Light nuclei and neutron matter with chiral EFT Hamiltonians

Stefano Gandolfi

Los Alamos National Laboratory (LANL)

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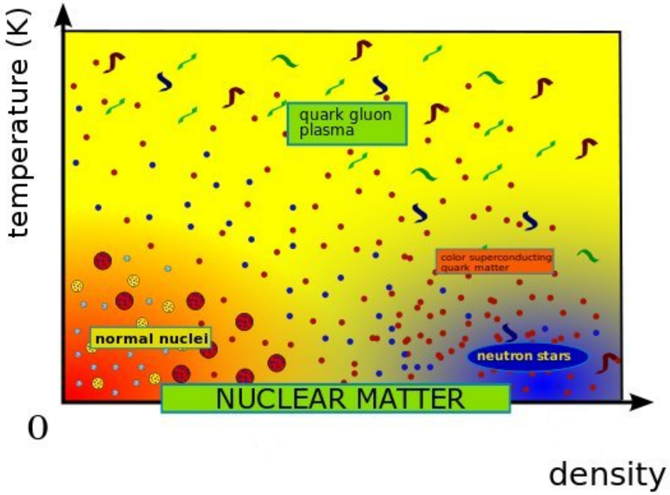
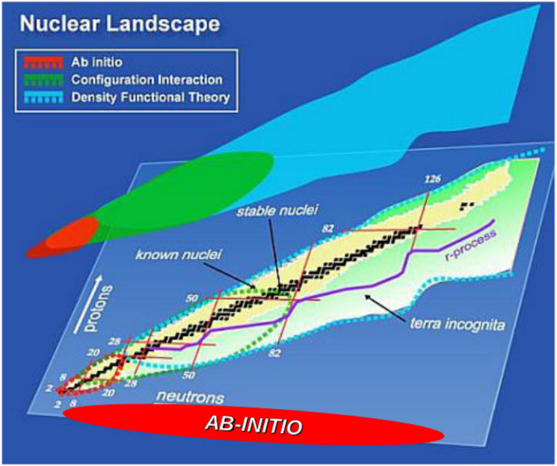
Homogeneous neutron matter



Nuclei



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Outline

- The Hamiltonian and Quantum Monte Carlo methods
- Nuclei and neutron matter with phenomenological Hamiltonians
- Two-body chiral Hamiltonians
- Chiral three-body forces, "technical" issues and open questions
- Results: A=3,4 binding energies, radii, neutron-^{4}He scattering and neutron matter
- Conclusions

Nuclear Hamiltonian

Model: non-relativistic nucleons interacting with an effective nucleon-nucleon force (NN) and three-nucleon interaction (TNI).

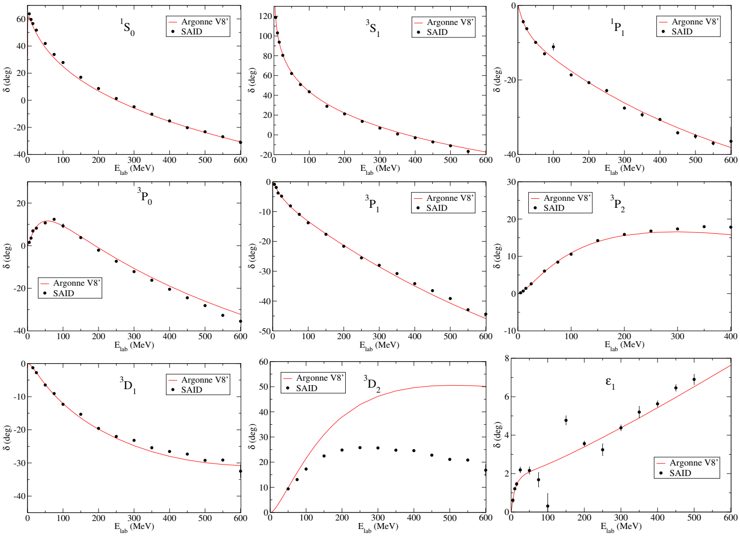
$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{A}\nabla_i^2 + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk}$$

v_{ij} NN fitted on scattering data. Sum of operators:

$$v_{ij} = \sum O_{ij}^{p=1,8} v^p(r_{ij})\,, \quad O_{ij}^p = (1, \vec{\sigma}_i \cdot \vec{\sigma}_j, S_{ij}, \vec{L}_{ij} \cdot \vec{S}_{ij}) \times (1, \vec{\tau}_i \cdot \vec{\tau}_j)$$

- NN: Argonne AV8' and AV18. NNN: Urbana UIX and IL7.
- Local chiral forces up to N^2LO (Gezerlis, Tews, et al. PRL (2013), PRC (2014)).

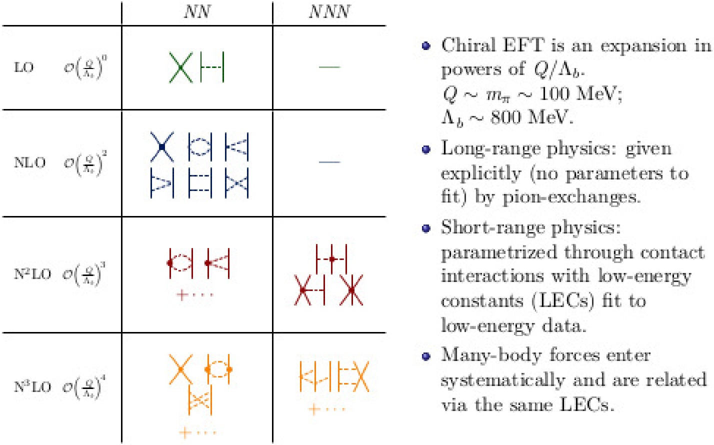
Phase shifts, AV8'



Difference AV8′-AV18 less than 0.2 MeV per nucleon up to A=12.

