Eccentric-orbit energy fluxes to 6PN order via analytic functions and spectral integration

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Motivation: EMRIs, GSF/BHP with eccentric orbits

Extreme-mass-ratio inspirals (EMRIs)

- eLISA sources
- Small mass ratio $\mu \ll M \quad \rightarrow \quad$ adiabatic inspiral
- Good application of black hole perturbation (BHP) theory

- Self-consistent approach: local gravitational self-force (GSF)

GSF/BHP applications

- More accurate long-term inspirals
- Tidal invariants

- EOB calibration
- PN comparison

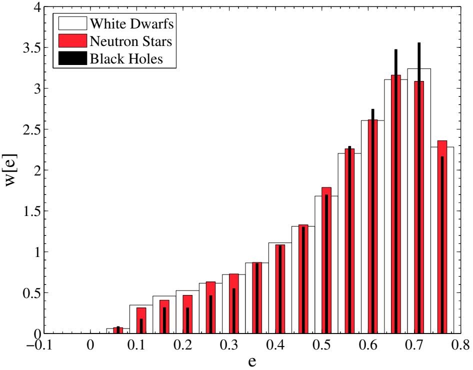
EMRIs likely very eccentric

- Peak $e \simeq 0.7$ (eLISA ingress)

Small eccentricity interesting too

- advLIGO, advVIRGO, KAGRA

Hopman and Alexander (2005)

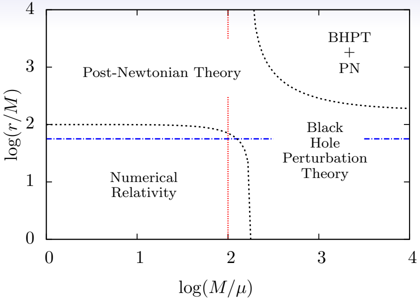


Comparisons between GSF/BHP and PN theory

Comparisons of BHP and PN theory

e.g.,

- Poisson (1993), Cutler etal (1993, 1994)
- Poisson and Sasaki (1995)
- Tagoshi & Nakamura (1994)
- Tagoshi, Shibata, Tanaka, Sasaki (1996)



Comparisons of GSF with PN theory

- Detweiler (2008)
- Sago, Barack, Detweiler (2008)
- Blanchet, Detweiler, Le Tiec, and Whiting (2010,2011)

High accuracy GSF/BHP vs PN theory

- Fujita (2012) [Mano, Suzuki, Takasugi (1996)]
- Bini and Damour (2013, 2014)
- Shah, Friedman, and Whiting (2014)
- Shah (2014)
- Fujita (2014)
- Johnson-McDaniel, Shah, and Whiting (2015)
- Kavanagh, Ottewill, and Wardell (2015)
- Akcay, Le Tiec, Barack, Sago, Warburton (2015)
- Forseth, CRE, Hopper (in preparation)

Extension to eccentric orbits

New results: energy flux at infinity to $\sim$ 6PN order

Simultaneously: $\mu/M \ll 1$ and $x = (\omega M)^{2/3} \ll 1$

Determine PN eccentricity-dependent coefficients

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$$\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^5 \Big[\mathcal{I}_0(e_t) + x\,\mathcal{I}_1(e_t) + x^{3/2}\,\mathcal{K}_{3/2}(e_t) + x^2\,\mathcal{I}_2(e_t)$$
$$+ x^{5/2}\,\mathcal{K}_{5/2}(e_t) + x^3\,\mathcal{I}_3(e_t) + x^3 \ln x\,\mathcal{I}_{3L}(e_t) + x^3\,\mathcal{K}_3(e_t)$$

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$$\mathcal{I}_0(e_t) = \frac{1}{(1-e_t^2)^{7/2}}\left(1 + \frac{73}{24}\,e_t^2 + \frac{37}{96}\,e_t^4\right)$$

Peters-Mathews (1963)

$$\mathcal{I}_1(e_t) = \frac{1}{(1-e_t^2)^{9/2}}\left(-\frac{1247}{336} + \frac{10475}{672}e_t^2 + \frac{10043}{384}e_t^4 + \frac{2179}{1792}e_t^6\right)$$

Full 3PN expressions in Arun et al. 2008; review by Blanchet 2014 (LRR)

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Very high precision calculations

Do calculations at large radii: $r \sim 10^{20} M$

Compute to 200, 600, or 1000's of digits of accuracy

Why? (See also following talks by Shah and Johnson-McDaniel)

Analytically known conservative circular-orbit PN terms:

(Shah, Friedman, and Whiting 2014)

$$\Delta U = \frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{-3872 + 123\pi^2}{96 r^4} + \frac{-592384 - 196608\gamma_E + 10155\pi^2 - 393216\ln(2)}{7680 r^5} + \frac{64\ln(r)}{5r^5} + \frac{-956\ln(r)}{105 r^6} + \frac{-13696\pi}{525 r^{6.5}} + \frac{-51256\ln(r)}{567 r^7} + \frac{81077\pi}{3675 r^{7.5}} + \frac{27392\ln^2(r)}{525 r^8} + \frac{82561159\pi}{467775 r^{8.5}} + \frac{-27016\ln^2(r)}{2205 r^9} + \frac{-11723776\pi\ln(r)}{55125 r^{9.5}} + \frac{-4027582708\ln^2(r)}{9823275 r^{10}} + \frac{99186502\pi\ln(r)}{1157625 r^{10.5}} + \frac{23447552\ln^3(r)}{165375 r^{11}}$$

Compare numerical result at $r = 10^{10} M$:

$$\Delta U = -1.000000000200000000050000000276879\cdots \times 10^{-10}$$

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Part 1: Method

- Mano-Suzuki-Takasugi (MST) analytic function expansions (1996)
  see Sasaki and Tagoshi 2003/06 (LRR)

- Extended homogeneous solutions (Barack, Ori, Sago 2008)
- EHS with RWZ formalism (Hopper, CRE 2010)

- Fast spectral source integration (SSI)
  Hopper, Forseth, Osburn, CRE (2015), arXiv:1506.04742

MST $\rightarrow$ RWZ $\rightarrow$ EHS $\rightarrow$ SSI (Mathematica)

