Evolution of a Dissipative Self-Gravitating Particle Disk or Terrestrial Planet Formation

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Outline

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Planetesimal Dynamics

- Viscous stirring
- Dynamical friction
- Orbital repulsion

Planetesimal Accretion

- Runaway growth of planetesimals
- Oligarchic growth of protoplanets
- Giant impacts
Introduction
Terrestrial Planets

Planets
- Mercury, Venus, Earth, Mars

Alias
- rocky planets

Orbital Radius
- $\simeq 0.4-1.5$ AU (inner solar system)

Mass
- $\sim 0.1-1 \ M_\oplus$

Composition
- rock (mantle), iron (core)

Close-in super-Earths are most common!
Semimajor Axis–Mass

“two mass populations”
Orbital Elements

Semimajor Axis–Eccentricity (●), Inclination (○)

"nearly circular coplanar"
**Terrestrial Planet Formation**

**Act 1** Dust to planetesimals (gravitational instability/binary coagulation)

**Act 2** Planetesimals to protoplanets (runaway-oligarchic growth)

**Act 3** Protoplanets to terrestrial planets (giant impacts)
Planetesimal Disks

Disk Properties

• many-body (particulate) system
• rotation
• self-gravity
• dissipation (collisions and accretion)

Planet Formation as Disk Evolution

• evolution of a dissipative self-gravitating particulate disk
• velocity and spatial evolution ↔ mass evolution
Question

How does a dissipative self-gravitating particulate disk evolve?
Planetesimal Dynamics
Question

How do particle orbits \((a, e, i)\) evolve?
**Terminology**

**Random Velocity**
- deviation velocity from a non-inclined circular orbit

\[ v_{\text{ran}} \simeq \left( e^2 + i^2 \right)^{1/2} v_K \]

\[ \sigma_R \propto \sigma_e, \quad \sigma_z \propto \sigma_i \]

\( e \): eccentricity, \( i \): incination, \( v_K \): Kepler circular velocity

**Hill (Roche/Tidal) Radius**
- radius of the potential well of an orbiting body

\[ r_H = \left( \frac{m}{3M_c} \right)^{1/3} a \]

\( M_c \): central body mass, \( m \): orbiting body mass, \( a \): semimajor axis
Disk Properties

Dynamics

- central gravity dominant (nearly Keplerian orbit)
- differential rotation (shear velocity)
- “collisional” system (evolution by two-body encounters)

Structure

- disk thickness $\propto$ velocity dispersion ($\sigma_z \propto \sigma_i$)
Equation of Motion

\[
\frac{dv_i}{dt} = -GM_c \frac{x_i}{|x_i|^3} + \sum_{j \neq i}^{N} Gm_j \frac{x_j - x_i}{|x_j - x_i|^3} + f_{\text{gas}} + f_{\text{col}}
\]

- **central gravity** (dominant) ⇒ nearly Keplerian orbits
- mutual interaction ⇒ random velocity
- gas drag ⇒ random velocity
- collision ⇒ random velocity

mutual interaction + gas drag/collision ⇒ equilibrium random velocity
Equation of Motion

\[ \frac{d\mathbf{v}_i}{dt} = -G M_c \frac{\mathbf{x}_i}{|\mathbf{x}_i|^3} + \sum_{j \neq i}^{N} G m_j \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|^3} \]

- central gravity (dominant) ⇒ nearly Keplerian orbits
- mutual interaction ⇒ random velocity

Two-Body Relaxation
Two-Body Relaxation

Elementary Process

- Two-body gravitational scattering

Chandrasekhar’s Two-Body Relaxation Time

- Timescale to forget the initial orbit

\[
 t_{\text{relax}} \equiv \frac{v^2}{dv^2/dt} \approx \frac{1}{n\pi r_g^2 v \ln \Lambda} = \frac{v^3}{n\pi G^2 m^2 \ln \Lambda}
\]

- \(n\): number density, \(r_g\): gravitational radius, \(\ln \Lambda\): Coulomb logarithm

(Chandrasekhar 1949)
Relaxation of Particulate Disks

Viscous Stirring (Disk Heating)