Nuclear Hamiltonian

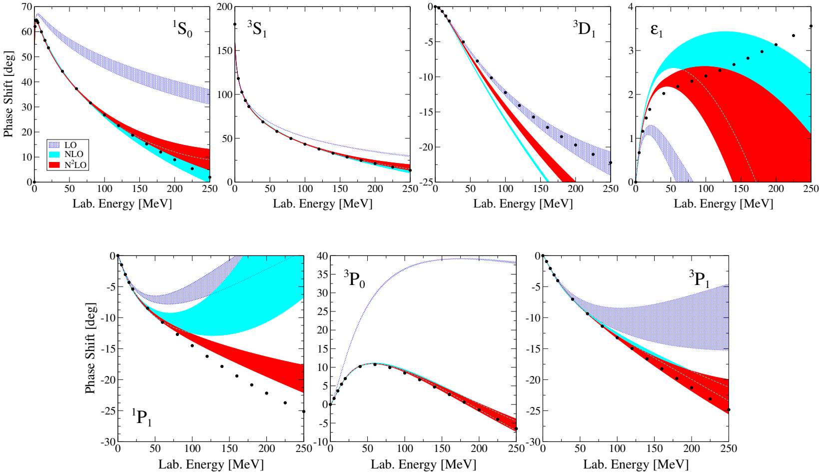
Chiral interactions permit to understand the evolution of theoretical uncertainties with the increasing of A.



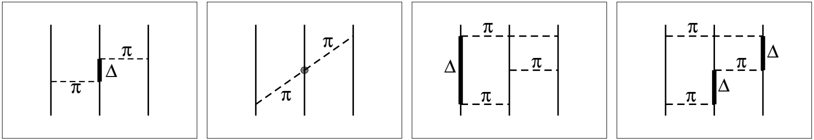
Slide by Joel Lynn, Scidac NUCLEI meeting 2014.

Nuclear Hamiltonian

Phase shifts, LO, NLO and N^2LO with R_0=1.0 and 1.2 fm:

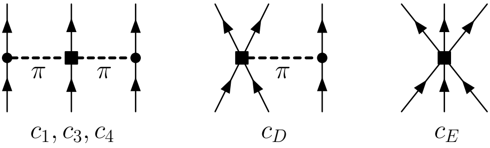


Urbana–Illinois V_{ijk} models processes like



$+$ short-range correlations (spin/isospin independent).

Chiral forces at N^{2}LO:



Nuclear Hamiltonians

Advantages:

- Argonne interactions fit phase shifts up to high energies. At $\rho = \rho_0$, $k_F \simeq 330$ MeV. Two neutrons have $E_{CM} \simeq 120$ MeV, $E_{LAB} \simeq 240$ MeV. $\rightarrow$ accurate up to (at least) 2-3ρ_0. Provide a very good description of several observables in light nuclei.
- Interactions derived from chiral EFT can be systematically improved. Changing the cutoff probes the physics and energy scales entering into observables. They are generally softer, and make most of the calculations easier to converge.

Disadvantages:

- Phenomenological interactions are phenomenological, not clear how to improve their quality. Systematic uncertainties hard to quantify.
- Chiral interactions describe low-energy (momentum) physics. How do they work at large momenta, (i.e. e and ν scattering)?

Important to consider both and compare predictions

Variational wave function

$$E_0 \le E = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\int dr_1 \dots dr_N \, \psi^*(r_1 \dots r_N) H \psi^*(r_1 \dots r_N)}{\int dr_1 \dots dr_N \, \psi^*(r_1 \dots r_N) \psi^*(r_1 \dots r_N)}$$

$\rightarrow$ Monte Carlo integration. Variational wave function:

$$|\Psi_T\rangle = \left[\prod_{i<j} f_c(r_{ij})\right] \left[\prod_{i<j<k} f_c(r_{ijk})\right] \left[1 + \sum_{i<j,p} \prod_k u_{ijk} f_p(r_{ij}) O_{ij}^p\right] |\Phi\rangle$$

where O^p are spin/isospin operators, f_c, u_{ijk} and f_p are obtained by minimizing the energy. About 30 parameters to optimize.

$|\Phi\rangle$ is a mean-field component, usually HF. Sum of many Slater determinants needed for open-shell configurations.

BCS correlations can be included using a Pfaffian.

Quantum Monte Carlo

Propagation in imaginary time:

$$H\,\psi(\vec{r}_1\ldots\vec{r}_N) = E\,\psi(\vec{r}_1\ldots\vec{r}_N) \qquad \psi(t) = e^{-(H-E_T)t}\psi(0)$$

Ground-state extracted in the limit of $t \to \infty$.

Propagation performed by

$$\psi(R,t) = \langle R|\psi(t)\rangle = \int dR'\,G(R,R',t)\psi(R',0)$$

- Importance sampling: $G(R,R',t) \to G(R,R',t)\,\Psi_I(R')/\Psi_I(R)$
- Constrained-path approximation to control the sign problem. Unconstrained-path calculation possible in several cases (exact).

GFMC includes all spin-states of nucleons in the w.f., nuclei up to A=12
AFDMC samples spin states, bigger systems, less accurate than GFMC

Ground–state obtained in a **non-perturbative way.** Systematic uncertainties within 1-2 %.

The Sign problem in one slide

Evolution in imaginary-time:

$$\psi_I(R')\Psi(R',t+dt) = \int dR\, G(R,R',dt)\frac{\psi_I(R')}{\psi_I(R)}\psi_I(R)\Psi(R,t)$$

note: $\Psi(R,t)$ must be positive to be "Monte Carlo" meaningful.

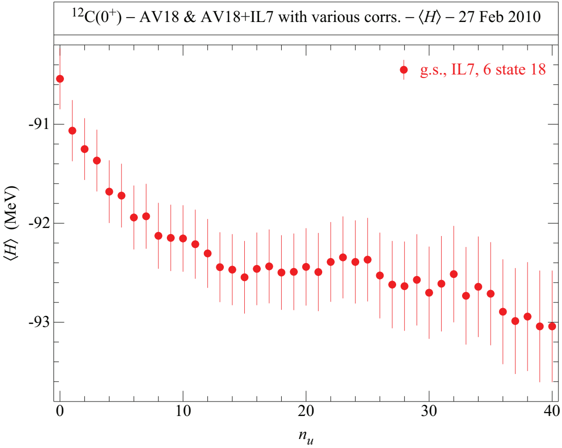
Fixed-node approximation: solve the problem in a restricted space where $\Psi > 0$ (Bosonic problem) $\Rightarrow$ upperbound.