Solution procedure overview

1. Analytic function expansions (MST) for Teukolsky equation ($a = 0$)

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r-M)\frac{d}{dr} + U_{l\omega}(r)\right] R_{lm\omega}(r) = 0,$$

2. Derive RWZ mode functions $\hat{X}^{-}_{lm\omega}$ and $\hat{X}^{+}_{lm\omega}$

$$\hat{X}^{\rm RW}_{lm\omega} = r^3 \left[\left(\frac{d}{dr}\right) - \frac{i\omega}{f}\right]^2 \left(\frac{1}{r^2} R_{lm\omega}\right)$$

$$\hat{X}^{\rm Z}_{lm\omega} = \left[\lambda(\lambda+1) + \frac{9M^2 f}{r\left[\lambda r + 3M\right]}\right] \hat{X}^{\rm RW}_{lm\omega} + 3Mf\frac{d\hat{X}^{\rm RW}_{lm\omega}}{dr}$$

3. Solve source problem in the RWZ formalism

$$\left[\frac{d^2}{dr_*^2} + \omega_{mn}^2 - V_l(r)\right] X_{lmn}(r) = Z_{lmn}(r) \qquad \text{find} \rightarrow \qquad C^{\pm}_{lmn}$$

4. Find TD solution via EHS

$$\Psi^{\pm}_{lm}(t,r) = \sum_n C^{\pm}_{lmn}\, \hat{X}^{\pm}_{lmn}\, e^{-i\omega_{mn}t}$$

MST analytic function expansion overview

Use Mano, Suzuki, and Takasugi (1996) (MST) method (earlier Leaver)

- $R^{\rm in}_{lm\omega}$ is an expansion (a_n) in hypergeometric functions $p_{n+\nu}$

$$R^{\rm in}_{lm\omega} = e^{-i\omega r_*}\, r^2\, k(r) \sum_{n=-\infty}^{\infty} a_n p_{n+\nu}(r)$$

- Outer solution $R^{\rm up}_{lm\omega}$ from expansion (b_n) in Coulomb wave functions

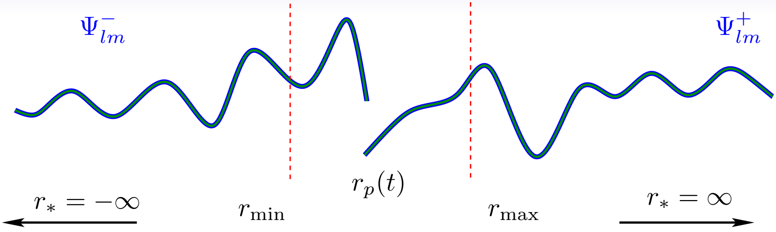
Free parameter: the renormalized angular momentum ν

- Eigenvalue of ν makes recurrence for a_n and b_n convergent $n \to \pm\infty$

Well understood procedure [see Sasaki & Tagoshi (2006) (LRR)]

Implemented in *Mathematica*

Standard source integration



Singular TD (RWZ) source

$$S_{lm}(t,r) = G_{lm}(t)\ \delta[r - r_p(t)] + F_{lm}(t)\ \delta'[r - r_p(t)]$$

Fourier transform: $\mathcal{F}[S_{lm}] = Z_{lmn}(r)$

Integrate for normalization coefficients

$$C^{\pm}_{lmn} = W^{-1}_{lmn} \int_{r_{\min}}^{r_{\max}} \hat{X}^{\mp}_{lmn}(r') Z_{lmn}(r') dr'/f(r')$$

Fast spectral source integration (SSI)

Common step: reverse t and r integrations

$$\bar{E}^{\pm}_{lmn}(t) = \int_{r_{\min}}^{r_{\max}} dr' \, \frac{1}{f(r')} \hat{X}^{\mp}_{lmn}(r') \, \bar{S}_{lm}(t, r') \qquad \text{then,}$$

$$C^{\pm}_{lmn} = \frac{1}{W_{lmn} \, T_r} \int_0^{T_r} dt \, \bar{E}^{\pm}_{lmn}(t) \, e^{in\Omega_r t}$$

Usual procedure: ODE or numerical integration for $C^{\pm}_{lmn}$

- Impossible number of RK steps ($\sim 10^{22}$!) for high accuracy (10^{-200})

Spectral source integration (SSI): replace integral with finite sum

$$C^{\pm}_{lmn} = \frac{1}{N W_{lmn}} \sum_{k=0}^{N-1} \bar{E}^{\pm}_{lmn}(t_k) \, e^{in\Omega_r t_k} \qquad \text{(see arXiv:1506:04742)}$$

Samples of $\bar{E}^{\pm}_{lmn}(t_k)$ at high precision $\Longrightarrow$ $C^{\pm}_{lmn}$ at high precision

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Samples of $\bar{E}^{\pm}_{lmn}(t_k)$ at high precision $\implies$ $C^{\pm}_{lmn}$ at high precision

Fast spectral source integration - why it works

- Start with normal FD decomposition – using Fourier series
- Function like $\bar{E}_{lmn}(t)$ are periodic and C^∞
- Fourier series coefficients fall off exponentially
- Truncate Fourier series to N total coefficients

- Truncated Fourier series is a bandlimited function (trig polynomial)
- Bandlimited function represented by discrete samples (sampling theorem)
- Discrete samples plus periodicity $\rightarrow$ finite number of samples (N)
- Whittaker-Shannon interpolation reproduce BL function from samples
- Fourier series replaced with DFT
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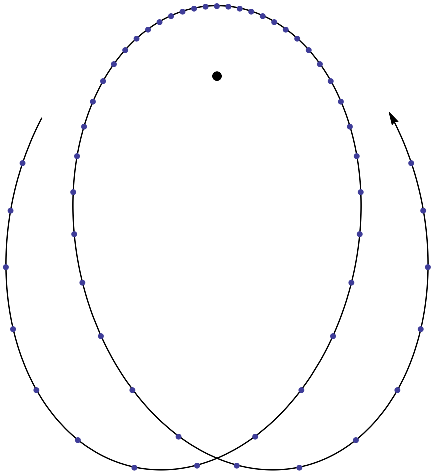
Fast spectral source integration - orbit integration

