi_{\text{2-halo}}(|\boldsymbol{x}-\boldsymbol{x}'|) = \int dM_1\, n_{\rm halo}(M_1) \int dM_2\, n_{\rm halo}(M_2) \times \int d^3\boldsymbol{x}_1 \frac{\rho_{\rm halo}(\boldsymbol{x}-\boldsymbol{x}_1;\, M_1)}{\rho_{\rm m}} \int d^3\boldsymbol{x}_1\, \frac{\rho_{\rm halo}(\boldsymbol{x}'-\boldsymbol{x}_2;\, M_2)}{\rho_{\rm m}}\, \xi_{\rm hh}(\boldsymbol{x}_1-\boldsymbol{x}_2;\, M_1, M_2). \tag{4.71}$$

Then, the corresponding Fourier counter-part (power spectrum) becomes

$$P(k) = P_{\text{1-halo}}(k) + P_{\text{2-halo}}(k)\,; \tag{4.72}$$

$$P_{\text{1-halo}}(k) = \int dM\, n_{\rm halo}(M) \left|\frac{\tilde{\rho}_{\rm halo}(k;\, M)}{\rho_{\rm m}}\right|^2, \tag{4.73}$$

$$P_{\text{2-halo}}(k) = \int dM_1\, n_{\rm halo}(M_1) \int dM_2\, n_{\rm halo}(M_2)\, \frac{\tilde{\rho}_{\rm halo}(k;\, M_1)}{\rho_{\rm m}} \frac{\tilde{\rho}_{\rm halo}(k;\, M_2)}{\rho_{\rm m}}\, P_{\rm hh}(k;\, M_1, M_2). \tag{4.74}$$

where the quantities $\tilde{\rho}_{\rm halo}$ and $P_{\rm hh}$ are the Fourier transform of the density profile and correlation function, respectively.

Given the explicit functional forms of $n_{\rm halo}$, $\rho_{\rm halo}$, and $\xi_{\rm hh}$, the above equations describe the nonlinear matter clustering reasonably well. In particular, halo model description is powerful to predict the statistics at small scales, where the perturbation theory treatment cannot reach. For more detail, see Ref. [20] for applications and extension to the galaxy clustering (for recent interesting applications, see Ref. [29] to the redshift-space distortions, Refs. [65, 36] to the non-Gaussian covariance).

4.5 Galaxy/halo bias

So far, we have focused on the dark matter clustering on large scales. However, the fundamental observables of the large-scale structure by the galaxy redshift surveys are the galaxies. To be precise, what we can observe/measure is the number density fluctuations of galaxies, which differs from the mass density fluctuations we have so far considered. In this respect, the galaxy distribution may be a biased tracer of large-scale matter distribution. Connecting the dark matter clustering with galaxy clustering is important and crucial issues, and needs to be addressed[5].

$$\delta_{\rm gal}(\boldsymbol{x}) = \frac{n_{\rm gal}(\boldsymbol{x})}{\overline{n}_{\rm gal}} - 1 \qquad \longleftrightarrow \qquad \delta(\boldsymbol{x}) = \frac{\rho_{\rm m}(\boldsymbol{x})}{\rho_{\rm m}} - 1. \tag{4.75}$$

Prescriptions for galaxy bias[6]

- Linear bias: the simplest prescription that has been first invented by Ref. [35]:

$$\delta_{\rm gal}(\boldsymbol{x}) = b\,\delta_{\rm m}(\boldsymbol{x}). \tag{4.76}$$

- Nonlinear bias : the second simplest prescription (e.g., [24])

$$\delta_{\rm gal}(\boldsymbol{x}) = \sum_n \frac{b_n}{n!}\Big[\{\delta_{\rm m}(\boldsymbol{x})\}^n - \langle\{\delta_{\rm m}(\boldsymbol{x})\}^n\rangle\Big]. \tag{4.77}$$

A more general prescription of the bias may be non-local, nonlinear and stochastic bias. But, generic prescription of it looks intractable.

Toy models for bias

Representative models of galaxy biasing are halo bias and peak bias. These are classified as the Lagrangian bias model, and the density fields are defined in Lagrangian space:

$$\text{Halo bias:} \qquad 1 + \delta_{\rm halo}(\boldsymbol{q};\,M) = \frac{\hat{n}_{\rm halo}(\boldsymbol{q};\,M)}{n_{\rm halo}(M)}, \tag{4.78}$$

$$\text{Peak bias:} \qquad 1 + \delta_{\rm peak}(\boldsymbol{q};\,\nu_{\rm c}) = \frac{\hat{n}_{\rm peak}(\boldsymbol{q};\,\nu_{\rm c})}{n_{\rm peak}(\nu_{\rm c})}, \tag{4.79}$$

where $\hat{n}_{\rm halo}$ and $\hat{n}_{\rm peak}$ are the random fields, and $n_{\rm halo}$ and $n_{\rm peak}$ represent their mean, i.e., $\langle\hat{n}_{\rm halo}\rangle = n_{\rm halo}$ and $\langle\hat{n}_{\rm peak}\rangle = n_{\rm peak}$. Explicit expression for $\hat{n}_{\rm halo}$ is (see Ref. [47]):

$$\hat{n}_{\rm halo}(\boldsymbol{q};\,M) = -2\frac{\rho_{\rm m}}{M}\frac{\partial}{\partial M}\Theta\big[\delta(\boldsymbol{q}|M) - \delta_{\rm crit}\big] \tag{4.80}$$