If Ψ is complex:

$$|\psi_I(R')||\Psi(R',t+dt)| = \int dR\, G(R,R',dt)\left|\frac{\psi_I(R')}{\psi_I(R)}\right||\psi_I(R)||\Psi(R,t)|$$

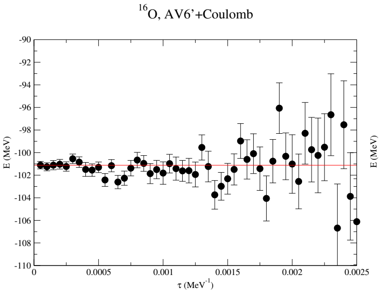
Constrained-path approximation: project the wave-function to the real axis. Extra weight given by $\cos\Delta\theta$ (phase of $\frac{\Psi(R')}{\Psi(R)}$), $Re\{\Psi\} > 0 \Rightarrow$ not necessarily an upperbound.

GFMC unconstrained-path propagation:



Changing the trial wave function gives same results.

AFDMC unconstrained-path propagation:

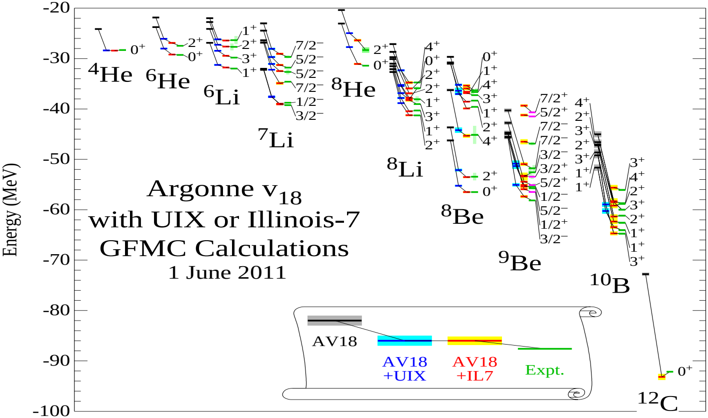


The difference between CP and UP results is mainly due to the presence of LS terms in the Hamiltonian. Same for heavier systems.

Work in progress to improve Ψ to improve the constrained-path.

Phenomenological Hamiltonians
(Argonne plus Urbana/Illinois)

Light nuclei spectrum computed with GFMC

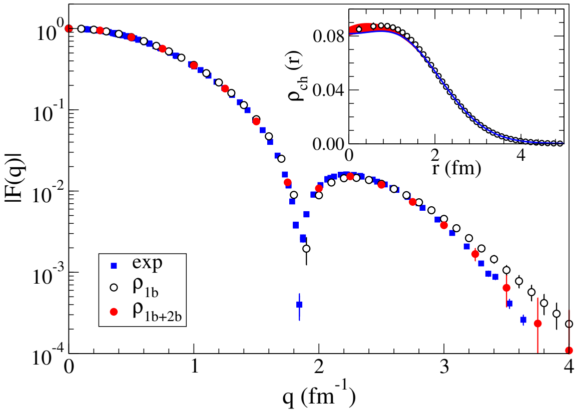


Carlson, Gandolfi, Pederiva, Pieper, Schiavilla, Schmidt, Wiringa, RMP (2015)

Also radii, densities, matrix elements, ...

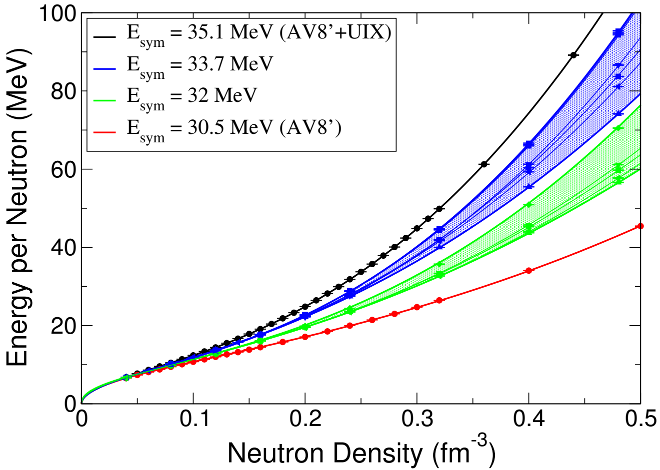
Charge form factor of ^{12}C

$$|F(q)| = \langle\psi|\rho_q|\psi\rangle \qquad \rho_q = \sum_i \rho_q(i) + \sum_{i<j} \rho_q(ij)$$



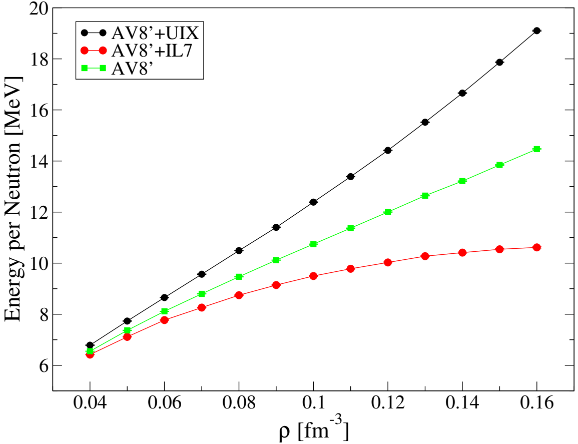
Lovato, Gandolfi, Butler, Carlson, Lusk, Pieper, Schiavilla, PRL (2013)

Model uncertainty vs E_{sym} uncertainty:



Gandolfi, Carlson, Reddy, PRC (2012)

Neutron matter and the deficiencies of three-body forces



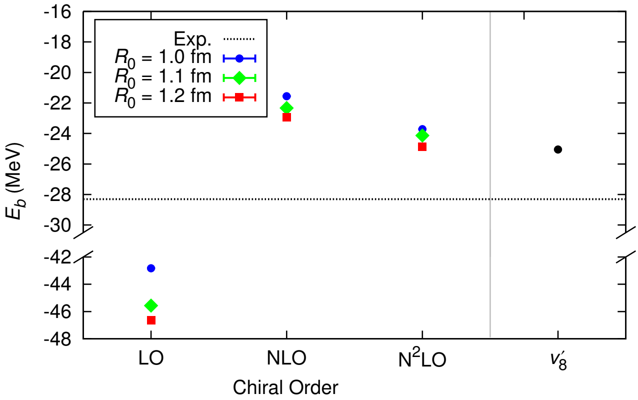
Maris, Vary, Gandolfi, Carlson, Pieper, PRC (2013)

Note: AV8'+UIX and (almost) AV8' are stiff enough to support observed neutron stars. AV8'+IL7 too soft. → How to reconcile with nuclei???

Local chiral forces

^{4}He energy with chiral two-body interactions.