- increase of random velocity $v_{\text{ran}} (e \text{ and } i)$

Dynamical Friction

- equiparation of random energy $mv_{\text{ran}}^2 \propto m(e^2 + i^2)$
Viscous Stirring

- increase of $e$ and $i$ ($\sigma_e > \sigma_i$)
- diffusion in $a$
Viscous Stirring

- $\sigma_e, \sigma_i \propto t^{1/4}$ (two-body relaxation timescale)
- $\sigma_e/\sigma_i = \sigma_R/\sigma_z \simeq 2$ (anisotropic velocity dispersion)
Viscous Stirring

Elementary Process

• two-body scattering: shear velocity $\rightarrow$ random velocity

Timescale

$$t_{VS} \equiv \frac{\sigma^2}{d\sigma^2/dt} \simeq \frac{\sigma^3}{n\pi G^2m^2 \ln \Lambda} \Rightarrow t_{VS} \propto \sigma^4 \Rightarrow \sigma \propto t^{1/4}$$

$$n \propto (\text{thickness})^{-1} \propto \sigma^{-1}$$

(Ida & Makino 1992; EK & Ida 1992)

Anisotropic Velocity Dispersion

• $\sigma_e/\sigma_i \propto$ shear strength
  (cf. $\sigma_R/\sigma_z \simeq 1.4$ for the Galactic disk)

(Ida, EK, & Makino 1993)
Dynamical Friction

- decrease of $e_M$ and $i_M$ (↔ increase of local $e$ and $i$)
- almost constant $a_M$
Dynamical Friction

\[ e_M, i_M \rightarrow 0 \] (non-inclined circular orbit)
Dynamical Friction

Chandrasekhar’s Formula
A large particle with $M$ and $v_M$ in a swarm of small particles with $m$ and $v_m$

$$\frac{1}{v_M} \frac{dv_M}{dt} \sim \frac{G^2 M mn_m}{v_M^3}$$

$(v_M > v_m)$

(Chandrasekhar 1949)

Application to a Particulate Disk

$$\frac{1}{e_M} \frac{de_M}{dt} \sim \frac{G^2 M mn_{s,m}}{2ia e_M^3 a^3 \Omega^3} \sim \frac{G^2 M \Sigma}{e_M^4 a^4 \Omega^3}$$

$(\Sigma = mn_{s,m}, v_M \simeq e_M a \Omega, n_m \simeq n_{s,m}/2ai_M, i_m < i_M)$

$$t_{DF} \equiv \frac{e_M}{de_M/dt} \sim \frac{e_M^4 a^4 \Omega^3}{G^2 M \Sigma}$$
- expansion of orbital separation $b$: $b = 3r_H \rightarrow b \approx 8r_H$  
- keeping $e_M$ and $i_M$ small
Orbital Repulsion Mechanism

1. protoplanet-protoplanet scattering \((e_M, b \uparrow)\)
2. dynamical friction from planetesimals \((e_M \downarrow, a_M \text{ constant})\)

\[ b \gtrsim 5r_H \]

(EK & Ida 1995)
Two-Body Relaxation of Particulate Disks

Disk Evolution:

• Viscous stirring
  - $\sigma_e, \sigma_i \propto t^{1/4}$ (≃ disk)
  - $\sigma_e/\sigma_i \approx 2$ (≃ differential rotation)

• Dynamical friction
  - $e, i \propto m^{-1/2}$ (≃ energy equipartition)

• Orbital repulsion
  - $b \gtrsim 5r_H$ (≃ scattering, dynamical friction)

All these elementary processes control the basic dynamics and structure of particulate disks!
Planetesimal Accretion
How does particle mass distribution evolve by accretion?
Planetesimals

Mass (size)

- \( m \sim 10^{18} \text{ g} \) (\( r \sim 1 \text{ km} \))

Surface density distribution

\[
\Sigma_{\text{solid}} = \Sigma_1 \left(\frac{a}{1 \text{ au}}\right)^{-\alpha} \text{ g cm}^{-2}
\]