[5]For comprehensive review on galaxy bias, see V. Desjacques, D. Jeong, and F. Schmidt [22].

[6]Examples presented here are regarded as Eulerian local bias. Another class of local bias prescription in terms of the Lagrangian quantities is called Lagrangian local bias.

with Θ being the Heaviside step function. The quantity $\delta_{\rm crit}$ is the critical density determined by the spherical collapse model [see Eq. (4.6) or (4.8)], and $\delta(\boldsymbol{q}|M)$ is the linearly extrapolated density field smoothed over the radius $\{M/(4\pi\,\rho_{\rm m}/3)\}^{1/3}$ with top-hat filter. For the peak, it is defined in terms of the smoothed linear density field with Gaussian filter [47]:

$$\hat{n}_{\rm peak}(\boldsymbol{q};\,\nu_{\rm c}) = \frac{3^{3/2}}{R_*^3}\,\delta_{\rm D}(\nu-\nu_{\rm c})\delta_{\rm D}(\vec{\eta})\Theta(\lambda_3)|\det(\zeta_{ij})| \tag{4.81}$$

with $R_* = \sqrt{3}\sigma_1/\sigma_2$, $\nu = \delta/\sigma$, $\eta_i = \nabla_i\delta/\sigma_1$, and $\zeta_{ij} = \nabla_i\nabla_j\delta/\sigma_2$. Here, σ_n is defined by $\sigma_n^2 = \langle(\nabla^n\delta)^2\rangle$.

The halo/peak density fields in Lagrangian space are given as nonlinear and scale-dependent functions of linear density field. Moreover, they are mapped into Eulerian space:

$$1+\delta_{\rm X}(\boldsymbol{x}) = \int d^3\boldsymbol{q}\,[1+\delta_X(\boldsymbol{q})]\,\delta_{\rm D}\big(\boldsymbol{x}-\boldsymbol{q}-\boldsymbol{\psi}(\boldsymbol{q})\big), \qquad ({\rm X}={\rm halo},\ {\rm peak}). \tag{4.82}$$

This further induces non-locality of the bias through the gravitational evolution (see Ref [49] for some attempts).

Practical bias parameterization ?

For practical application to observed galaxy power spectrum or correlation function (specifically BOSS), the bias prescription recently used in the literature is [58, 48, 17]

$$\delta_{\rm gal}(\boldsymbol{x}) = b_1\,\delta(\boldsymbol{x}) + \frac{b_2}{2}\left[\delta(\boldsymbol{x})^2 - \langle\delta(\boldsymbol{x})^2\rangle\right] + \frac{1}{2}\,b_{s^2}\left[s(\boldsymbol{x})^2 - \langle s(\boldsymbol{x})^2\rangle\right] + \cdots \tag{4.83}$$

with the non-local field s defined by

$$s(\boldsymbol{x})^2 = s_{ij}(\boldsymbol{x})s^{ij}(\boldsymbol{x})\ ; \quad s_{ij}(\boldsymbol{x}) = \left(\nabla_i\nabla_j\,\nabla^{-2} - \frac{1}{3}\delta_{ij}\right)\,\delta(\boldsymbol{x}) \tag{4.84}$$

Note that when you go to third-order, another non-local correction arises from the coupling between the tidal fields of density and velocity, whose coefficient is called $b_{3\rm nl}$ [58]. At the end, on top of the cosmological parameters of our interest, there additionally appear 5 free parameters in the galaxy power spectrum, which have to be determined simultaneously from the measured power spectrum [10][7]:

$$\begin{aligned} P_{\rm gal}(k) = b_1^2\,P_{\delta\delta}(k) &+ 2b_2b_1\,P_{b2,\delta}(k) + 2b_{s2}b_1\,P_{bs2,\delta}(k) + 2b_{3\rm nl}b_1\,\sigma_3^2(k)\,P_{\rm lin}(k) \\ &+ b_2^2\,P_{b22}(k) + 2\,b_2b_{s2}\,P_{b2s2}(k) + b_{s2}^2\,P_{bs22}(k) + N, \end{aligned} \tag{4.85}$$

where $P_{\delta\delta}$ and $P_{\rm lin}$ are the nonlinear and linear matter power spectrum. The definitions and expressions for other power spectra are given in Ref. [58, 10].

