Low-Lying Mode Contribution to the Quenched Meson Correlators in the $\epsilon$-Regime

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We present a quenched calculation of meson correlators using the overlap fermion with very small masses, 2.6–13 MeV. In this region, the pion Compton wavelength is larger than our lattice size ($L \simeq 1.23$ fm), and the system is in the so-called $\epsilon$-regime of chiral perturbation theory. We find that the scalar and pseudo-scalar correlators are accurately approximated by a few hundred low-lying fermion eigenmodes in this regime, whereas the axial-vector correlator contains significant contributions from higher eigenmodes. We also compute the disconnected pseudo-scalar correlator, which is well saturated with low-lying modes. Matching these lattice data with the one-loop expressions for the correlators in quenched chiral perturbation theory, we evaluate the decay constant $F_\pi$ and the chiral condensate $\Sigma$, as well as the parameters $m_0^2$ and $\alpha$, which describe the artifacts of the quenched approximation.

§1. Introduction

Chiral perturbation theory (ChPT) provides a systematic method to calculate low energy dynamics of QCD, though it contains unknown parameters at each order of the expansion in the pion mass squared, $m_\pi^2$, and momentum squared, $p_\pi^2$. At lowest order, these parameters are the pion decay constant, $F_\pi$, and the chiral condensate, $\Sigma$, and there are ten other low energy constants in the next-to-leading order. It is one of the important tasks of lattice QCD to calculate these low energy constants non-perturbatively from first principles. In the standard approach, however, this is very difficult because one must use large enough lattices, satisfying $m_\pi L \gg 1$, to avoid possible finite size effects. A computation in which the chiral limit $m_\pi \to 0$ and the continuum limit are taken while satisfying this condition would be prohibitively time consuming, even with today’s fastest supercomputers, especially when the quarks are treated dynamically.

In the so-called $\epsilon$-regime, in which the linear extent of the space-time box is smaller than the pion Compton wavelength, $L \ll 1/m_\pi$ (but larger than the QCD scale, $1/\Lambda_{QCD} \ll L$, which insures that the pion can be treated as a point particle and that other heavier hadrons are decoupled), the chiral Lagrangian is still applicable, except that the expansion parameter is given by $\epsilon^2 \sim m_\pi/\Lambda \sim p_\pi^2/\Lambda^2$, where $\Lambda$ is the cutoff scale of the chiral Lagrangian, approximately 1 GeV. An important observation is that the low energy constants in the chiral Lagrangian are defined at
the cutoff scale in the same way for both the standard and $\epsilon$-regimes. Therefore, one can determine the low energy constants in the $\epsilon$-regime and use them in the standard ChPT. In this way, one can avoid the problem of the chiral limit, while keeping a large lattice volume. Analytic calculations of the meson correlation functions in the $\epsilon$-regime have been made for both quenched and unquenched theories.$^5$–$^7$

In the study of the chiral regime of lattice QCD, chiral symmetry plays an essential role. First, one must treat the pions near the massless limit, which appear as a result of spontaneous chiral symmetry breaking. Furthermore, it is known that the effects of fermion zero modes become important in the $\epsilon$-regime,$^8$ and the correlators depend strongly on the topological charge of the background gauge field.$^5$,6) Lattice fermion formulations preserving chiral symmetry$^9$ (by satisfying the Ginsparg-Wilson relation$^{10}$) are now commonly used (but only in the quenched approximation). In this work we use Neuberger’s overlap-Dirac operator.$^{11}$,12) With this formulation, there is no fundamental problem involved in approaching the chiral limit, as required in the study of the $\epsilon$-regime, but the computational cost of inverting the overlap-Dirac operator increases for small quark masses.

In the chiral regime, the meson correlators are strongly affected by the low-lying fermion modes, especially by the chiral zero modes. In this work we explicitly study the effects of such low-lying modes, using the eigenmode decomposition of the fermion propagator. In the quenched approximation, we find that the connected scalar and pseudo-scalar meson correlators are reproduced to 98–99.9% accuracy (depending on the channel) with only 200 lowest-lying eigenmodes on a $10^3 \times 20$ lattice with small quark masses ($2.6 \text{ MeV} \lesssim m \lesssim 13 \text{ MeV}$). Such accuracy obtained with only low-lying eigenmodes was previously found in Refs. 13) and 14), but it should be better in the $\epsilon$-regime. (In our study, $L$ satisfies $m_\pi L \sim 0.6$ and $\Lambda_{QCD} L \sim 3$.) An advantage of such an eigenmode decomposition is that the meson correlators can be averaged over space-time points without much extra computational cost. The statistical fluctuations originating from local bumps in the zero-mode wave function are suppressed by space-time averaging, and thus we can avoid the large noise found in Ref. 15). The low-mode averaging was also used in recent studies.$^{16}$,17)

Matching our numerical data for the axial-vector, scalar and pseudo-scalar correlators with the quenched ChPT (QChPT) expressions, we extract the leading-order low-energy constants $\Sigma$ and $F_\pi$, as well as the parameters appearing as artifacts of the quenching, $\alpha$ and $m_0$. The axial-vector correlator is most sensitive to $F_\pi$, while $\Sigma$ is precisely determined by the connected scalar and pseudo-scalar correlators. We also investigate the chiral condensates and the disconnected (hairpin) correlators for the pseudo-scalar channel. In general, we find good agreement between the lattice data and the QChPT predictions for the topological charge $|Q| = 0$ and 1 sectors, but in the sectors with larger topological charge, these results deviate significantly, which may suggest the breakdown of the $\epsilon$-expansion in the QChPT for large $|Q|$.