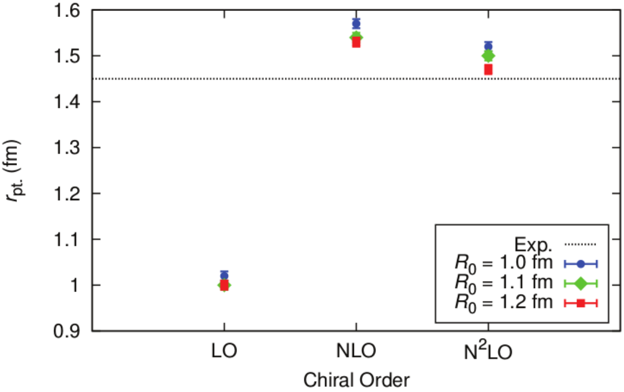
Binding energy of ^{4}He with only two-body interactions:



Lynn, Carlson, Epelbaum, Gandolfi, Gezerlis, Schwenk, PRL (2014).

^{4}He radii with chiral two-body interactions.

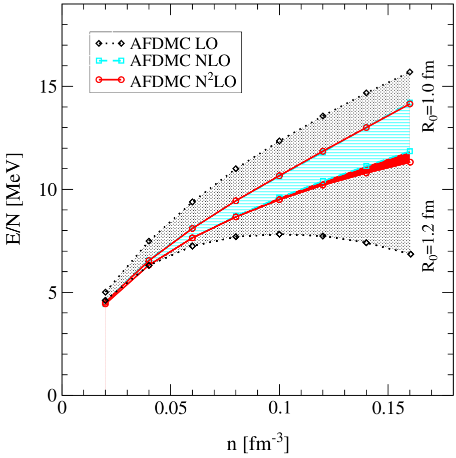
Charge radius of ^{4}He with only two-body interactions:



Lynn, Carlson, Epelbaum, Gandolfi, Gezerlis, Schwenk, PRL (2014).

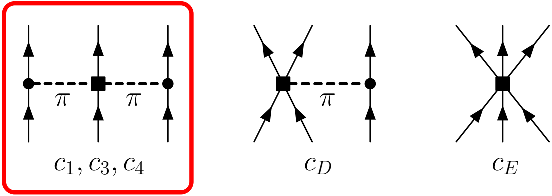
Neutron matter

Equation of state of neutron matter using NN chiral forces:



Gezerlis, Tews, *et al.*, PRL (2013), PRC (2014)

Chiral three-body forces, issue (I)

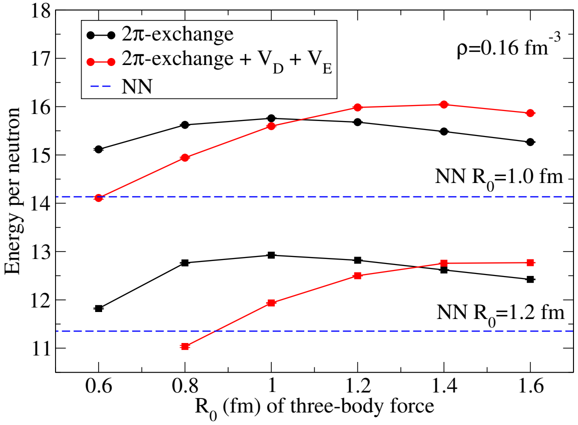


For a finite cutoff, there are "additional" V_D and V_E diagrams coming from Fourier transforming 2π exchange.

Usually they are effectively reabsorbed trough the fit of c_D and c_E, but often neglected in existing neutron matter calculations.

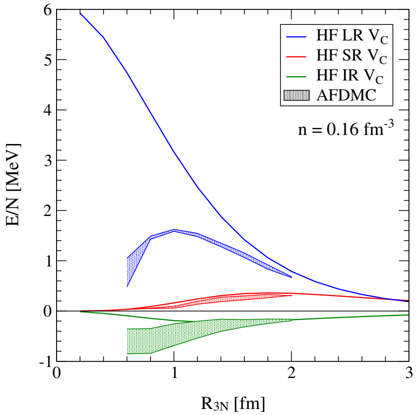
Neutron matter with chiral forces

Contribution of the "additional" V_D and V_E terms, with c_D=c_E=0. AFDMC calculations.



Note: Contribution of FM (2π exchange) about 0.9 MeV with AV8'+UIX.

Hartree-Fock and AFDMC contributions of 2π exchange and the additional V_D and V_E terms, with $c_D = c_E = 0$.

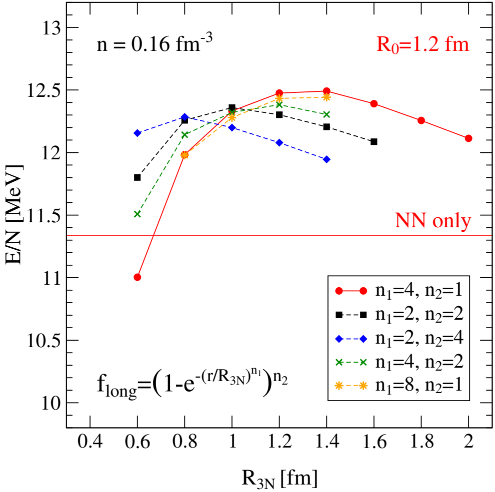


Tews, Gandolfi, Gezerlis, Schwenk, arXiv:1507.05561.

AFDMC bands obtained by using R=1.0 and 1.2 fm in the NN cutoff.

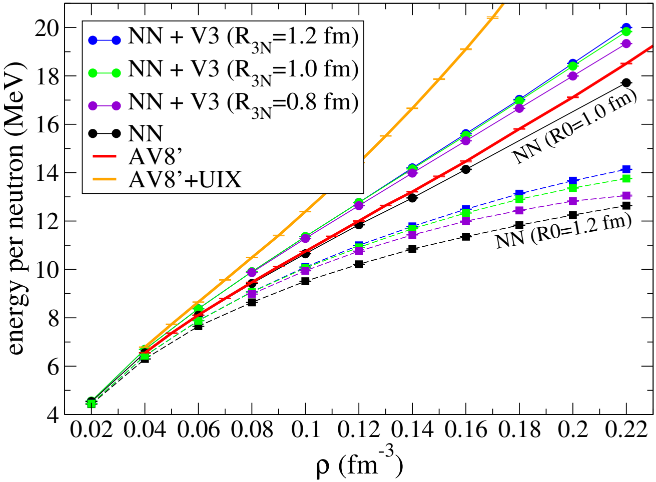
Neutron matter with chiral forces

Exploring the form of the regulator and the cutoff:



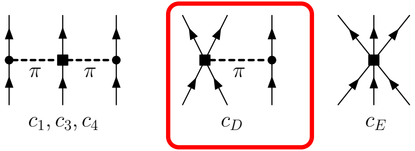
Tews, Gandolfi, Gezerlis, Schwenk, arXiv:1507.05561.