[7]In redshift space, one more free parameter arises from the Finger-of-God damping factor.

Chapter 5

Statistical properties of large-scale structure

While the observed large-scale matter inhomogeneities are supposed to have evolved from a small Gaussian random fluctuation, their statistical nature exhibit non-Gaussianity through the processes of nonlinear structure formation, including gravitational evolution and galaxy biasing.

Here, we summarize several ways of characterization of non-Gaussian nature of large-scale structure, together with theoretical predictions.

5.1 One-point statistics

One-point statistics are the simplest way to characterize (non-)Gaussian nature of density fields. A simple example for one-point statistics is the moment. It measures the amplitude of (weighted) density field, or counts the (weighted) local number of galaxies, and take an average over the observed volume.

Moment

$$\langle\{\delta(\boldsymbol{x})\}^n\rangle, \quad (n = 1,\, 2,\, \cdots) \tag{5.1}$$

Because of the statistical isotropy, the moment defined above does not depend on the location. By definition, it gives zero for $n = 1$, but generally becomes non-zero for $n \geq 2$. For Gaussian density field, because of Wick's theorem, odd moments (i.e., $n = 1,\, 3,\, 5, \cdots$) are all zero, and even moments are expressed in terms of $n = 2$ [see Eqs. (4.53) and (4.54)]. We then have $\langle\delta^{2n}\rangle = (2n-1)!!\,\langle\delta^2\rangle^n$ for integer n.

Cumulant

To characterize the non-Gaussian properties, instead of moment, cumulant is sometimes useful. The n-th order cumulant, denoted by $\langle\delta^n\rangle_c$, is defined by the n-th order moment subtracting the *un-connected* part expressed in terms of the lower-order moment. For instance, for the zero-mean density field, the second- and third-order cumulants are equal to the second- and third-order moments, i.e., $\langle\delta^2\rangle_c = \langle\delta^2\rangle$ and $\langle\delta^3\rangle_c = \langle\delta^3\rangle$, but the fourth-order cumulant is given by

$$\langle\delta^4\rangle_c = \langle\delta^4\rangle - 3\langle\delta^2\rangle^2. \tag{5.2}$$

To derive the general relation between moments and cumulants, it is convenient to introduce the moment and cumulant generating functions, $\mathcal{M}(t)$ and $\mathcal{C}(t)$:

$$\mathcal{M}(t) \equiv \sum_n \frac{(it)^n}{n!}\langle\delta^n\rangle, \qquad \mathcal{C}(t) \equiv \sum_n \frac{(it)^n}{n!}\langle\delta^n\rangle_c, \tag{5.3}$$

Then, these generating functions are related each other through

$$\mathcal{C}(t) = \ln\mathcal{M}(t) \tag{5.4}$$

Thus, general relation between the n-th order cumulant and moments is obtained:

$$\langle\delta^n\rangle_c = \frac{1}{i^n}\frac{\partial^n\mathcal{C}(t)}{\partial t^n}\bigg|_{t=0} = \frac{1}{i^n}\frac{\partial^n\ln\mathcal{M}(t)}{\partial t^n}\bigg|_{t=0} \tag{5.5}$$

Using this general formula, we systematically obtain the higher-order cumulants expressed in terms of lower-order cumulants and moment:

$$\begin{aligned}
\langle\delta^5\rangle_c &= \langle\delta^5\rangle - 10\langle\delta^3\rangle_c\langle\delta^2\rangle_c, \\
\langle\delta^6\rangle_c &= \langle\delta^6\rangle - 15\langle\delta^4\rangle_c\langle\delta^2\rangle_c - 10\langle\delta^3\rangle_c^2 - 15\langle\delta^2\rangle_c^3, \\
\langle\delta^7\rangle_c &= \langle\delta^7\rangle - 21\langle\delta^5\rangle_c\langle\delta^2\rangle_c - 35\langle\delta^4\rangle_c\langle\delta^3\rangle_c - 105\langle\delta^3\rangle_c\langle\delta^2\rangle_c^2, \\
&\vdots
\end{aligned}$$

For a perturbative calculation with Gaussian initial conditions, the leading-order results of the n-th order cumulant becomes of the order of $\langle\delta_0^2\rangle^{n-1}$, with δ_0 being the initial (linear) density field. It is thus convenient to characterize the cumulants with

$$S_n \equiv \frac{\langle\delta^n\rangle_c}{\langle\delta^2\rangle^{n-1}}. \tag{5.6}$$