The axial-vector correlator has been calculated in recent works.$^{15}$,17) Bietenholz et al.$^{15}$ considered a relatively larger quark mass ($\sim 21 \text{ MeV}$) and found that the correlator is fit well with the QChPT formula for a sufficiently large lattice size ($L > 1.1 \text{ fm}$). They also pointed out that the signal is very noisy for $|Q| = 0$. Using the low-mode averaging technique,$^{16}$,17) Giusti et al. pushed the quark mass down to
10 MeV and found an encouraging agreement between the value of $F_\pi$ measured in the $\epsilon$-regime and that obtained from the standard measurement. Their result is $F_\pi = 102(4)$ MeV in the $\epsilon$-regime. There is another interesting work by the same authors in which they investigated the divergent (as $\sim 1/m^2$) contributions of the zero mode in the massless limit and matched them with the theoretical predictions of QChPT.

This paper is organized as follows. In §2, we review the quenched chiral perturbation theory (QChPT) in the $\epsilon$-regime, following the treatment of Damgaard et al.\textsuperscript{5) We describe the details of our simulation in §3 and study the low-lying eigenmode dominance in §4. In §5 we present our results for the chiral condensate and meson correlators and a comparison with the QChPT. Conclusions are given in §6.

§2. Quenched chiral perturbation theory in the $\epsilon$-regime

In this section, we briefly review quenched chiral perturbation theory (QChPT) in the $\epsilon$-regime\textsuperscript{5} and summarize the formulae relevant to our analysis of meson correlation functions.

The partition function of QChPT with $N_v$ valence quarks is written

$$Z(\theta, M) = \int dU \exp \left( -\int d^4 x \mathcal{L}_M^\theta (x) \right),$$

where the Lagrangian $\mathcal{L}_M^\theta$ is given by

$$\mathcal{L}_M^\theta (x) = \frac{F_\pi^2}{4} \text{Str}[\partial_\mu U (x)^{-1} \partial_\mu U (x)] - \frac{m \Sigma}{2} \text{Str}[U_\theta U (x) + U (x)^{-1} U_\theta^{-1}]$$

$$+ \frac{m_\theta^2}{2N_c} \Phi (x)^2 + \frac{\alpha}{2N_c} \partial_\mu \Phi (x) \partial_\mu \Phi (x)$$

(2.2)

to leading order in the $m^2_\pi$ and $p^2_\pi$ expansion. The field variable $U(x)$ is integrated over a sub-manifold of the super-group $GL(N_v|N_v)$, the maximally symmetric Riemannian sub-manifold, which is characterized by a matrix of the form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A \in U(N_v), \quad D \in GL(N_v)/U(N_v),$$

(2.3)

and Grassmannian $N_v \times N_v$ matrices $B$ and $C$. Here Str denotes the super-trace. The mass term corresponds to the choice of the mass matrix $M = (m I_v + m \bar{I}_v)$, with $I_v$ and $\bar{I}_v$ the identity matrix in the fermion-fermion and boson-boson blocks, respectively. The effect of the CP violating $\theta$ term enters through $U_\theta \equiv \exp(i \theta/N_v) I_{N_v} + \bar{I}_{N_v}$. In the quenched approximation, the singlet field $\Phi (x) \equiv F_\pi \text{Str}[\ln U (x)]$ does not decouple, and the couplings $m_\theta^2$ and $\alpha$ are introduced.\textsuperscript{19)}

The $\epsilon$-regime\textsuperscript{2)–4) is realized when the quark mass is small enough that the pion Compton wavelength, $\sim 1/m_\pi$, is larger than the linear extent of the space-time, $L$. The systematic expansion is then reorganized, and the expansion parameter is given by $\epsilon^2 \sim m_\pi/4\pi F_\pi \sim 1/(LF_\pi)^2$. Unlike in the standard ChPT, in the QChPT,
the zero mode of the pion gives an important contribution and one must explicitly integrate out the constant mode of \( U(x) \). This is done by writing

\[
U(x) = U_0 \exp \left( i \frac{\sqrt{2} \xi(x)}{F_\pi} \right)
\]

and integrating over the constant mode \( U_0 \). The partition function at fixed topological charge \( Q \) is obtained by Fourier transforming (2.1), and we have

\[
Z_Q(M) \equiv \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta e^{iQ\theta} Z(\theta, M) = \frac{1}{\sqrt{2\pi \langle Q^2 \rangle}} e^{-Q^2/2\langle Q^2 \rangle} \int dU_0 d\xi \left( e^{iQ\theta} \right) \exp \left[ \frac{m\Sigma V}{2} \text{Str}(U_0 + U_0^{-1}) \right]
\]

\[+ \int d^4x \left( -\frac{1}{2} \text{Str}(\partial_\mu \xi \partial_\mu \xi) - \frac{m_0^2}{2N_c} (\text{Str}\xi)^2 - \frac{\alpha}{2N_c} (\partial_\mu \text{Str}\xi)^2 \right) + O(\epsilon^4) \],

(2.5)

where \( dU_0 \) denotes the Haar measure of the maximally Riemannian sub-manifold of \( \text{Gl}(N_v|N_v) \). The topological charge is distributed in the form of a Gaussian with variance

\[
\frac{\langle Q^2 \rangle}{V} = \frac{F^2_\pi m_0^2}{2N_c}
\]

in the quenched theory, which holds exactly in the \( V \to \infty \) limit. This contrasts sharply with the full theory, which predicts \( \langle Q^2 \rangle = m\Sigma V/N_f \) for \( N_f \) flavors.