Equation of state of neutron matter at N^2LO.



Note: $c_D = c_E = 0$ (they will be non-zero in a few slides).

Chiral three-body forces, issue (II)



In the Fourier transformation of V_D two possible operator structures arise:

$$V_{D1} = \frac{g_A c_D m_\pi^2}{96\pi \Lambda_\chi F_\pi^4} \sum_{i<j<k} \sum_{cyc} \tau_i \cdot \tau_k \left[X_{ik}(r_{kj})\delta(r_{ij}) + X_{ik}(r_{ij})\delta(r_{kj}) - \frac{8\pi}{m_\pi^2}\sigma_i \cdot \sigma_k \delta(r_{ij})\delta(r_{kj}) \right]$$

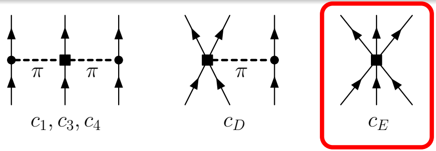
$$V_{D2} = \frac{g_A c_D m_\pi^2}{96\pi \Lambda_\chi F_\pi^4} \sum_{i<j<k} \sum_{cyc} \tau_i \cdot \tau_k \left[X_{ik}(r_{ik}) - \frac{4\pi}{m_\pi^2}\sigma_i \cdot \sigma_k \delta(r_{ik}) \right] \left[\delta(r_{ij}) + \delta(r_{kj}) \right]$$

$$X_{ij}(r) = T(r)S_{ij} + Y(r)\sigma_i \cdot \sigma_j$$

Navratil (2007), Tews et al (2015), Lynn et al (2015).

Equivalent only in the limit of an infinite cutoff. Implications in real life?

Chiral three-body forces, issue (III)



Equivalent forms of operators entering in V_E (or combinations of them):

$$1, \quad \sigma_i \cdot \sigma_j, \quad \tau_i \cdot \tau_j, \quad \sigma_i \cdot \sigma_j \tau_i \cdot \tau_j, \quad \sigma_i \cdot \sigma_j \tau_i \cdot \tau_k, \quad [(\sigma_i \times \sigma_j) \cdot \sigma_k][(\tau_i \times \tau_j) \cdot \tau_k]$$

Epelbaum et al (2002). We investigated three choices:

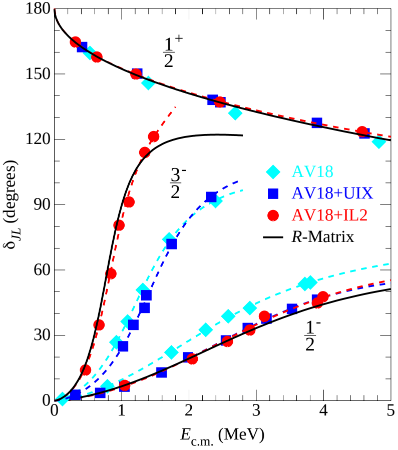
$$V_{E\tau} = \frac{c_E}{\Lambda_\chi F_\pi^4} \sum_{i<j<k} \sum_{\text{cyc}} \tau_i \cdot \tau_k \delta(r_{kj}) \delta(r_{ij})$$

$$V_{E1} = \frac{c_E}{\Lambda_\chi F_\pi^4} \sum_{i<j<k} \sum_{\text{cyc}} \delta(r_{kj}) \delta(r_{ij})$$

$$V_{E\mathcal{P}} = \frac{c_E}{\Lambda_\chi F_\pi^4} \sum_{i<j<k} \sum_{\text{cyc}} \mathcal{P}_{S,T=1/2} \delta(r_{kj}) \delta(r_{ij})$$

Qualitative differences expected, i.e. consider ^{4}He vs neutron matter!

Coefficients c_D and c_E fit to reproduce the binding energy of ^{4}He and neutron-^{4}He scattering. $\rightarrow$ more information on T=3/2 part of three-body interaction.

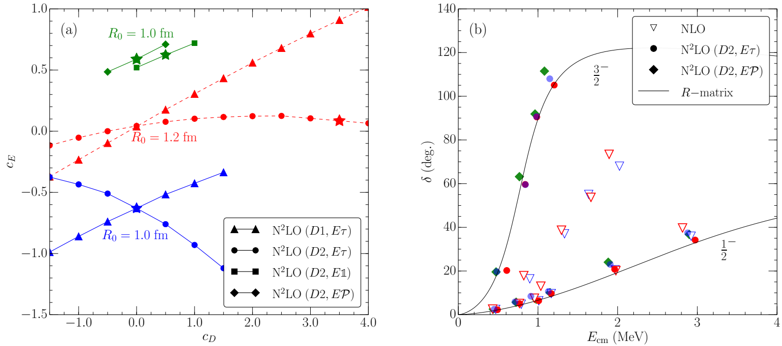


GFMC neutron-^{4}He results using Argonne Hamiltonians.

Nollett, Pieper, Wiringa, Carlson, Hale, PRL (2007).

Regulator: $\delta(r) = \frac{1}{\pi\Gamma(3/4)R_0^3}\exp(-r/R_0)^4$

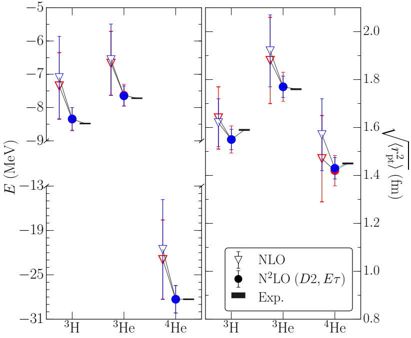
Cutoff R_0 taken consistently with the two-body interaction.



No fit to both observables can be obtained for $R_0 = 1.2$ fm and V_{D1}

Lynn, Tews, Carlson, Gandolfi, Gezerlis, Schmidt, Schwenk (2015).