1 \( \leq \Sigma_1 \leq 100 \), \( 1/2 \leq \alpha \leq 5/2 \)

standard protosolar disk: \( \Sigma_1 \sim 10 \), \( \alpha = 3/2 \) (Hayashi 1981)
Growth Mode

\[
\frac{d}{dt} \left( \frac{M_1}{M_2} \right) = \frac{M_1}{M_2} \left( \frac{1}{M_1} \frac{dM_1}{dt} - \frac{1}{M_2} \frac{dM_2}{dt} \right)
\]

relative growth rate: \( \frac{1}{M} \frac{dM}{dt} \propto M^p \)

orderly growth \( p < 0 \)

runaway growth \( p > 0 \)
Collisional Cross-Section

Gravitational focusing

\[ R_{gf} = R \left( 1 + \frac{2GM}{Rv_{rel}^2} \right)^{1/2} = R \left( 1 + \frac{v_{esc}^2}{v_{rel}^2} \right)^{1/2} \]

Collisional cross-section

\[ S_{gf} = \pi R_{gf}^2 = \pi R^2 \left( 1 + \frac{v_{esc}^2}{v_{rel}^2} \right) \]
Growth Rate

Test body: \( M, R, v_{\text{esc}} \)

Field bodies: \( n \) (number density), \( m \)

\[
\frac{dM}{dt} \approx n\pi R^2 \left( 1 + \frac{v_{\text{esc}}^2}{v_{\text{rel}}^2} \right) v_{\text{rel}} m \Rightarrow \frac{1}{M} \frac{dM}{dt} \propto M^{1/3} v_{\text{ran}}^{-2}
\]

\[
\left( v_{\text{rel}} \approx v_{\text{ran}}, n \propto v_{\text{ran}}^{-1}, v_{\text{esc}} \propto M^{1/3}, R \propto M^{1/3}, v_{\text{rel}} < v_{\text{esc}} \right)
\]

Random velocity controls
- the growth mode
- the growth timescale
Runaway Growth of Planetesimals

self-gravity of planetesimals dominant for random velocity

\[ v_{\text{ran}} \neq f(M) \]

\[ \frac{1}{M} \frac{dM}{dt} \propto M^{1/3} v_{\text{ran}}^{-2} \propto M^{1/3} \]

runaway growth!

(EK & Ida 2000)
Runaway Growth of Planetesimals

\[ M_{\text{max}}, \langle m \rangle (10^{23} \text{ g}) \]

\[ t (\text{yr}) \]

**solid**: \( M_{\text{max}} \), **dashed**: \( \langle m \rangle \)  

(EK & Ida 2000)
Runaway Growth of Planetesimals

\[ \frac{d \log n_c}{d \log m} \simeq -\frac{11}{8} \]

\( n_c \)

\( m \left( 10^{23} \text{ g} \right) \)

dotted: 0 yr, dashed: 10^5 yr, solid: 2 \times 10^5 yr

(EK & Ida 2000)
Oligarchic Growth of Protoplanets

\[ \Sigma_1 = 10, \alpha = 3/2 \]

Slowdown of runaway scattering of planetesimals by a protoplanet with \( M \gtrsim 100m \)

\[ v_{\text{ran}} \propto r_H \propto M^{1/3} \]

\[ \downarrow \]

\[ \frac{1}{M} \frac{dM}{dt} \propto M^{1/3} v_{\text{ran}}^{-2} \propto M^{-1/3} \]

orderly growth!

(Ida & Makino 1993)

Orbital repulsion

orbital separation: \( b \simeq 10r_H \)

(EK & Ida 1998)

(aU)
**Protoplanets**

Assumptions

- no radial migration
- 100% accretion efficiency

Isolation mass

$$M_{\text{iso}} \simeq 2\pi ab\Sigma_{\text{solid}} = 0.16 \left( \frac{b}{10r_H} \right)^{3/2} \left( \frac{\Sigma_1}{10} \right)^{3/2} \left( \frac{a}{1 \text{AU}} \right)^{(3/2)(2-\alpha)} M_{\oplus}$$

Growth time

$$t_{\text{grow}} \simeq 3.2 \times 10^5 \left( \frac{f_{\text{gas}}}{240} \right)^{-2/5} \left( \frac{b}{10r_H} \right)^{1/10} \left( \frac{\Sigma_1}{10} \right)^{-9/10} \left( \frac{a}{1 \text{AU}} \right)^{(9\alpha+16)/10} \text{yr}$$