We note that in the quenched approximation, the Gaussian approximation of the Fourier transform with respect to \( \theta \) is valid only for small topological charge, satisfying \( |Q| \ll \langle Q^2 \rangle \). (See the Appendix for details.) Therefore, all the results given below are valid only for small \( |Q| \). We investigate how this breakdown of the effective theory occurs using the lattice data.

In the following, we consider \( N_v = 1 \) and 2, as we are interested in a system with two light quarks. All the results are obtained by carrying out a perturbation of the \( \xi \) fields and an exact integration over the zero mode \( U_0 \), which can be written in terms of Bessel functions.

At the tree level, the scalar condensate is given

\[
- \langle \bar{\psi} \psi \rangle_Q = \Sigma_Q(\mu) = \Sigma(\mu) (I_{|Q|}(\mu) K_{|Q|}(\mu) + I_{|Q|+1}(\mu) K_{|Q|+1}(\mu) + \Sigma |Q|/\mu),
\]

(2.7)

with \( \mu \equiv m\Sigma V \). The functions \( I_{|Q|}(\mu) \) and \( K_{|Q|}(\mu) \) represent the modified Bessel functions. The \( \mu \) dependence of \( \Sigma_Q(\mu) \) is shown in Fig. 1. Near the massless limit, it behaves asymptotically as \( \Sigma_Q(\mu) \to \Sigma |Q|/\mu \) for \( |Q| > 0 \). The one-loop correction does not change its functional form, and we have

\[
\Sigma_Q^{1\text{-loop}}(\mu) = \Sigma_{\text{eff}}' (I_{|Q|}(\mu') K_{|Q|}(\mu') + I_{|Q|+1}(\mu') K_{|Q|+1}(\mu')) + \Sigma_{\text{eff}} \frac{|Q|}{\mu'},
\]

(2.8)
Fig. 1. $\Sigma_Q(\mu)$ in different topological sectors for $\Sigma^{1/3} = 270$ MeV.

but the parameters $\mu$ and $\Sigma$ are shifted to $\mu'$ and $\Sigma_{\text{eff}}$:

\[
\begin{align*}
\mu' &\equiv m\Sigma_{\text{eff}}V, \\
\Sigma_{\text{eff}} &\equiv \Sigma \left(1 + \frac{m_0^2 \bar{G}(0) + \alpha \bar{\Delta}(0)}{N_c F_\pi^2}\right).
\end{align*}
\]

Here, the parameters $\bar{G}(0)$ and $\bar{\Delta}(0)$ are ultraviolet divergent tadpole integrals, where we have

\[
\begin{align*}
\bar{G}(x) &\equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^4}, \\
\bar{\Delta}(x) &\equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2},
\end{align*}
\]

which need to be renormalized. In our analysis, they are to be determined by matching with lattice data.

Let us define the flavor-singlet meson operators:

\[
\begin{align*}
S_0^0(x) &\equiv \bar{\psi}(x)I_{N_v}\psi(x), \\
P_0^0(x) &\equiv \bar{\psi}(x)i\gamma_5I_{N_v}\psi(x).
\end{align*}
\]

Adding these operators to the QCD Lagrangian as source terms,

\[
\mathcal{L} \rightarrow \mathcal{L} + s(x)S_0^0(x) + p(x)P_0^0(x),
\]

corresponds to the substitution

\[
M \rightarrow M + s(x)I_{N_v} + ip(x)I_{N_v}
\]
in the effective theory. Their two-point correlation functions are obtained by differentiating the generating functional with respect to \( s(x) \) and \( p(x) \). To \( O(\epsilon^2) \), the results are

\[
\langle S^0(x)S^0(0) \rangle_Q = C_S^0 + \frac{\Sigma^2}{2F_\pi^2} \left[ \frac{a_+}{N_c} (m_0^2 G(x) + \alpha \Delta(x)) - \Delta(x) \frac{a_+ + a_- - 4}{2} \right],
\]

\[
\langle P^0(x)P^0(0) \rangle_Q = C_P^0 - \frac{\Sigma^2}{2F_\pi^2} \left[ \frac{a_+}{N_c} (m_0^2 G(x) + \alpha \Delta(x)) - \Delta(x) \frac{a_+ + a_- + 4}{2} \right],
\]

(2.17, 2.18)

where

\[
a_+ = 4 \left[ \left( \frac{\Sigma Q(\mu)}{\Sigma} \right)' + 1 + \frac{Q^2}{\mu^2} \right],
\]

\[
a_- = 4 \left[ -1 \frac{\Sigma Q(\mu)}{\mu} + 1 + \frac{Q^2}{\mu^2} \right],
\]