A=3, 4 nuclei at N2LO

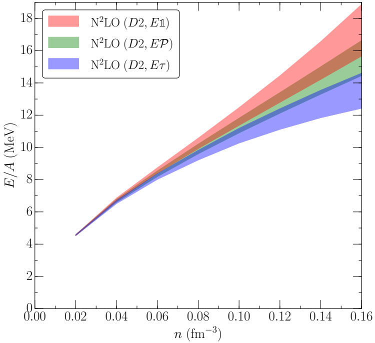


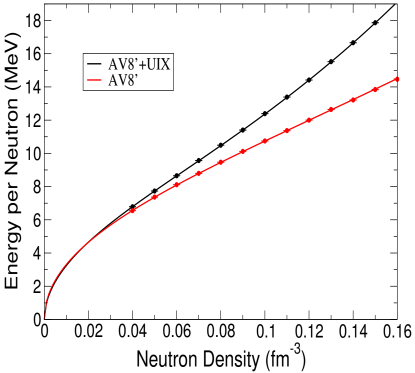
Error quantification: define $Q = \max\left(\frac{p}{\Lambda_b},\ \frac{m_\pi}{\Lambda_b}\right)$ and calculate:

$$\Delta(N2LO) = \max\left(Q^4|\hat{O}_{LO}|,\ Q^2|\hat{O}_{LO} - \hat{O}_{NLO}|,\ Q|\hat{O}_{NLO} - \hat{O}_{N2LO}\right)$$

Epelbaum, Krebs, Meissner (2014).

EOS of pure neutron matter at N2LO, R_0=1.0 fm.
Error quantification estimated as previously.





Lynn, Tews, Carlson, Gandolfi, Gezerlis, Schmidt, Schwenk (2015).

Summary

- Ab-initio QMC methods useful to study nuclear systems in a coherent framework using phenomenological and local chiral forces.
- Spectrum of nuclei and other properties well reproduced with Argonne Hamiltonians. Still problems in describing neutron matter.
- Many ambiguities regarding the choice of three-body operators. Effect in heavier nuclei and nuclear matter?
  Provocation: Same issue for NN???
- (some) local chiral interaction describe A=3,4,5 and neutron matter.

Acknowledgments:

- Joe Carlson (LANL)
- Joel Lynn, Ingo Tews, Achim Schwenk (Darmstadt)
- Alex Gezerlis (Guelph)
- Kevin Schmidt (ASU)
- Evgeny Epelbaum (Bochum)

Extra slides

Scattering data and neutron matter

Two neutrons have

$$k \approx \sqrt{E_{lab}\, m/2}\,, \qquad \rightarrow k_F$$

that correspond to

$$k_F \rightarrow \rho \approx (E_{lab}\, m/2)^{3/2}/2\pi^2\,.$$

E_{lab}=150 MeV corresponds to about 0.12 fm^{-3}.

E_{lab}=350 MeV to 0.44 fm^{-3}.

Argonne potentials useful to study dense matter above ρ_0=0.16 fm^{-3}

$$H\,\psi(\vec{r}_1\ldots\vec{r}_N) = E\,\psi(\vec{r}_1\ldots\vec{r}_N) \qquad \psi(t) = e^{-(H-E_T)t}\psi(0)$$

Ground-state extracted in the limit of $t \to \infty$.

Propagation performed by

$$\psi(R,t) = \langle R|\psi(t)\rangle = \int dR' G(R,R',t)\psi(R',0)$$

- Importance sampling: $G(R,R',t) \to G(R,R',t)\,\Psi_I(R')/\Psi_I(R)$
- Constrained-path approximation to control the sign problem. Unconstrained calculation possible in several cases (exact).

Ground–state obtained in a **non-perturbative way.** Systematic uncertainties within 1-2 %.

Overview

Recall: propagation in imaginary-time

$$e^{-(T+V)\Delta\tau}\psi \approx e^{-T\Delta\tau}e^{-V\Delta\tau}\psi$$

Kinetic energy is sampled as a diffusion of particles:

$$e^{-\nabla^2\Delta\tau}\psi(R) = e^{-(R-R')^2/2\Delta\tau}\psi(R) = \psi(R')$$

The (scalar) potential gives the weight of the configuration:

$$e^{-V(R)\Delta\tau}\psi(R) = w\psi(R)$$

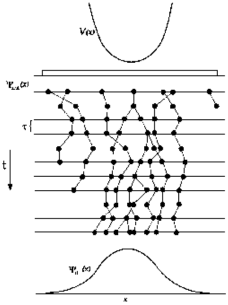
Algorithm for each time-step:

- do the diffusion: $R' = R + \xi$
- compute the weight w
- compute observables using the configuration R' weighted using w over a trial wave function ψ_T.

For spin-dependent potentials things are much worse!

Branching

The configuration weight w is efficiently sampled using the branching technique:



Configurations are replicated or destroyed with probability

$$int[w + \xi]$$

Note: the re-balancing is the bottleneck limiting the parallel efficiency.

GFMC and AFDMC

Because the Hamiltonian is state dependent, all spin/isospin states of nucleons must be included in the wave-function.

Example: spin for 3 neutrons (radial parts also needed in real life):

GFMC wave-function:

$$\psi = \begin{pmatrix} a_{\uparrow\uparrow\uparrow} \\ a_{\uparrow\uparrow\downarrow} \\ a_{\uparrow\downarrow\uparrow} \\ a_{\uparrow\downarrow\downarrow} \\ a_{\downarrow\uparrow\uparrow} \\ a_{\downarrow\uparrow\downarrow} \\ a_{\downarrow\downarrow\uparrow} \\ a_{\downarrow\downarrow\downarrow} \end{pmatrix}$$

A correlation like

$$1 + f(r)\sigma_1 \cdot \sigma_2$$

can be used, and the variational wave function can be very good. Any operator accurately computed.

AFDMC wave-function:

$$\psi = \mathcal{A}\left[\xi_{s_1}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\xi_{s_2}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\xi_{s_3}\begin{pmatrix} a_3 \\ b_3 \end{pmatrix}\right]$$

We must change the propagator by using the Hubbard-Stratonovich transformation:

$$e^{\frac{1}{2}\Delta t O^2} = \frac{1}{\sqrt{2\pi}}\int dx e^{-\frac{x^2}{2} + x\sqrt{\Delta t}O}$$

Auxiliary fields x must also be sampled. The wave-function is pretty bad, but we can simulate larger systems (up to $A \approx 100$). Operators (except the energy) are very hard to be computed, but in some case there is some trick!