Specifically, S_3 and S_4 are respectively called skewness and kurtosis, and the perturbative calculation at tree-level order leads to (see below)

$$S_3 = \frac{34}{7}, \tag{5.7}$$

$$S_4 = \frac{60712}{1323} \tag{5.8}$$

$$S_5 = \frac{200575880}{305613} \tag{5.9}$$

$$\vdots$$

Probability distribution function (PDF)

Probability distribution function (PDF) has whole information on the moments and cumulants of density field. Given the PDF $P(\delta)$, the n-th order moment is computed with

$$\langle\delta^n\rangle = \int_{-1}^{\infty} d\delta\,\delta^n\,P(\delta). \tag{5.10}$$

PDF is thus related to the moment/cumulant generating functions given at Eq. (5.3) through

$$\mathcal{M}(t) = \int_{-1}^{\infty} d\delta \sum_n \frac{(it)^n}{n!}\delta^n P(\delta) = \int_{-1}^{\infty} d\delta\, e^{i\delta\, t} P(\delta)$$
$$\Longrightarrow \qquad P(\delta) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\,\mathcal{M}(t)\, e^{-i\delta\, t} = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-i\delta\, t+\mathcal{C}(t)} \tag{5.11}$$

It is empirically known that the one-point PDFs measured from dark matter density field in N-body simulations and observed galaxy density fields reasonably agree well with lognormal distribution (e.g., [32, 18, 37])[1] :

$$P_{\rm LN}(\delta) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\Big[-\frac{\{\ln(1+\delta)+\sigma_1^2\}^2}{2\sigma_1^2}\Big]\frac{1}{1+\delta}, \qquad \sigma_1^2 = \ln(1+\sigma_{\rm nl}^2) \tag{5.13}$$

with $\sigma_{\rm nl}^2$ being the measured amplitude of $\langle\delta^2\rangle$. Note, however, that the skewness, kurtosis, and other higher-order cumulants predicted from Eq. (5.13) are slightly inconsistent with those obtained from perturbative calculation [see Eqs. (5.7)-(5.9)]. Nevertheless, the lognormal description of density fields is frequently used to model underlying density field of large-scale Lymann-α (neutral hydrogen distribution) and/or to create quickly mock galaxy data.

Edgeworth expansion

Eq. (5.11) implies that provided all the cumulants or moments, we can reconstruct the PDF. In practice, this is possible if the departure from the Gaussianity is small and the first several lower-order cumulants are known. A systematic method to reconstruct PDF is known as *Edgeworth expansion*, which has been first proposed in other subject of physics, and been later applied to cosmology (e.g., [34, 8, 59]). Suppose that the normalized amplitude, S_n, given at Eq. (5.6), is the order of unity. Denoting $\langle\delta^2\rangle$ by σ^2 and introducing the new variables $\nu \equiv \delta/\sigma$ and $x \equiv \sigma t$, one can rewrite Eq. (5.11) with

$$P(\delta) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\,\sigma}\, e^{-i\nu\, x - x^2/2} \exp\left[\sigma\,\frac{S_3}{3!}(ix)^3 + \sigma^2\,\frac{S_4}{4!}(ix)^4 + \sigma^3\,\frac{S_5}{5!}(ix)^5 + \cdots\right] \tag{5.14}$$

Recall that the support of the integrand is around $|x| \lesssim 1$, the exponent that contains a series of cumulants ($n \geq 3$) can be Taylor-expanded for small value of σ, and be truncated

[1] Lognormal density field δ with variance $\sigma_{\rm nl}$ is obtained from the transformation of the Gaussian density field, $\delta_{\rm G}$, with variance $\sigma_{\rm G}$ through

$$1+\delta = \frac{1}{(1+\sigma_{\rm nl}^2)^{1/2}} \exp\Big[\frac{\delta_{\rm G}}{\sigma_{\rm G}}[\ln(1+\sigma_{\rm nl}^2)]^{1/2}\Big]. \tag{5.12}$$

The corresponding one-point PDF is thus given by $P_{\rm LN}(\delta) = \frac{d\delta_{\rm G}}{d\delta}\, P_{\rm G}(\delta_{\rm G})$, leading to Eq. (5.13), where $P(\delta_{\rm G})$ is the Gaussian PDF, $P_{\rm G} = \exp[-(\delta_{\rm G}/\sigma_{\rm G})^2/2]/\sqrt{2\pi\,\sigma_{\rm G}^2}$.

at finite order. Thus, we obtain

$$P(\delta) \simeq \int_{-\infty}^{\infty} \frac{dx}{2\pi\,\sigma}\, e^{-i\,\nu\,x - x^2/2} \left[1 + \sigma\,\frac{S_3}{3!}(ix)^3 + \sigma^2\left\{\frac{S_4}{4!}(ix)^4 + \frac{S_3^2}{2\,(3!)^2}(ix)^6\right\} + \cdots\right]$$
$$= \int_{-\infty}^{\infty} \frac{dx}{2\pi\,\sigma}\, e^{-x^2/2} \left[1 + \sigma\,\frac{S_3}{3!}\left(-\frac{d}{d\nu}\right)^3 + \sigma^2\left\{\frac{S_4}{4!}\left(-\frac{d}{d\nu}\right)^4 + \frac{S_3^2}{2\,(3!)^2}\left(-\frac{d}{d\nu}\right)^6\right\} + \cdots\right] e^{-i\,\nu\,x}$$