(2.19, 2.20)

and the constant terms are given by

\[
C_S^0 = \frac{\Sigma_{\text{eff}}^2}{4} a_+^{1\text{-loop}} = \Sigma_{\text{eff}}^2 \left[ \left( \frac{\Sigma Q(\mu')}{\Sigma_{\text{eff}}} \right)' + 1 + \frac{Q^2}{\mu^2} \right],
\]

\[
C_P^0 = -\frac{\Sigma_{\text{eff}}^2}{4} a_-^{1\text{-loop}} = \Sigma_{\text{eff}}^2 \left[ \frac{1}{\mu'} \frac{\Sigma Q(\mu')}{\Sigma_{\text{eff}}} - \frac{Q^2}{\mu^2} \right].
\]

(2.21, 2.22)

Note that the prime denotes differentiation with respect to \( \mu \):

\[
\left( \frac{\Sigma Q(\mu)}{\Sigma} \right)' = I_{|Q|}(\mu) K_{|Q|}(\mu) - I_{|Q|+1}(\mu) K_{|Q|-1}(\mu) - \frac{|Q|}{\mu^2}.
\]

(2.23)

For flavor non-singlet mesons, we need an \( N_v = 2 \) super-group integral, which is also described by the Bessel functions. The non-singlet operators are given by

\[
S^a(x) \equiv \bar{\psi}(x) (\tau^a/2) I_{N_V} \psi(x),
\]

\[
P^a(x) \equiv \bar{\psi}(x) (\tau^a/2) i \gamma_5 I_{N_V} \psi(x),
\]

(2.24, 2.25)

with the Pauli matrices \( \tau^a \). To \( O(\epsilon^2) \), the two-point functions are given by

\[
\langle S^a(x)S^a(0) \rangle_Q = C_S^a + \frac{\Sigma^2}{2F_\pi^2} \left[ \frac{c_-}{N_c} (m_0^2 G(x) + \alpha \Delta(x)) - \Delta(x) b_- \right],
\]

\[
\langle P^a(x)P^a(0) \rangle_Q = C_P^a - \frac{\Sigma^2}{2F_\pi^2} \left[ \frac{c_+}{N_c} (m_0^2 G(x) + \alpha \Delta(x)) - \Delta(x) b_+ \right],
\]

(2.26, 2.27)

where

\[
b_{\pm} = 2 \left( 1 + \frac{Q^2}{\mu^2} \right),
\]

(2.28)
\[ b_- = \frac{2 Q^2}{\mu^2}, \quad (2.29) \]
\[ c_+ = 2 \left( \frac{\Sigma_Q(\mu)}{\Sigma} \right)', \quad (2.30) \]
\[ c_- = -2 \frac{1}{\mu} \frac{\Sigma_Q(\mu)}{\Sigma}, \quad (2.31) \]

and
\[ C^a_S = \frac{\Sigma_{\text{eff}}^2}{2} \left( \frac{\Sigma_Q(\mu')}{\Sigma_{\text{eff}}} \right)', \quad (2.32) \]
\[ C^a_P = \frac{\Sigma_{\text{eff}}^2}{2} \left( \frac{\Sigma_Q(\mu')}{\mu' \Sigma_{\text{eff}}} \right). \quad (2.33) \]

For the flavor non-singlet axial-vector current
\[ A^a_\mu(x) = \bar{\psi}(x) (\tau^a/2) i\gamma_\mu \gamma_5 \psi(x), \quad (2.34) \]
the correlator is obtained as\(^6\)
\[ \langle A^a_0(x) A^a_0(0) \rangle_Q = -\frac{F_\pi}{V} - 2 m \Sigma_Q(\mu) \Delta(x). \quad (2.35) \]

Here, an important observation is that the axial-current correlator does not involve the parameters that arise as artifacts of the quenching, i.e. \(m_0^2\) and \(\alpha\). We also note that the constant term is proportional to \(F_\pi\) rather than \(\Sigma\). Therefore, this channel is suitable for the extraction of \(F_\pi\), whereas for the pseudo-scalar and scalar correlators, \(F_\pi\) appears only in the coefficients of the \(\Delta(x)\) and \(G(x)\) terms.

In the lattice calculation, we compute the correlators using a zero spatial momentum projection. It is therefore convenient to define the functions
\[ h_1(|t/T|) = \frac{1}{T} \int d^3x \Delta(x) = \frac{1}{2} \left[ \left( \frac{|t|}{T} - \frac{1}{2} \right)^2 - \frac{1}{12} \right], \quad (2.36) \]
\[ h_2(|t/T|) = -\frac{1}{T^3} \int d^3x G(x) = \frac{1}{24} \left[ \frac{t^2}{T^2} \left( \frac{|t|}{T} - 1 \right)^2 - \frac{1}{30} \right]. \quad (2.37) \]

In the \(\epsilon\)-regime the correlators do not decay in the usual exponential form, \(\exp(-Mt)\), with the mass gap \(M\) in a large volume. Instead, it becomes a simple quadratic function for the single pole \(\Delta(x)\) and a quartic function for the double pole \(G(x)\) integral.