Example orbit: $e = 0.7$ and $p = 50M$

- Only 22 points are required to integrate orbital motion
- Achieves double precision accuracy ($\sim 10^{-15}$)
- Interpolate anywhere in between

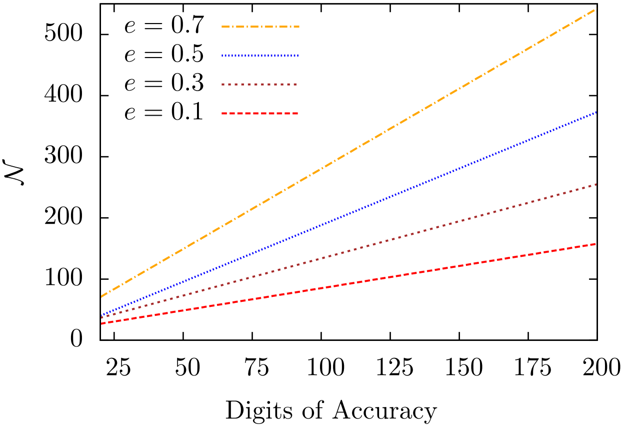
Factor of 50% to 100% more points for source integration

Number of points scales with accuracy requirement



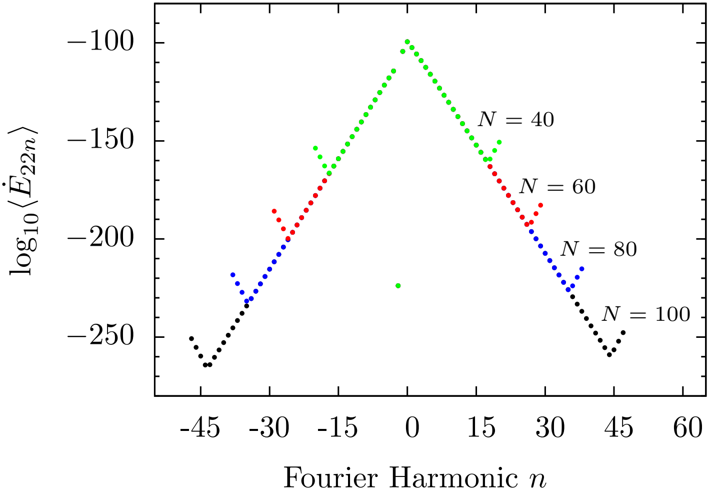
Fast spectral source integration - scaling

Scaling of sampling versus precision goal and eccentricity



Fast spectral source integration - DFT vs FS

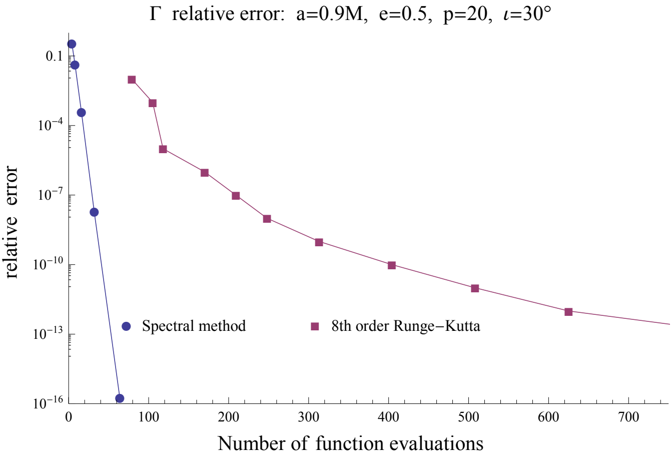
Flux $l = 2$, $m = 2$ from orbit with $p = 10^{20}M$, $e = 0.01$



SSI for generic Kerr - in progress

Kerr generic orbit spectral integration first

Kerr source integration – in the works



Part 1 Conclusion on new SSI method

Fast spectral source integration (SSI) for biperiodic eccentric orbits

- MST and EHS can be combined
- Sample points on orbit 100's to few 1000's
- High precisions in fluxes and orbit achieved $(\mathcal{D} = 200+)$
- Works in RWZ gauge and Lorenz gauge

New: SSI working for triperiodic Kerr orbits

Part 2: Energy flux to 6PN order

Overlap region: $\mu/M \ll 1$ and $x = (\omega M)^{2/3} \ll 1$

Verify known flux (green) to 3PN order (Arun et al. 2008)

Determine new PN eccentricity functions (red) at 3.5PN and beyond

$$
\begin{aligned}
\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 x^5 \Big[& \mathcal{I}_0(e_t) + x\,\mathcal{I}_1(e_t) + x^{3/2}\,\mathcal{K}_{3/2}(e_t) + x^2\,\mathcal{I}_2(e_t) \\
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$$

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Known 3PN: mix of instantaneous and hereditary terms

- Blanchet (2014) gives:

$$\langle \dot{E} \rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^5 \Big[\mathcal{I}_0(e_t) + x\mathcal{I}_1(e_t) + x^{3/2}\mathcal{K}_{3/2}(e_t) + x^2\mathcal{I}_2(e_t) + x^{5/2}\mathcal{K}_{5/2}(e_t) + x^3\mathcal{I}_3(e_t) + x^3\mathcal{K}_3(e_t) \Big]$$

- $x \equiv \Omega_\phi^{2/3} = \mathcal{O}(1/R)$

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↑ Circular flux

- $x \equiv \Omega_\phi^{2/3} = \mathcal{O}(1/R)$

Known 3PN: mix of instantaneous and hereditary terms

Peters-Mathews enhancement

- Blanchet (2014) gives:

$$\langle \dot{E} \rangle = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 x^5 \Big[\mathcal{I}_0(e_t) + x\mathcal{I}_1(e_t) + x^{3/2}\mathcal{K}_{3/2}(e_t) + x^2\mathcal{I}_2(e_t) + x^{5/2}\mathcal{K}_{5/2}(e_t) + x^3\mathcal{I}_3(e_t) + x^3\mathcal{K}_3(e_t) \Big]$$

- $x \equiv \Omega_\phi^{2/3} = \mathcal{O}(1/R)$

Known 3PN: mix of instantaneous and hereditary terms

1PN instantaneous correction

- Blanchet (2014) gives:

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Known 3PN: mix of instantaneous and hereditary terms

1.5PN tail correction, etc.