Performing the integral and using the definition of Hermite polynomials (see Appendix A.3), we arrive at

$$P\,(\delta) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\nu^2/2} \left[1 + \sigma\,\frac{S_3}{6}\,H_3(\nu) + \sigma^2\left\{\frac{S_4}{24}H_4(\nu) + \frac{S_3^2}{72}H_6(\nu)\right\} + \cdots\right] \tag{5.15}$$

with $\nu = \delta/\sigma$.

While the Edgeworth expansion gives an accurate description for non-Gaussian PDF around $\delta/\sigma \lesssim 0.5$, it does not preserve positivity, leading to an undesirable behavior at high-density tails. To remedy this, the other types of expansion scheme have been proposed (e.g., [19, 25]).

Tree-level summation

The perturbative calculation of one-point statistics has a remarkable property that allows us to systematically derive the cumulants hierarchy of all order at tree-level [5, 2, 4]. Here, we briefly give a sketch on how to compute cumulants at all order at tree-level.

First we define the vertices ν_n, which is given as the angular average of n-th order PT kernel:

$$\nu_n \equiv n! \int \frac{d^2\widehat{\Omega}_1 \cdots d^2\widehat{\Omega}_n}{(4\pi)^n}\, F_n(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_n). \tag{5.16}$$

Notice that this corresponds to the following amplitude:

$$\nu_n = \frac{\langle \delta_n \,\{\delta_1\}^n \rangle}{\sigma_{\rm lin}^{2n}}, \qquad (\,\sigma_{\rm lin}^2 = \langle \delta_1^2 \rangle\,). \tag{5.17}$$

We then defined the following generation functions:

$$\mathcal{G}_\delta(\tau) \equiv \sum_{n=1} \nu_n\,\frac{(-\tau)^n}{n!}, (\nu_1 = 1)$$
$$\varphi(y) \equiv -\sum_{p=1} S_p\,\frac{(-y)^p}{p!}, \qquad (S_1 = S_2 = 1). \tag{5.18}$$

Note that the function $\varphi(y)$ is related to the cumulant generating function $\mathcal{C}(t)$ through $\varphi(y) = -\sigma^2\,\mathcal{C}(-i\,y/\sigma^2)$ or $\mathcal{C}(t) = -\varphi(-it\,\sigma^2)/\sigma^2$.

It is shown that the above two generating functions are related through (e.g., [5, 2, 4]):

$$\varphi(y) = y\,\mathcal{G}_\delta(\tau(y)) + \frac{1}{2}\,\tau^2(y), \qquad \tau = -y\,\frac{d}{d\tau}\mathcal{G}_\delta(\tau). \tag{5.19}$$

Expanding the $\mathcal{G}_\delta$ and φ around small values of τ and y, these transcendental equations give the relation between S_n and ν_n:

$$S_3 = 3\,\nu_2, \quad S_4 = 4\nu_3 + 12\nu_2^2, \quad S_5 = 5\nu_4 + 60\nu_3\nu_2 + 60\nu_2^3, \cdots . \tag{5.20}$$

Thus, provided the vertex amplitude ν_n, we can compute the tree-level cumulants as well as the one-point PDF. A remarkable property is that the function $\mathcal{G}_\delta$ is indeed obtained as the solution of spherical collapse model[2]:

$$\mathcal{G}_\delta(\tau(\theta)) = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}, \qquad \tau(\theta) = -\frac{3}{5}\left[\frac{3}{4}(\theta-\sin\theta)\right]^{2/3}, \quad (0\le\theta\le 2\pi) \tag{5.21}$$

for $\tau<0$, and

$$\mathcal{G}_\delta(\tau(\theta)) = \frac{9}{2}\frac{(\sinh\theta-\theta)^2}{(\cosh\theta-1)^3}, \qquad \tau(\theta) = \frac{3}{5}\left[\frac{3}{4}(\sinh\theta-\theta)\right]^{2/3}, \quad (0\le\theta\le \infty) \tag{5.22}$$

for $\tau>0$. Expanding the function above around small value of θ, one systematically obtains the vertex amplitude:

$$\nu_2 = \frac{34}{7}, \quad \nu_3 = \frac{682}{189} \simeq 3.6, \quad \nu_4 = \frac{446,440}{43,659} \simeq 10.2, \quad \nu_4 = \frac{8,546,480}{243,243} \simeq 35.1, \cdots . \tag{5.23}$$

Combining Eq. (5.20) with Eq. (5.23), we arrive at the tree-level perturbative predictions of S_n given at Eqs. (5.7)–(5.9).