Propagator

We first rewrite the potential as:

$$V = \sum_{i<j}[v_\sigma(r_{ij})\vec{\sigma}_i\cdot\vec{\sigma}_j + v_t(r_{ij})(3\vec{\sigma}_i\cdot\hat{r}_{ij}\vec{\sigma}_j\cdot\hat{r}_{ij} - \vec{\sigma}_i\cdot\vec{\sigma}_j)] =$$

$$= \sum_{i,j}\sigma_{i\alpha}A_{i\alpha;j\beta}\sigma_{j\beta} = \frac{1}{2}\sum_{n=1}^{3N}O_n^2\lambda_n$$

where the new operators are

$$O_n = \sum_{j\beta}\sigma_{j\beta}\psi_{n,j\beta}$$

Now we can use the HS transformation to do the propagation:

$$e^{-\Delta\tau\frac{1}{2}\sum_n\lambda O_n^2}\psi = \prod_n\frac{1}{\sqrt{2\pi}}\int dx e^{-\frac{x^2}{2}+\sqrt{-\lambda\Delta\tau}xO_n}\psi$$

Computational cost $\approx (3N)^3$.

Three-body forces

Three-body forces, Urbana, Illinois, and local chiral N^{2}LO can be exactly included in the case of neutrons.

For example:

$$O_{2\pi} = \sum_{cyc}\left[\{X_{ij}, X_{jk}\}\{\tau_i \cdot \tau_j, \tau_j \cdot \tau_k\} + \frac{1}{4}[X_{ij}, X_{jk}][\tau_i \cdot \tau_j, \tau_j \cdot \tau_k]\right]$$
$$= 2\sum_{cyc}\{X_{ij}, X_{jk}\} = \sigma_i \sigma_k f(r_i, r_j, r_k)$$

The above form can be included in the AFDMC propagator.

What about three-body forces?

The full inclusion of three-body forces for nuclei/nuclear matter in AFDMC is not possible. Ideas:

- Reduce $V_3 \rightarrow V_2(\rho)$ in the AFDMC propagator, and calculate perturbatively:

$$\delta_3 = \frac{\langle\psi|V_3 - V_2(\rho)|\psi\rangle}{\langle\psi|\psi\rangle}$$

- "Partially" include three-body terms in the propagator: some of them can be treated exactly. Example, Fujita-Miyazawa:

$$O_{2\pi} = \sum_{cyc}\left[\{X_{ij}, X_{jk}\}\{\tau_i\cdot\tau_j, \tau_j\cdot\tau_k\} + \frac{1}{4}[X_{ij}, X_{jk}][\tau_i\cdot\tau_j, \tau_j\cdot\tau_k]\right]$$

$$\Rightarrow O_{2\pi}^{eff} = \alpha\sum_{cyc}\left[\{X_{ij}, X_{jk}\}\{\tau_i\cdot\tau_j, \tau_j\cdot\tau_k\}\right]$$

and calculate the difference perturbatively.

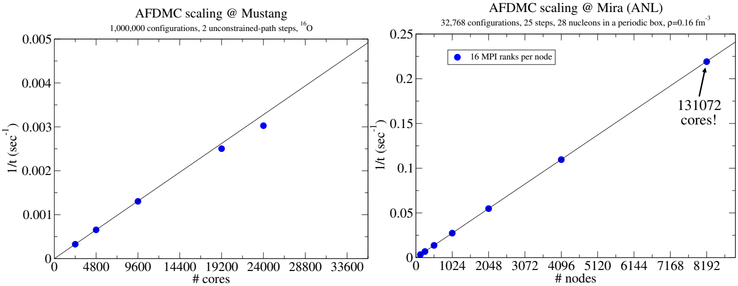
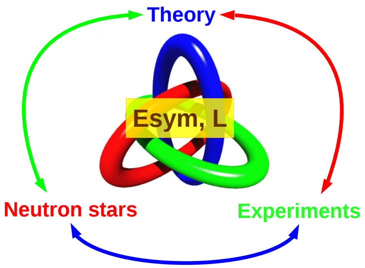


Figure : Efficiency of the AFDMC code. On Mustang we tested the unconstrained-path AFDMC for the ^{16}O (left panel), and the constrained-path version using fewer configurations on Mira for 28 nucleons in a box (right panel).

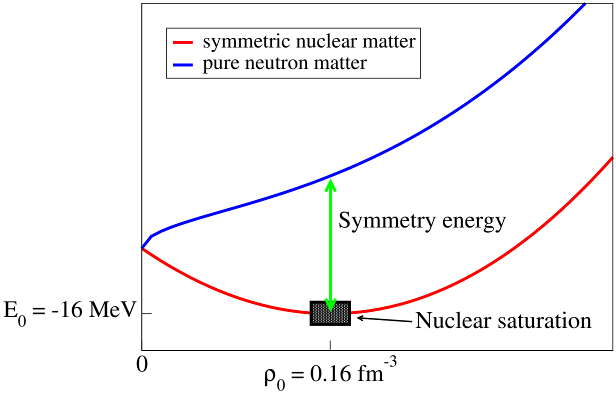
Neutron matter equation of state

Neutron matter is an "exotic" system. Why do we care?

- EOS of neutron matter gives the symmetry energy and its slope.
- The three-neutron force ($T = 3/2$) very weak in light nuclei, while $T = 1/2$ is the dominant part. No direct $T = 3/2$ experiments available.
- Determines radii of neutron stars.



What is the Symmetry energy?



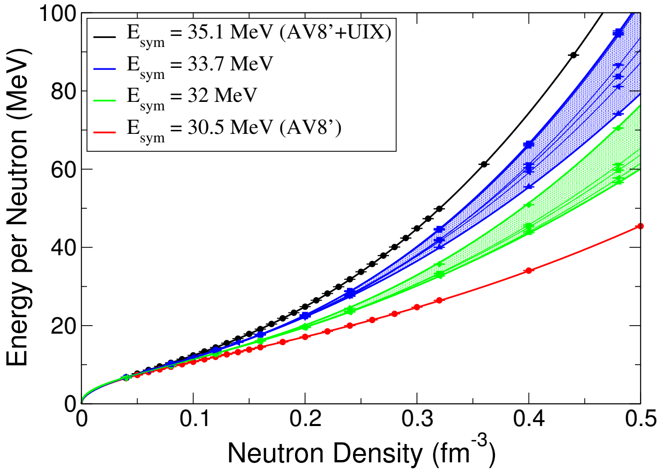
Assumption from experiments:

$$E_{SNM}(\rho_0) = -16MeV\ , \quad \rho_0 = 0.16fm^{-3}\ , \quad E_{sym} = E_{PNM}(\rho_0) + 16$$

At ρ_0 we access E_{sym} by studying PNM.