- Blanchet (2014) gives:

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- $x \equiv \Omega_\phi^{2/3} = \mathcal{O}(1/R)$

Instantaneous contributions

- Peters-Mathews (1963):

$$\mathcal{I}_0(e_t) = \frac{1}{(1-e_t^2)^{7/2}} \left(1 + \frac{73}{24} e_t^2 + \frac{37}{76} e_t^4\right)$$

- 1PN:

$$\mathcal{I}_1(e_t) = \frac{1}{(1-e_t^2)^{9/2}} \left(-\frac{1247}{336} + \frac{10475}{672} e_t^2 + \frac{10043}{384} e_t^4 + \frac{2179}{1792} e_t^6\right)$$

- 2PN and 3PN also have closed forms

- Note: factors that are singular as $e_t \to 1$

Hereditary contributions

- 1.5PN:

$$\mathcal{K}_{3/2}(e_t) = 4\pi\varphi(e_t) = 4\pi\left(1 + \frac{2335}{192}e_t^2 + \mathcal{O}(e_t^4)\right)$$

- 2.5PN:

$$\mathcal{K}_{5/2}(e_t) = -\frac{8191}{672}\pi\psi(e_t) = -\frac{8191}{672}\pi\left(1 - \frac{22988}{8191}e_t^2 + \mathcal{O}(e_t^4)\right)$$

- 3PN:

$$\mathcal{K}_3(e_t) = -\frac{116761}{3675}\kappa(e_t) + \left[\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma - \frac{1712}{105}\ln\left(\frac{4x^{3/2}}{x_0}\right)\right]F(e_t)$$

[Arun, et. al. 2008]

- Enhancement functions φ, ψ, and κ not closed!
- Given through first three terms only!

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Hereditary contributions

- We had to discover how to expand φ, ψ, κ to arbitrary order in e_t
- 1.5PN tail:

$$\mathcal{K}_{3/2}(e_t) = \frac{4\pi}{(1-e_t^2)^5}\left(1 + \frac{1375}{192}e_t^2 + \frac{3935}{768}e_t^4 + \frac{10007}{36864}e_t^6 + \cdots\right)$$

- 2.5PN tail:

$$\mathcal{K}_{5/2}(e_t) = \frac{-8191\pi}{672(1-e_t^2)^6}\left(1 - \frac{72134}{8191}e_t^2 + \frac{19817891}{524224}e_t^4 + \frac{62900483}{4718016}e_t^6 + \cdots\right)$$

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- Discovered singular factors
- Found almost arbitrarily deep expansions

Definitions of eccentricity

- e_t is "time eccentricity" in quasi-Keplerian (QK) representation in MH gauge
- Only defined through 3PN order
- We use Darwin e
- Developed expansion for e_t in powers of e:

$$e_t^2 = e^2 - 6e^2 x + \frac{\left[-15e^6 + \left(15\sqrt{1-e^2} - 19\right)e^2 + \left(34 - 15\sqrt{1-e^2}\right)e^4\right]x^2}{-e^4 + 2e^2 - 1} + \cdots$$

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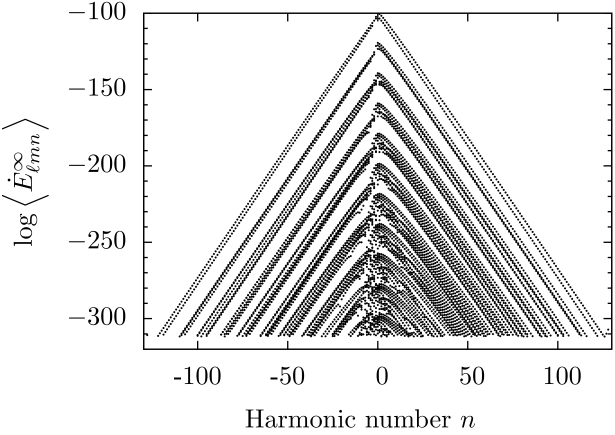
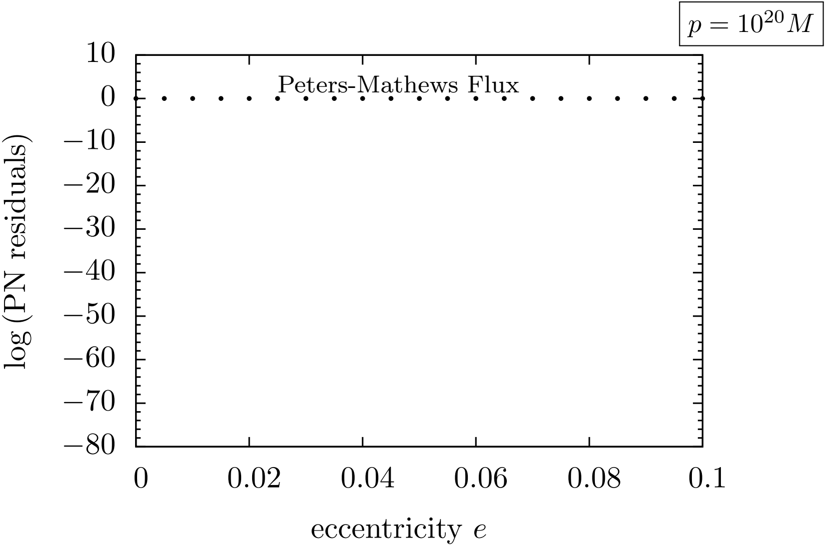
Energy fluxes from *one* orbit

Orbit: $p = 10^{20}M$, $e = 0.1$, digits of accuracy $\mathcal{D} = 200$, total modes > 1000