Note finally that the perturbative predictions presented here are all un-smoothed case. In practical application, we may have to introduce the filter function $W(\boldsymbol{x})$ and measure the smoothed density field. This smoothing procedure is known to change the amplitude of cumulants and S_n. For Gaussian filter function, analytic expression is only known for the skewness in power-law power spectrum (e.g., [33, 42, 43]), but there is a systematic way to derive tree-level predictions for top-hap filter function [3], and a construction of vertex generating function is also known, as similarly to the one presented above [5, 4].

5.2 Two-point statistics

The two-point statistics are the second simplest quantities characterizing the clustering properties of large-scale structure, and power spectrum and correlation function are the most representative quantities frequently used in the cosmological analysis.

Correlation function and power spectrum

Although we have already discussed and studied both the power spectrum and correlation function, we here summarize their relation:

$$\xi(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,P(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad P(\boldsymbol{k}) = \int d^3\boldsymbol{x}\,\xi(\boldsymbol{x})\,e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}} \tag{5.24}$$

[2]The result comes from the fact that the vertex defined at Eq. (5.16) is the n-th order kernel of spherically symmetric density field, and replacing $-\tau$ with D_+, the generating function $\mathcal{G}_\delta$ can be viewed as a Taylor-expanded form of the solution of spherically symmetric density field.

Imposing the statistical isotropy, we have

$$\xi(x) = \int \frac{dk\,k^2}{2\pi^2}\,P(k)\,\frac{\sin(kx)}{k\,x}, \qquad P(k) = \int \frac{dx\,x^2}{2\pi^2}\,\xi(x)\,\frac{\sin(kx)}{k\,x} \tag{5.25}$$

For discrete sources like galaxies, the correlation function, defined as $\xi(\boldsymbol{r}) = \langle\delta(\boldsymbol{x})\delta(\boldsymbol{x}+\boldsymbol{r})\rangle$, can be also quantified as an excess probability of pair of discrete sources, and is measured with a direct pair count. Denoting the probability for finding an object in both the volume elements dV_1 and dV_2 at separation x by dP_{12}, we have

$$dP_{12}(x) = \overline{n}^2\,\left[1+\xi(x)\right]\,dV_1 dV_2. \tag{5.26}$$

Alternatively, we may first choose at random an object from the sample, and measure the probability of finding a neighbor at distance r in dV:

$$dP(r) = \overline{n}\,\left[1+\xi(r)\right]\,dV. \tag{5.27}$$

These are the basis to estimate the correlation function from N-body and observational data . Note that the correlation function needs to consistently satisfy the so-called integral constraint:

$$\int d^3\boldsymbol{r}\,\xi(r) = 0, \tag{5.28}$$

which comes from the fact that $P(k) = 0$ at $k \to 0$. The integral constraint indicates that the positivity of the correlation function is not necessarily ensured. Given the fact that nonlinear gravity leads to a strong enhancement of the correlation function with positive values at small separation, the sign of the correlation function becomes eventually flipped at large scales, and it takes the negative values. The integral constraint is also important for measuring the correlation function, and it indicates that without correctly estimating mean number density, the measured results may be biased.

There are tremendous amount of studies on two-point statistics both from theoretical and observational point-of-view, which I cannot cover. I here give several properties for nonlinear evolution of power spectrum.

Tips for nonlinear power spectrum

- In standard cosmological model, the matter power spectrum starts to deviate from linear theory prediction at $k \gtrsim 0.1$ around $z \lesssim 3-5$, and the amplitude of power spectrum becomes significantly amplified. The size of deviation gradually increases as decreasing the redshift, and is typically, e.g., at $k = 0.2\,h\,\mathrm{Mpc}^{-1}$, is $2-3\%$ (z=3), 10% (z=1), and 20% (z=0).
- Perturbation theory (PT) prediction can describe the nonlinear power spectrum to some extent. For instance, at $z = 1$, the standard (resummed) PT prediction at one-loop order reproduces the results of N-body simulation at a percent level, up to $k \simeq 0.12$ (0.17) $h\,\mathrm{Mpc}^{-1}$. At two-loop order, a percent level agreement will be further improved up to $k \simeq 0.22\,h\,\mathrm{Mpc}^{-1}$ (in both standard and resummed PT predictions[3]).

[3] Recalling the convergence of standard PT expansion, a remarkable agreement of the standard PT at $z = 1$ is thought to be coincidence, and should be taken with a great care.

- One notable PT prediction, especially for resumed PT such as multi-point propagator expansion, is that low-k behavior of higher-order corrections is always dominated by the terms which scale as k^4, irrespective of the initial power spectrum[4]. This is a sort of universal and generic feature of gravity-induced nonlinear evolution (see Sec. 24 of Peebles' textbook for intuitive explanation).