As the above expressions show, the meson correlators in the \(\epsilon\)-regime are quite sensitive to the topological charge and the fermion mass. Hence they provide a good way of testing the lattice simulations in the \(\epsilon\)-regime. Furthermore, the parameters \(F_\pi\), \(\Sigma\), \(m_0\) and \(\alpha\) can be extracted from the fitting of these correlators. The parameter \(\Sigma\) always appears in association with the quark mass \(m\). This is natural, because only the combination \(m\Sigma\) is independent of the renormalization scale and
scheme. The numbers we extract for \( \Sigma \) in the following analysis should be understood as results obtained in the lattice regularization at a scale of \( 1/a \). To relate them with a conventional scheme, such as the \( \overline{\text{MS}} \) scheme, requires perturbative or non-perturbative matching, which is beyond the scope of this paper.

§3. Lattice simulations

We generated the gauge link variables setting \( \beta = 5.85 \) in the quenched approximation on a \( 10^3 \times 20 \) lattice. The lattice spacing was \( a = 0.123 \) fm, which is obtained from the Sommer scale \( r_0 = 0.5 \) fm using the interpolation formula given in Ref. 21). The linear extent of the lattice is therefore approximately 1.23 fm. We employ the overlap-Dirac operator defined by

\[
D_m = \left( 1 - \frac{\bar{a}m}{2} \right) D + m, \tag{3.1}
\]

\[
D = \frac{1}{\bar{a}} \left( 1 + \gamma_5 \text{sgn}(H_W) \right), \tag{3.2}
\]

with the kernel \( H_W \) built with the Wilson-Dirac operator \( D_W \),

\[
H_W = \gamma_5 (a D_W - 1 - s). \tag{3.3}
\]

The parameter \( s \) controls the negative mass given to \( a D_W \) and we choose \( s = 0.6 \) for \( \beta = 5.85 \) in order to minimize the number of low-lying modes in \( H_W \). Here the quantity \( \text{sgn}(H_W) \) is the sign function of the large sparse Hermite matrix \( H_W \), and \( \bar{a} \) is defined as \( \bar{a} = a/(1+s) \). This overlap-Dirac operator satisfies the Ginsparg-Wilson relation

\[
\gamma_5 D + D \gamma_5 = \bar{a} D \gamma_5 D \tag{3.4}
\]

exactly, and the \( \gamma_5 \)-hermiticity, \( D^\dagger = \gamma_5 D \gamma_5 \), is also satisfied. In the actual implementation of this operator, we approximate the sign function \( \text{sgn}(H_W) \) using a Chebyshev polynomial of degree 100–200 after subtracting the 60 lowest-lying eigenmodes of \( \text{sgn}(H_W) \) exactly. The relative magnitude of the error resulting from this approximation of the sign function is then on the order of \( 10^{-12} \), and safely ignored in our numerical results.

One of the essential points of this work is to use the eigenmode decomposition of the fermion propagator. For this purpose, we calculate the 100 lowest (non-zero) eigenvalues of \( P_{\pm} D P_{\mp} \) and their eigenfunctions, as well as zero-mode eigenfunctions of \( P_{\mp} D P_{\mp} \) for negative (and positive) topological charge. We used the numerical package ARPACK,\(^{22}\) which implements the implicit restarted Arnoldi method. The chiral projection operator \( P_{\pm} \equiv (1 \pm \gamma_5)/2 \) is applied in order to reduce the rank of the matrix. The eigenvalues and eigenvectors of the original matrix \( D \) can be reconstructed from those for the chirally projected operators. We thus obtain \( 200 + |Q| \) eigenmodes of \( D \) for each gauge configuration. Note that these \( 200 + |Q| \) eigenvalues cover more than 15% of the circle in the complex space of the eigenvalues of \( D \), as shown in Fig. 2. The topological charge is obtained from the number of zero-modes and their chirality. The number of configurations for each topological sector is given in Table I. We analyze the gauge configurations of the \( |Q| \leq 3 \) sectors.
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Fig. 2. The lowest 202 eigenvalues of the overlap-Dirac operator for $\beta = 5.85$ on a $10^3 \times 20$ lattice of topological charge $Q = -2$. The eigenvalues cover a $\pi/3$ arc of the circle.

Table I. Number of configurations in each topological sector.

<table>
<thead>
<tr>
<th>$Q$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td># of confs.</td>
<td>20</td>
<td>45</td>
<td>44</td>
<td>24</td>
</tr>
</tbody>
</table>

When the exact inverse of the overlap-Dirac operator is needed, we use the techniques described in Ref. 23). For a given source vector $\eta$, we solve the equation

$$D_m \psi = \eta$$  (3.5)

by separating the left- and right-handed components as $\psi = P_- \psi + P_+ \psi$ and solving the two equations

$$P_- \psi = (P_- D_m^\dagger D_m P_-)^{-1} P_- D_m^\dagger \eta,$$  (3.6)

$$P_+ \psi = (P_+ D_m P_+)^{-1} (P_+ \eta - P_+ D_m P_- \psi)$$  (3.7)

consecutively. (The above equations apply for positive $Q$, and the same procedure applies with the exchange $P_+ \leftrightarrow P_-$ for negative $Q$.) We use the conjugate gradient (CG) algorithm to invert the chirally projected matrices using the low-mode preconditioning with the 20 lowest eigenmodes. With this low-mode preconditioning, the CG solver becomes approximately one order of magnitude faster (We need only 20–40 iterations for each $P_\pm \psi$) for our smallest quark mass, 0.0016, which corresponds to 2.6 MeV in physical units.
§4. Low-mode dominance for the meson correlators