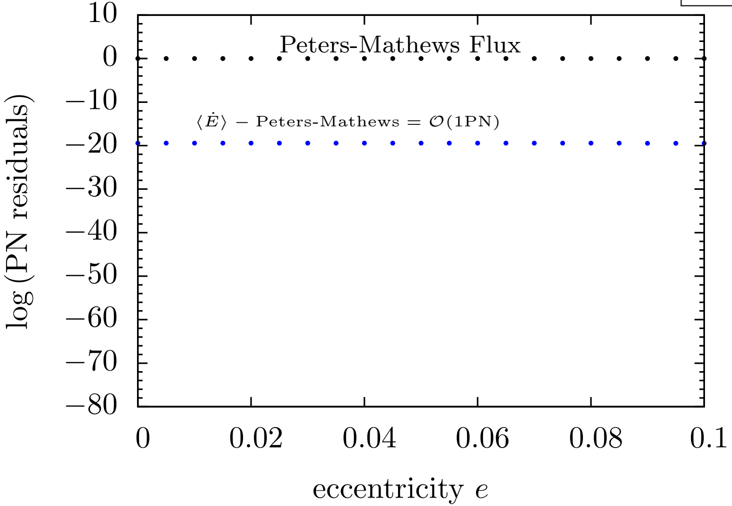
Whole dataset: ~ 1700 orbits; 51 radii and 33 eccentricities

Normalize fluxes to Peters-Mathews result

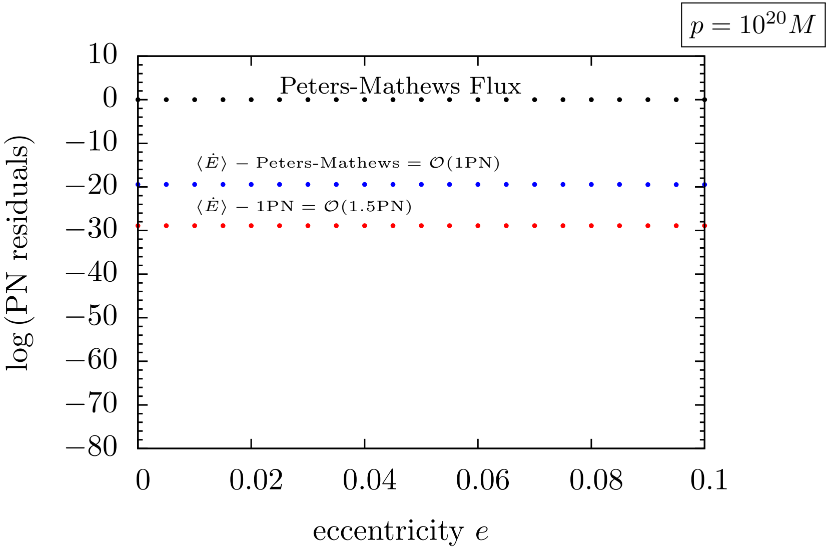


$\ell = 2$ quadrupole flux

$p = 10^{20} M$



$\ell \leq 3$ quadrupole and octupole flux



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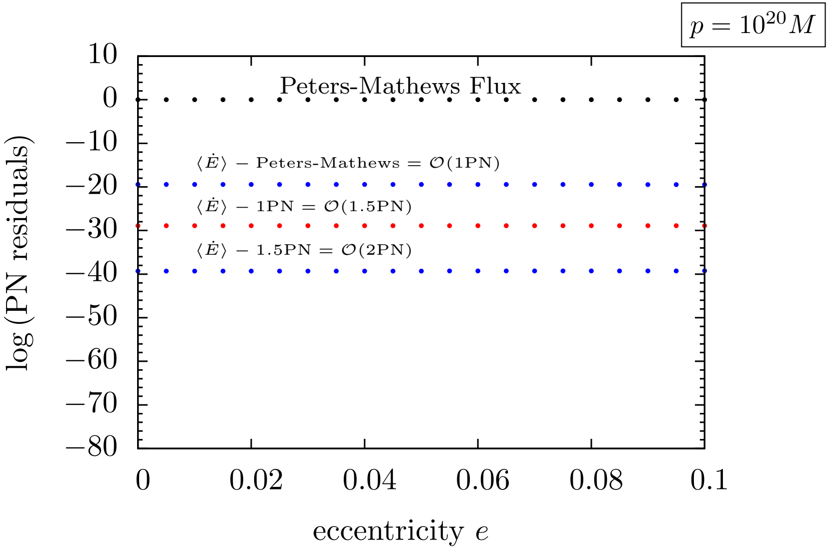
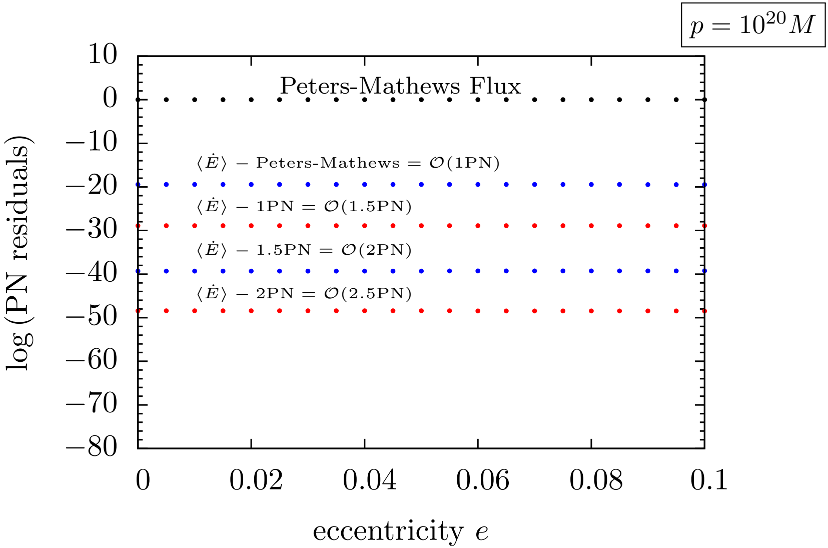