- Beyond weakly nonlinear regime (e.g., $k > 0.2\,h\,\mathrm{Mpc}^{-1}$ at $z = 1$), we need fully non-perturbative technique for prediction. One powerful approach is to employ N-body simulation. With a suite of simulation data in various cosmological models, *fitting formula* (e.g., [61, 66]), which gives an analytic expression on how the power spectrum is modulated in a scale-dependent manner for given linear power spectrum, and emulator code (e.g., [27, 40, 28]), which simply interpolates the numerical data of power spectrum in different models, have been developed, and been frequently used in the literature. Accuracy of these prediction is more or less $2-5\%$.

- At nonlinear regime, the power spectrum is strongly enhanced by more than one order of magnitude, but the enhancement is slightly reduced (to some extent), and it seems to approach (to some extent) universal trend predicted by stable clustering hypothesis under the assumption of self-similarity (e.g., see Sec. 73 of Peebles' textbook, and [26, 55]). These phenomena are kinematically related to the formation and merging of halos.

5.3 Three-point statistics

Three-point correlation function and bispectrum

Similar to the two-point statistics, the three-point correlation function, defined as $\zeta(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \langle\delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\delta(\boldsymbol{x}_3)\rangle$, is related to the bispectrum $B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ through

$$\zeta(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \int\prod_{i=1}^{3}\frac{d^3\boldsymbol{k}_i}{(2\pi)^3}\,B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,\delta_{\mathrm{D}}(\boldsymbol{k}_{123})\,\exp\Big[i\big(\boldsymbol{k}_1\cdot\boldsymbol{x}_1+\boldsymbol{k}_2\cdot\boldsymbol{x}_2+\boldsymbol{k}_3\cdot\boldsymbol{x}_3\big)\Big]. \tag{5.29}$$

In contrast to the two-point statistics, the relation now involves six-dimensional integrals, and it apparently looks difficult to handle. However, in real space, we can take advantage of the statistical isotropy and the constraint on the triangular shape ($\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3=0$). Thus, the bispectrum is characterized as a function of $|\boldsymbol{k}_1|$, $|\boldsymbol{k}_2|$, and $\theta_{12}\equiv\cos^{-1}(\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2)$. We can thus expand the bispectrum with Legendre polynomials:

$$B(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \sum_\ell B_\ell(k_1,k_2)\mathcal{P}_\ell(\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2) \tag{5.30}$$

Similarly, the three-point correlation function is expanded as

$$\zeta(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \sum_\ell \zeta_\ell(x_1,x_2)\mathcal{P}_\ell(\hat{\boldsymbol{x}}_1\cdot\hat{\boldsymbol{x}}_2). \tag{5.31}$$

[4]For standard PT expansion, on the other hand, the contribution $P_{1n}\simeq\langle\delta_1\delta_n\rangle$, is always dominant at low-k, and it asymptotically scales as $k^2\,P_0(k)$

Then, from Eq. (5.29), we obtain [64]

$$\zeta_\ell(x_1, x_2) = (-1)^\ell \int \frac{dk_1 dk_2\, k_1^2 k_2^2}{(2\pi)^2}\, j_\ell(k_1 x_1)\, j_\ell(k_2 x_2)\, B_\ell(k_1, k_2). \tag{5.32}$$

Eq. (5.32) greatly simplifies the relation between three-point correlation function and bispectrum.

Reduced bispectrum

At leading order, the PT prediction of bispectrum lead to $B \propto P_0\, P_0$ [see Eq. (4.60)]. It is thus convenient to characterize the shape of bispectrum by introducing the following normalization:

$$Q \equiv \frac{B(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)}{P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1)} \tag{5.33}$$

In similar manner, reduced amplitude of three-point point correlation function is defined. With the definition above, leading-order PT prediction of Q only depends on the shape of initial power spectrum. Several properties of the reduced amplitude Q are summarized as follows:

- For fixed wavenumbers k_1 and k_2, the amplitude Q plotted as function of the opening angle $\theta \equiv \cos^{-1}(\hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2)$ shows a characteristic shape, that is, as increasing θ, it first decreases and after reaching the minimum at $\theta \sim \pi/2$, and it turns to increases. Note that $Q(\theta = 0) = Q(\theta = \pi)$.
- The shape of the reduced amplitude Q becomes flattened when the nonlinear gravity is significant. At strongly nonlinear regime (or large values of k_1 and k_2), the amplitude approaches a constant value.
- The galaxy bias also changes the shape of the bispectrum. In particular, the nonlinearity of the bias, if we expand $\delta_g = b_1\delta + (b_2/2)\delta^2 + \cdots$, leads to a flattened shape, i.e., $Q \rightarrow (Q + b_2/b_1)/b_1$.