(EK & Ida 2002, 2012)
Isolation Mass of Protoplanets

Standard protosolar disk
\[ \Sigma_1 = 10, \ \alpha = \frac{3}{2} \]

Terrestrial planet zone
\[ M_{iso} \approx 0.1M_\oplus \]

Final formation stage
- large planets: impacts among protoplanets
- small planets: leftover protoplanets

(EK & Ida 2000)
What is the final state of disk evolution?
Giant Impacts among Protoplanets

- Protoplanets gravitationally perturb each other to become orbitally unstable after gas dispersal ($t_{\text{gas}} \lesssim 10^7$ yr)

$$\log t_{\text{inst}} \simeq c_1 (b/r_H) + c_2$$

(e.g., Chambers+ 1996; Yoshinaga, EK & Makino 1999)
Timescale of Orbital Instability

\[ \log t_{\text{inst}} \simeq c_1 \left( \frac{b_{\text{ini}}}{r_H} \right) + c_2 \]

(Yoshinaga, EK & Makino 1999)
Example Runs

\[ \Sigma_1 = 10, \alpha = 3/2, b = 10r_H \]

\[ r = 0.1-0.3 \text{ au} \]

\[ r = 0.5-1.5 \text{ au} \]

\[ N : 24 \rightarrow 5 \]
compact, dynamically cold

\[ N : 16 \rightarrow 3 \]
sparse, dynamically hot
System Parameters

Mass Distribution

- most massive: $M_1/M_{\text{tot}} (0.51)$
- dispersion: $\sigma_M/\bar{M} (0.85)$

Orbital Structure

- mass-weighted orbital elements: $\langle a \rangle_M, \langle e \rangle_M, \langle i \rangle_M (0.90 \text{ au, 0.022, 0.034})$
- mean orbital separation: $\tilde{b} = b/r_H (43)$
- mean eccentricity: $\tilde{e} = ea/r_H (10)$
- angular momentum deficit (AMD): $(0.0018)$

$$D = \frac{\sum_j M_j \sqrt{a_j} \left(1 - \sqrt{1 - e_j^2 \cos i_j}\right)}{\sum_j M_j \sqrt{a_j}} \approx \frac{\sum_j M_j (e_j^2 + i_j^2)/2}{\sum_j M_j} \quad (\text{Hill's approximation})$$

(solar system terrestrial planets)
System Radius Dependence

\[ \Sigma_1 = 10, \alpha = 2, b = 10r_H, \langle e^2 \rangle^{1/2} = 0.01, r = 0.05-0.15, 0.1-0.3, 0.2-0.6, 0.5-1.5 \text{ au} \]

\[ \frac{d \log \langle \tilde{b} \rangle}{d \log \tilde{r}_p} \approx -0.3 \]

(EK+ in prep.)
Orbital Architecture by Giant Impacts

Key Parameter

- physical to Hill radius ratio: \( \tilde{r}_p = \frac{r_p}{r_H} = \left(\frac{9M_*}{4\pi\rho}\right)^{1/3} \left(\frac{1}{a}\right) \)

Large \( \tilde{r}_p \) Effects

- relatively weak scattering and effective collisions \( \rightarrow \)
  smaller \( e \), less mobility \( \rightarrow \)
  local accretion \( \rightarrow \)
  dynamically cold compact system
\[ \Sigma_1 = 10, 30, 100, \alpha = 3/2 - 5/2, b = 5 - 15r_H, r = 0.05 - 0.15, 0.1 - 0.3, 0.2 - 0.6, 0.5 - 1.5 \text{ au} \]

\[ \langle \tilde{e}_{\text{fin}} \rangle \text{ increases with } \langle \tilde{b}_{\text{fin}} \rangle \text{ (with decreasing } \tilde{r}_p) \]

\[ \frac{d \log \langle \tilde{e} \rangle}{d \log \langle \tilde{b} \rangle} \simeq 2 \]

(EK+ in prep.)
Summary

Planetesimal System
• Dissipative self-gravitating particle disk

Planetesimal Dynamics
• Viscous stirring
• Dynamical friction
• Orbital repulsion

Planetesimal Accretion
• Runaway growth of planetesimals
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• Giant impacts