The inverse of the overlap-Dirac operator $D$ can be decomposed into the contributions from each eigenmode with eigenvalue $\lambda_i$ and eigenvector $v_i(x)$ as

$$D^{-1}_m(x, y) = \sum_{i=1}^{N_{\text{low}}} \frac{1}{(1 - \bar{a}m/2)\lambda_i + m} v_i(x)v_i(\bar{y}) + \Delta D^{-1}_m(x, y). \quad (4.1)$$

Here, the eigenmode decomposition is incomplete, and the sum is truncated at some cutoff $N_{\text{low}}$, which we set $N_{\text{low}} = 200 + |Q|$. The additional term $\Delta D^{-1}_m(x, y)$ represents the contribution from higher eigenmodes.

We conjecture that the low energy physics is dominated by the low-lying eigenmodes. Near the massless limit, the lowest-lying eigenmodes are dominant, as they are enhanced by a factor of $1/\lambda_i$, and in particular, the zero modes give a divergent contribution in the quenched approximation. (In the unquenched case, their appearance is suppressed by the fermion determinant.) Sensitivity to the gauge field topology comes mainly from these low-lying eigenmodes. Therefore, the eigenmode decomposition (4.1) is expected to give a good approximation of the low energy physics, even if we ignore the higher-mode contribution $\Delta D^{-1}_m(x, y)$. The cutoff $N_{\text{low}}$ must be large enough for such an approximation to cover all the relevant low-lying modes, and it depends on the pion mass and the physical volume of the system. For massive fermions, the eigenmodes for which $\lambda_i \lesssim m$ become equally important, and we need a larger $N_{\text{low}}$ than in the massless limit.

We note here that, in contrast to the situation discussed above, the short distance physics could be affected by the higher eigenmodes. This can be understood by considering the defining equation $\sum_z D_m(x, z)D^{-1}_m(z, y) = \sum_{i=1}^{N_{\text{low}}} v_i(x)v_i(\bar{y}) + D_m(x, z)\Delta D^{-1}_m(z, y) = \delta(x, y)$ at $x = y$. Because the sum $\sum_{i=1}^{N_{\text{low}}} v_i(x)v_i(\bar{y})$ approaches 1 monotonically as $N_{\text{low}}$ approaches the size of the matrix, $N_{\text{max}}$, the contribution from the remaining term, $\Delta D^{-1}_m(x, y)$, becomes significant for $N_{\text{low}}$ much smaller than $N_{\text{max}}$. However, such a short distance correlation described by $\Delta D^{-1}_m(x, y)$ should be insensitive to the gauge field topology.

4.1. Connected correlators

First, let us consider the “connected” meson correlators

$$M_i(t)M_i(0)_{\text{conn}} \equiv \sum_{\bar{x}} \bar{\psi}\Gamma_i\psi(\bar{x}, t)\bar{\psi}\Gamma_i\psi(\bar{0}, 0)_{\text{conn}}$$

$$= -\sum_{\bar{x}} \text{tr} \left( \Gamma_i D^{-1}_m(\bar{x}, t; \bar{0}, 0)\Gamma_i D^{-1}_m(\bar{0}, 0; \bar{x}, t) \right), \quad (4.2)$$

where $M_i$ denotes the local operator corresponding to the pseudo-scalar (PS), scalar (S), vector (V) and axial-vector (AV) currents, and $\Gamma_i$ denotes the corresponding gamma matrix, with $\Gamma_{\text{PS}} = i\gamma_5$, $\Gamma_{\text{S}} = 1$, $\Gamma_{\text{V}} = i\gamma_0$, and $\Gamma_{\text{AV}} = i\gamma_5\gamma_0$.

We calculate these correlators in two ways, one employing an exact calculation with the conjugate gradient (CG) method, and the other employing the low-mode
Table II. Comparison of the low-mode approximated “connected” correlators with the exact ones.
The approximation employs the quark propagator (4.3) with $200+|Q|$ low-lying modes for $m = 0.0016, 0.0048, 0.008$. The maximum deviation in the range $7 \leq |t| \leq 13$ is shown. The number of configurations in each case is given in Table I.

| $m$ = 0.008 ($\sim 13$ MeV) | $|Q| = 0$ | $|Q| = 1$ | $|Q| = 2$ | $|Q| = 3$ |
|--------------------------|-----------|-----------|-----------|-----------|
| scalar                   | 2.01%     | 1.49%     | 0.46%     | 0.44%     |
| pseudo-scalar            | 1.19%     | 0.48%     | 0.28%     | 0.22%     |
| vector                   | 41.6%     | 258%      | 259%      | 94.6%     |
| axial-vector             | 28.4%     | 40.5%     | 24.3%     | 24.2%     |
| $m$ = 0.0048 ($\sim 7.7$ MeV) | $|Q| = 0$ | $|Q| = 1$ | $|Q| = 2$ | $|Q| = 3$ |
| scalar                   | 1.57%     | 0.65%     | 0.16%     | 0.18%     |
| pseudo-scalar            | 1.06%     | 0.33%     | 0.10%     | 0.08%     |
| vector                   | 28.9%     | 198%      | 110%      | 104%      |
| axial-vector             | 26.2%     | 38.6%     | 19.7%     | 18.3%     |
| $m$ = 0.0016 ($\sim 2.6$ MeV) | $|Q| = 0$ | $|Q| = 1$ | $|Q| = 2$ | $|Q| = 3$ |
| scalar                   | 1.41%     | 0.08%     | 0.03%     | 0.03%     |
| pseudo-scalar            | 0.95%     | 0.06%     | 0.04%     | 0.02%     |
| vector                   | 22.5%     | 149%      | 157%      | 166%      |
| axial-vector             | 21.5%     | 32.6%     | 20.4%     | 34.7%     |