$\ell \leq 4$ flux



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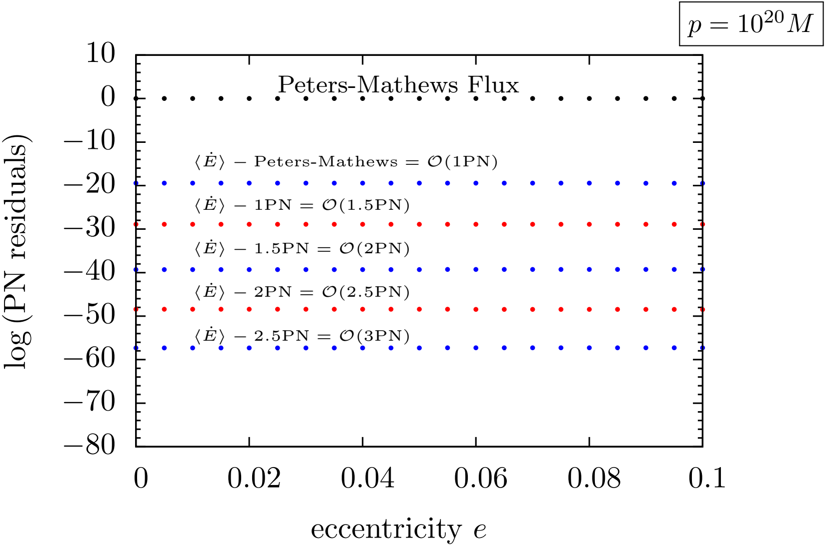
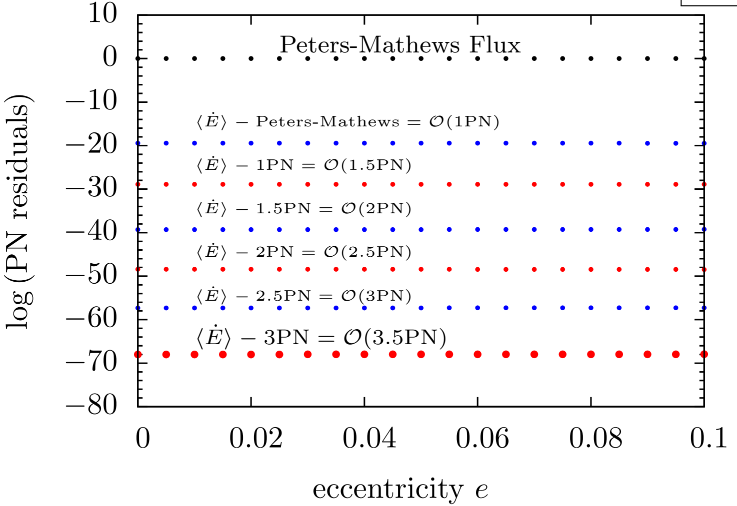


$\ell \leq 5$ flux

$p = 10^{20} M$



Fitting new terms: example 3.5PN

- For a given e, fit over terms in x:

$$\langle \dot{E} \rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^5 \left(\mathcal{I}_0 + \cdots + \mathcal{L}_{7/2} x^{7/2} + \mathcal{L}_4 x^4 + \cdots \right)$$

Then, fit coefficients over e.

- For 3.5PN, we expect the form

$$\mathcal{L}_{7/2} = -\frac{16285\pi}{504(1-e^2)^7} \left(1 + a_2 e^2 + a_4 e^4 + \cdots \right).$$

- Use integer relation algorithm (PSLQ) to find rational and/or irrational parameters $\{a_2, a_4, \ldots\}$.

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- Fit gives

$$a_2 = 13.7525630692866646197932657865111042881997748439259031828881383686418994985160167843598422334113482417705119883034494121793$$

- This agrees with

$$\frac{21500207}{1563360}$$

to 108 digits.

- Likelihood that this is coincidence is 10^{-93}

- See Abhay's and Nathan's talks

Fitting new terms: example 3.5PN

- Fit gives

$$
\begin{aligned}
a_2 = 13.&7525630692866646197932657865111042881997748439259031 8\\
&2888138368641899498516016784359842233411348241770511 9\\
&883034494121793
\end{aligned}
$$

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3.5PN

- In terms of Darwin e:

$$
\begin{aligned}
\mathcal{L}_{7/2} = -\frac{16285\pi}{504(1-e^2)^7}\Bigg(1 &+ \frac{21500207}{1563360}e^2 + \frac{3345329}{1563360}e^4 - \frac{111594754909}{1350743040}e^6 \\
&- \frac{82936785623}{1800990720}e^8 - \frac{11764982139179}{3457902182400}e^{10} \\
&- \frac{216868426237103}{311211196416000}e^{12} - \frac{30182578123501193}{813298593300480000}e^{14} \\
&- \frac{351410391437739607}{1561533299136921600}e^{16} - \frac{10065633193333377521717}{6745823852271501312000}e^{18} \\
&- \frac{1384335564976030365 91}{131754372114677760000 0}e^{20} - \frac{16836217054749609972406421}{67364621317273603276800000}e^{22} \\
&- \frac{2077866815397007172515220959}{109130686533983237308416000 0}e^{24} + \cdots\Bigg)
\end{aligned}
$$

- Likelihood that e^{24} coefficient is coincidence: 10^{-9}

4PN – Log term

- Now to 4PN (Log)...

$$\langle \dot{E} \rangle = \frac{32}{5} \left(\frac{\mu}{M} \right)^2 x^5 \left(\mathcal{I}_0 + \cdots + \mathcal{L}_{7/2} x^{7/2} + \mathcal{L}_4 x^4 + \cdots \right)$$

- 4PN enhancement $\mathcal{L}_4$ has a piece multiplying $\log(x)$, which we found:

$$\mathcal{L}_4^{\log} = \frac{232597}{8820(1-e^2)^{15/2}} \left(1 + \frac{14770533}{465194} e^2 + \frac{142278179}{930388} e^4 + \frac{318425291}{1860776} e^6 + \frac{1256401651}{29772416} e^8 + \frac{64986219}{59544832} e^{10} \right)$$

- Closed-form!

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- Circular term is known

$$-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_E - \frac{1369}{126}\pi^2 + \frac{39931}{294}\log(2) - \frac{47385}{1568}\log(3)$$

- Coefficient on e^2 is

$$-\frac{128412398137}{23543520} + \frac{4923511}{2940}\gamma_E - \frac{104549}{252}\pi^2 - \frac{343177}{252}\log(2) + \frac{55105839}{15680}\log(3)$$

- Likely need to fit (l, m)-mode by (l, m)-mode (Johnson-McDaniel, Shah, Whiting 2015)

4PN – non-Log term

- Now to 4PN (non-Log)...