Unlike the two-point statistics, measurement of bispectrum or three-point correlation function has been so far restricted to small scales, where the accurate theoretical prediction is difficult. But, with future galaxy surveys that will uncover a gigantic cosmological volume, we will be able to measure the three-point statistics at weakly nonlinear regime, where we can give a robust theoretical prediction. Since the bispectrum and three-point correlation function are known to have rich cosmological information, combination of two- and three-point statistics helps to break parameter degeneracy, and hence to tighten the constraints on cosmological model parameters.

Appendix A

Useful formulas

A.1 Fourier transformation

$$A(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, A(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.1}$$

$$A(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, A(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.2}$$

Dirac's delta-function:

$$\delta_{\rm D}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{A.3}$$

Relation between $\xi(r)$ and $P(k)$ (Wiener-Khintchine relation):

$$\xi(r) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, P(|\boldsymbol{k}|)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{A.4}$$

$$P(k) = \int d^3\boldsymbol{r}\, \xi(|\boldsymbol{r}|)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{A.5}$$

A.2 Legendre polynomials

$$(1-\mu^2)\frac{d^2\mathcal{P}_\ell(\mu)}{d\mu^2} - 2\mu\frac{d\mathcal{P}_\ell(\mu)}{d\mu} + \ell(\ell+1)\,\mathcal{P}_\ell(\mu) = 0 \tag{A.6}$$

$$\int_{-1}^{1} d\mu\, \mathcal{P}_\ell(\mu)\, \mathcal{P}_{\ell'}(\mu) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} \tag{A.7}$$

$$\ell\,\mathcal{P}_\ell(\mu) - (2\ell-1)\mu\,\mathcal{P}_{\ell-1}(\mu) + (\ell-1)\mathcal{P}_{\ell-2}(\mu) = 0 \tag{A.8}$$

$$(\mu^2-1)\frac{d\mathcal{P}_\ell(\mu)}{d\mu} = \ell\,\{\mu\,\mathcal{P}_\ell(\mu) - \mathcal{P}_{\ell-1}(\mu)\} = (\ell+1)\{\mathcal{P}_{\ell+1}(\mu) - \mu\mathcal{P}_{\ell-1}(\mu)\} \tag{A.9}$$

$$\begin{aligned}
\mathcal{P}_0(\mu) &= 0, \\
\mathcal{P}_1(\mu) &= \mu, \\
\mathcal{P}_2(\mu) &= \frac{1}{2}(3\mu^2 - 1), \\
\mathcal{P}_3(\mu) &= \frac{1}{2}(5\mu^3 - 3\mu), \\
\mathcal{P}_4(\mu) &= \frac{1}{8}(35\mu^4 - 30\mu^2 + 3),
\end{aligned} \tag{A.10}$$

$$\mathcal{P}_\ell(\hat{k} \cdot \hat{x}) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{k})\, Y^*_{\ell m}(\hat{x}). \tag{A.11}$$

A.3 Hermite polynomials

$$H_n(x) = e^{x^2/2} \left(-\frac{d}{dx} \right)^n e^{-x^2/2} \tag{A.12}$$

$$\frac{d^2 H_n(x)}{dx^2} - x \frac{dH_n(x)}{dx} + n\, H_n(x) = 0. \tag{A.13}$$

$$H_{n+1}(x) - x\, H_n(x) + n\, H_{n-1}(x) = 0, \tag{A.14}$$

$$\begin{aligned}
H_1(1) &= x, \\
H_2(x) &= x^2 - 1, \\
H_3(x) &= x^3 - 3x, \\
H_4(x) &= x^4 - 6x^2 + 3, \\
H_5(x) &= x^5 - 10x^3 + 15x.
\end{aligned} \tag{A.15}$$

A.4 Spherical Bessel functions

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + 1 - \frac{\ell(\ell+1)}{x^2} \right] j_\ell(x) = 0 \tag{A.16}$$

$$j_\ell(x) = x^n \left(-\frac{1}{x}\frac{d}{dx} \right)^n \frac{\sin x}{x} \tag{A.17}$$

$$
\begin{aligned}
j_0(x) &= \frac{\sin x}{x}, \\
j_1(x) &= \frac{\sin x - x\cos x}{x^2}, \\
j_2(x) &= \frac{(3-x^2)\sin x - 3x\cos x}{x^3}, \\
j_3(x) &= \frac{(15-6x^2)\sin x - x(15-x^2)\cos x}{x^4},
\end{aligned} \tag{A.18}
$$

$$
e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} = \sum_{\ell=0} i^\ell (2\ell+1)\, j_\ell(kx)\, \mathcal{P}_\ell(\hat{k}\cdot\hat{x}). \tag{A.19}
$$

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