where the higher eigenmode contributions are ignored. A comparison of the results of these two methods of calculation for $m = 0.0016, 0.0048, 0.008$ ($\ll 13$ MeV) is given in Table II, where the maximal difference between the two correlators in the region $7 \leq t \leq 13$ is listed for each operator and topological sector. For the scalar and pseudo-scalar mesons, the approximation (4.3) is quite good, yielding 98–99.9% accuracy, while the lowest $200+|Q|$ modes are not sufficient to reproduce the vector and axial vector correlators. Figure 3 displays the pseudo-scalar and axial-vector correlators for $m = 0.008$ and $|Q| = 1$. We observe very good agreement for the pseudo-scalar correlator over a long interval of $t$ (from 3 to 17), but the situation in the axial-vector case is much worse. This is probably because the axial-vector correlator is smaller in magnitude by a factor of $O(m)$, and therefore relative fluctuations from the higher-modes are large.24 Figure 4 shows how the approximate correlators containing only low-lying mode contributions converge to the full correlators. We find that the lowest $\sim 100$ eigenmodes suffice to approximate the full correlator to very good accuracy. The plot depicts the case for $m = 0.008$, but the accuracy of the approximation is even better for smaller quark masses.

An advantage of the low-mode approximation (4.3) is that $D_{m}^{-1}(x,y)$ for any $x$ and $y$ can be obtained without performing the CG inversion, so that one can easily
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Fig. 3. Pseudo-scalar (left) and axial-vector (right) correlators for $m = 0.008$ and $|Q| = 1$. The low-mode-approximated correlator (4·3) is compared with the corresponding exact one.

Fig. 4. Convergence of the connected pseudo-scalar correlator for one sample configuration in the case $m = 0.008$. The two plots correspond to $|Q| = 0$ (left) and $|Q| = 1$ (right). The rightmost points (around 120,000) correspond to the exact correlator obtained with the CG method.

Fig. 5. The pseudo-scalar correlator for $m = 0.008$ and $|Q| = 1$. The solid dots represent the data with the low-mode averaging, while open squares represent data that are not averaged.
average the source point over the space-time:

\[
\left\langle \sum_{\vec{x}} \bar{\psi} \Gamma_i \psi(\vec{x}, t) \bar{\psi} \Gamma_i \psi(\vec{0}, 0) \right\rangle_{\text{conn}} \rightarrow \frac{1}{T L^3} \sum_{\vec{x}_0, t_0} \left\langle \sum_{\vec{x}} \bar{\psi} \Gamma_i \psi(\vec{x}, t + t_0) \bar{\psi} \Gamma_i \psi(\vec{x}_0, t_0) \right\rangle_{\text{conn}}.
\]

This so-called low-mode averaging dramatically reduces the fluctuations of the low-lying modes, as shown in Fig. 5. This effect is also reported in Ref. 17). In practice, we average over only \((L/2)^3 \times (T/2)\) lattice points, where the site index is an even number for each direction. Note that even after the low-mode averaging, the error resulting from the truncation of the higher modes is negligible compared to the statistical error, \(\sim 15\%\) in the \(Q = 0\) sector and \(\sim 5\%\) in the \(Q \neq 0\) sectors in the range \(7 \leq t \leq 13\).

4.2. Chiral condensates

Consider the low-mode contribution to the scalar and pseudo-scalar condensates for a fixed topological charge \(Q\),

\[
\langle \bar{\psi} \psi(x) \rangle^Q = -\langle \text{tr} D_m^{-1}(x, x) \rangle^Q
\]

\[
= -\left\langle \text{tr} \left( \sum_{i=1}^{N_{\text{low}}} \frac{1}{(1-\bar{a}m/2)\lambda_i + m} v_i(x) v_i^\dagger(x) + \Delta D_m^{-1}(x, x) \right) \right\rangle^Q,
\]

\[
\langle \bar{\psi} \gamma_5 \psi(x) \rangle^Q = -\langle \text{tr} \gamma_5 D_m^{-1}(x, x) \rangle^Q
\]

\[
= -\left\langle \text{tr} \left( \sum_{i=1}^{N_{\text{low}}} \frac{1}{(1-\bar{a}m/2)\lambda_i + m} \gamma_5 v_i(x) v_i^\dagger(x) + \gamma_5 \Delta D_m^{-1}(x, x) \right) \right\rangle^Q.
\]