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Beyond 4PN: 4.5PN and 5PN

- Settle for mixed analytic and numerical form through 6PN order

$$
\begin{aligned}
\mathcal{L}_{9/2} = & -\frac{\pi}{(1-e^2)^8}\Bigg[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_{\rm E} - \frac{13696}{105}\log(2) + \bigg(\frac{5031659060513}{447068160} - \frac{418477}{252}\gamma_{\rm E} - \frac{1024097}{1260}\log(2) \\
& - \frac{702027}{280}\log(3)\bigg)e^2 + h_4 e^4 + h_6 e^6 + h_8 e^8 + h_{10} e^{10} + h_{12} e^{12} + h_{14} e^{14} + h_{16} e^{16} + h_{18} e^{18} + h_{20} e^{20} \\
& + h_{22} e^{22} + h_{24} e^{24} + h_{26} e^{26} + h_{28} e^{28} + h_{30} e^{30} + h_{32} e^{32} + h_{34} e^{34} + h_{36} e^{36}\Bigg], \\
\mathcal{L}_{9/2L} = & -\frac{\pi}{(1-e^2)^8}\bigg(\frac{3424}{105} + \frac{418477}{504}e^2 + \frac{32490229}{10080}e^4 + \frac{283848209}{96768}e^6 + \frac{1378010735}{2322432}e^8 + \frac{59600244089}{4644864000}e^{10} \\
& - \frac{482765917}{7962624000}e^{12} + \frac{532101153539}{29132587008000}e^{14} - \frac{576726373021}{199766310912000}e^{16} + \frac{98932878601597}{36245599451873280000}e^{18} + g_{20} e^{20} \\
& + g_{22} e^{22} + g_{24} e^{24} + g_{26} e^{26} + g_{28} e^{28} + g_{30} e^{30} + \cdots\bigg), \\
\mathcal{L}_5 = & \frac{1}{(1-e^2)^{17/2}}\Bigg[-\frac{2500861660823683}{2831932303200} - \frac{424223}{6804}\pi^2 + \frac{916628467}{7858620}\gamma_{\rm E} - \frac{83217611}{1122660}\log(2) + \frac{47385}{196}\log(3) \\
& + k_2 e^2 + k_4 e^4 + k_6 e^6 + k_8 e^8 + k_{10} e^{10} + k_{12} e^{12} + k_{14} e^{14} + k_{16} e^{16} + k_{18} e^{18} + k_{20} e^{20} + k_{22} e^{22} \\
& + k_{24} e^{24} + k_{26} e^{26} + \cdots\Bigg],
\end{aligned}
$$

Beyond 4PN: 5PN and 5.5PN

- Settle for mixed analytic and numerical form through 6PN order

$$\mathcal{L}_{5L} = \frac{1}{(1-e^2)^{17/2}}\Big(\frac{916628467}{15717240} + \frac{11627266729}{31434480}e^2 - \frac{84010607399}{10478160}e^4 - \frac{67781855563}{1632960}e^6 - \frac{87324451928671}{2011806720}e^8$$
$$- \frac{301503186907}{29804544}e^{10} - \frac{752883727}{1290240}e^{12} - \frac{22176713}{129024}e^{14} - \frac{198577769}{2064384}e^{16} - \frac{250595605}{4128768}e^{18}$$
$$- \frac{195002899}{4718592}e^{20} - \frac{280151573}{9437184}e^{22} - \frac{1675599991}{75497472}e^{24} + j_{26}e^{26} + j_{28}e^{28} + \cdots\Big),$$

$$\mathcal{L}_{11/2} = \frac{\pi}{(1-e^2)^9}\Big[\frac{8399309750401}{101708006400} + \frac{177293}{1176}\gamma_{\rm E} + \frac{8521283}{17640}\log(2) - \frac{142155}{784}\log(3)$$
$$+ \Big(\frac{9346787}{792} - \frac{7379257}{528}\gamma_{\rm E} + \frac{5529233}{528}\log(2) - \frac{13113409}{792}\log(3) + \frac{37057}{132}\log(5)\Big)e^2$$
$$+ m_4e^4 + m_6e^6 + m_8e^8 + m_{10}e^{10} + m_{12}e^{12} + m_{14}e^{14} + m_{16}e^{16} + m_{18}e^{18} + \cdots\Big],$$

$$\mathcal{L}_{11/2L} = \frac{\pi}{(1-e^2)^9}\Big(\frac{177293}{2352} + \frac{197515529}{35280}e^2 + \frac{22177125281}{451584}e^4 + \frac{362637121649}{3386880}e^6 + \frac{175129893794507}{2601123840}e^8$$
$$+ \frac{137611940506079}{13005619200}e^{10} + l_{12}e^{12} + l_{14}e^{14} + l_{16}e^{16} + \cdots\Big),$$

Beyond 4PN: 5.5PN and 6PN

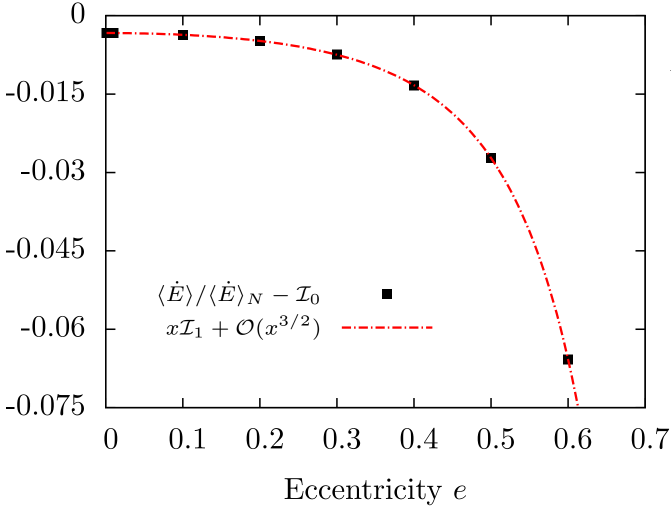
- Settle for mixed analytic and numerical form through 6PN order