Neutron matter

Model uncertainty vs E_{sym} uncertainty:

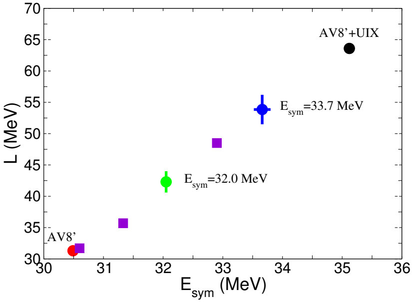


Gandolfi, Carlson, Reddy, PRC (2012)

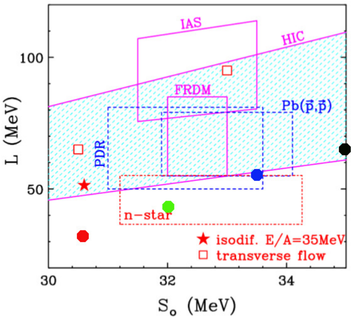
Neutron matter and symmetry energy

From the EOS, we can fit the symmetry energy around ρ_0 using

$$E_{sym}(\rho) = E_{sym} + \frac{L}{3}\frac{\rho - 0.16}{0.16} + \cdots$$



Gandolfi *et al.*, EPJ (2014)

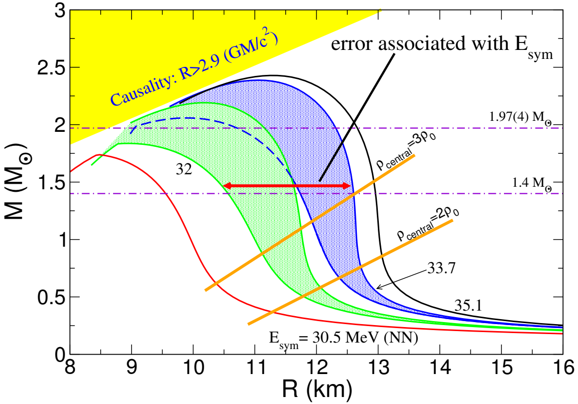


Tsang *et al.*, PRC (2012)

Very weak dependence to the model of 3N force for a given E_{sym}.
Knowing E_{sym} or L useful to constrain 3N! (within this model...)

Neutron star structure

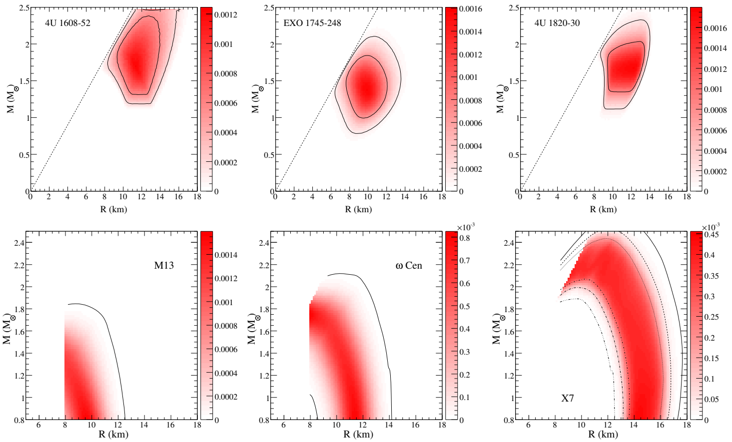
EOS used to solve the TOV equations.



Gandolfi, Carlson, Reddy, PRC (2012).

Accurate measurement of E_{sym} put a constraint to the radius of neutron stars, **OR** observation of M and R would constrain E_{sym}!

Neutron stars



Steiner, Lattimer, Brown, ApJ (2010)

Neutron star observations can be used to 'measure' the EOS and constrain E_{sym} and L. (Systematic uncertainties still under debate...)

Neutron star matter

Neutron star matter model:

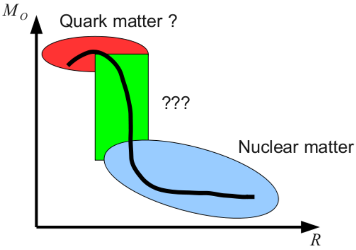
$$E_{NSM} = a\left(\frac{\rho}{\rho_0}\right)^{\alpha} + b\left(\frac{\rho}{\rho_0}\right)^{\beta}, \quad \rho < \rho_t$$

form suggested by QMC simulations,
contrast with the commonly used $E_{FG} + V$

and a high density model for $\rho > \rho_t$

i) two polytropes

ii) polytrope+quark matter model

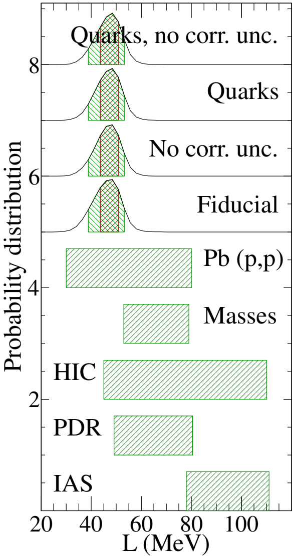


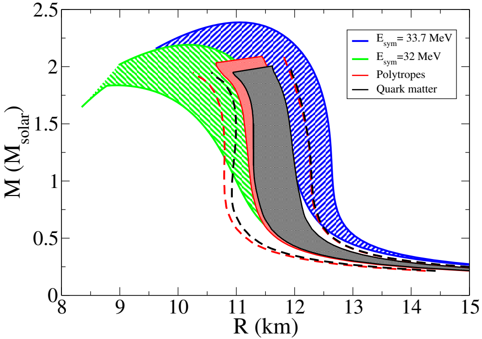
Neutron star radius sensitive to the EOS at nuclear densities!

Direct way to extract E_{sym} and L from neutron stars observations:

$$E_{sym} = a + b + 16\,, \quad L = 3(a\alpha + b\beta)$$

Neutron star matter really matters!





$32 < E_{sym} < 34\ MeV$

$43 < L < 52\ MeV$

Steiner, Gandolfi, PRL (2012).