$$\mathcal{L}_6 = \frac{1}{(1-e^2)^{19/2}}\Big(\frac{3803225263}{10478160}\pi^2 - \frac{27392}{105}\zeta(3) + \frac{1465472}{11025}\gamma_\mathrm{E}^2 - \frac{256}{45}\pi^4 - \frac{27392}{315}\gamma_\mathrm{E}\pi^2 - \frac{246137536815857}{157329572400}\gamma_\mathrm{E}$$
$$+ \frac{2067586193789233570693}{602387400044430000} + \frac{5861888}{11025}\log^2(2) - \frac{54784}{315}\pi^2\log(2) - \frac{271272899815409}{157329572400}\log(2)$$
$$+ \frac{5861888}{11025}\gamma_\mathrm{E}\log(2) - \frac{37744140625}{260941824}\log(5) - \frac{437114506833}{789268480}\log(3) + n_2 e^2 + n_4 e^4 + n_6 e^6 + n_8 e^8 + n_{10} e^{10}$$
$$+ n_{12} e^{12} + \cdots\Big),$$

$$\mathcal{L}_{6L} = \frac{1}{(1-e^2)^{19/2}}\Big(-\frac{246137536815857}{314659144800} - \frac{13696}{315}\pi^2 + \frac{1465472}{11025}\gamma_\mathrm{E} + \frac{2930944}{11025}\log(2) + p_2 e^2 + p_4 e^4 + p_6 e^6$$
$$+ p_8 e^8 + p_{10} e^{10} + p_{12} e^{12} + \cdots\Big),$$

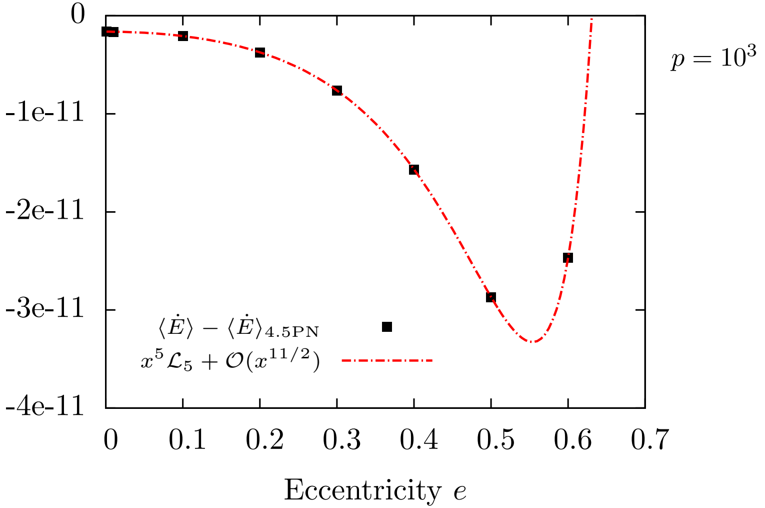
$$\mathcal{L}_{6L^2} = \frac{1}{(1-e^2)^{19/2}}\Big(\frac{366368}{11025} + \frac{189812971}{132300}e^2 + \frac{1052380631}{105840}e^4 + \frac{9707068997}{529200}e^6 + \frac{8409851501}{846720}e^8$$
$$+ \frac{4574665481}{3386880}e^{10} + \frac{6308399}{301056}e^{12} + \cdots\Big).$$

Assessing new results: subtraction of Peters-Mathews vs. 1PN flux expansion

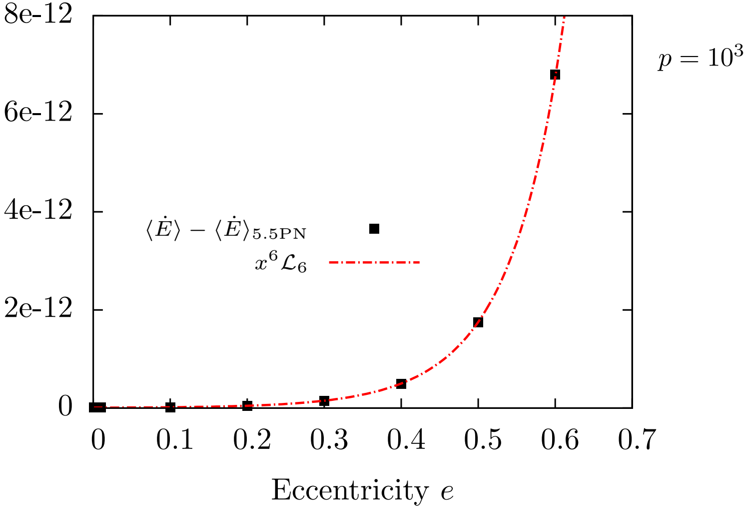


$p = 10^3$

Assessing new results: subtraction of 4.5PN flux vs. 5PN expansion

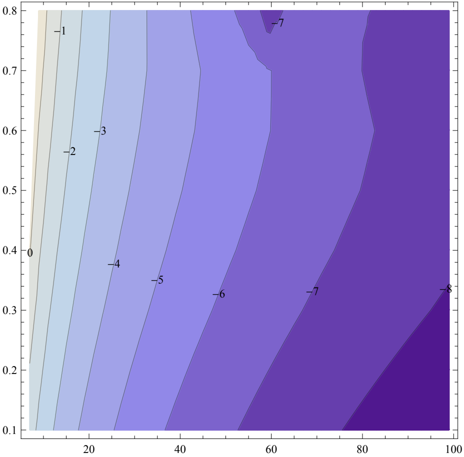


Assessing new results: subtraction of 5.5PN flux vs. 6PN expansion



Extending into strong-field region

- Compare 6PN expansion to direct RWZ calculation
- Robust to $e \simeq 0.7$



Part 2 conclusions

- Confirmed 3PN expansion for eccentric orbits (lowest order in ν)
- Found nearly arbitrary expansions for 1.5PN, 2.5PN, and 3PN tail terms
- Found mixed analytic and numerical terms for 3.5PN through 6PN
- Found eccentricity singular factors
- New expansion extrapolates to $e \gtrsim 0.7$ and $x \gtrsim 0.01$