For the scalar condensates, it is known that \(\Delta D_m^{-1}(x, x)\) includes unwanted additive ultraviolet divergences:

\[
\langle \text{tr} D_m^{-1}(x, x) \rangle^Q = \frac{6}{(1+s)a^3} + \frac{C_2 m}{a^2} + \frac{C_1 m^2}{a} + \Sigma Q(\mu') + O(\epsilon^4),
\]

\[Q \leq 10\]

Fig. 6. Low-mode domination of \(-\langle \bar{\psi} \psi \rangle^Q\). The data are averaged over 20 gauge configurations for \(Q = 0\) (left) and \(Q = 1\) (right).
where $C_2$ and $C_1$ are unknown constants. The first term here comes from the modification of the chiral symmetry in the Ginsparg-Wilson relation,\cite{10} $\gamma_5 D^{-1}(x, y) + D^{-1}(x, y)\gamma_5 = a\delta_{x,y}$, at $x = y$. If $N_{\text{low}}$ is large enough, the higher-mode contribution, $\Delta D_m^{-1}(x, x)$, should be insensitive to the link variables $U_\mu(y)$ separated from $x$ by a sufficiently large amount, and thus insensitive to the global structure of the gauge field configuration, for example, the topological charge. We, therefore, expect that such a contribution vanishes in the difference between different topological sectors,

\begin{equation}
-\langle (\bar{\psi}\psi(x))^Q - (\bar{\psi}\psi(x))^0 \rangle. \tag{4.8}
\end{equation}

In other words this difference should be accurately described by only the low-lying eigenmodes. In fact, we observe such a low-mode domination in Figs. 6 and 7, while the individual condensate $-\langle (\bar{\psi}\psi(x))^Q \rangle$ is not dominated by the low-modes.

For the pseudo-scalar condensate, we do not have to treat the higher modes because the condensate is determined by only the zero modes, and the contributions from other eigenmodes cancel, because of the orthogonality of different eigenvectors. As shown in Fig. 8, our data with $N_{\text{low}} = 200 + |Q|$ low modes agree very precisely
with the theoretical prediction,
\[ -\langle \bar{\psi} \gamma_5 \psi \rangle_Q = \frac{Q}{mV}. \]

4.3. Disconnected correlators

Here we discuss the expectation value of the “disconnected” diagrams for a fixed topological charge. As in the case of connected diagrams, we expect
\[
\langle M_i(t)M_i(0) \rangle_{\text{disc}}^Q = \left\langle \sum_{\vec{x}} \bar{\psi}_i \Gamma_i \psi(\vec{x}, t) \bar{\psi}_i \psi(\vec{0}, 0) \right\rangle^Q_{\text{disc}} = \left\langle \sum_{\vec{x}} \text{tr} \left( \Gamma_i D_{\text{m}}^{-1}(\vec{x}, t; \vec{x}, t) \right) \text{tr} \left( \Gamma_i D_{\text{m}}^{-1}(\vec{0}, 0; \vec{0}, 0) \right) \right\rangle^Q_{\text{disc}}.
\]

where \( x = (\vec{x}, t) \). We assume that the contribution of higher modes is not correlated with any local operator \( O(y) \) separated from \( x \) by a sufficiently large amount, i.e.
\[
\langle \Delta D_{\text{m}}^{-1}(x, x) O(y) \rangle^Q_{\text{disc}} \overset{t \gg 0}{\rightarrow} \langle \Delta D_{\text{m}}^{-1}(x, x) \rangle' \times \langle O(y) \rangle^Q,
\]

where the expectation value \( \langle \cdots \rangle' \) represents the insensitivity to the topological charge. We also use translational invariance, which implies \( \langle O(x) \rangle = \langle O(0) \rangle \). Unlike in the “connected” case, we cannot check the low-mode dominance by explicitly computing the exact correlators, because the computational cost is too large. However, for the pseudo-scalar disconnected diagram, the quantity \( \langle \text{tr} \gamma_5 \Delta D_{\text{m}}^{-1}(x, x) \rangle' \) vanishes, because \( \Delta D_{\text{m}}^{-1}(x, x) \) does not contain the zero modes.

In fact, as shown in Fig. 9, we obtain a good convergence with the lowest 200 eigenmodes for the pseudo-scalar disconnected correlators. Similar results were obtained previously in a study of the \( \eta' \) propagator with the Wilson fermion\(^{26}\) and with the overlap fermion\(^{27}\).
§5. Extraction of the low energy constants from the meson correlators

5.1. $F_\pi$ from the axial-vector correlator

The axial-vector current correlator (2.35) is most sensitive to $F_\pi$ and not contaminated by the parameters $m_0$ and $\alpha$. The problem, however, is that in this case low-mode dominance does not exist, and we have to solve the quark propagators exactly. Hence, the statistical noise could become a problem, as we cannot average the source point over space-time. Our strategy is to simultaneously treat data for different quark masses and different topologies in order to reduce such large statistical errors.

We use the local axial-current

$$A_\mu^a(x) = \bar{\psi}(x)\gamma_5\gamma_\mu(\tau^a/2)\psi(x),$$

constructed from the overlap fermion field $\psi(x)$. Because it is not the conserved current corresponding to the lattice chiral symmetry, (finite) renormalization is needed to relate it to the continuum axial-vector current. To calculate the $Z_A$ factor non-perturbatively, we follow the method applied in Refs. 28) and 29); that is, we calculate

$$aR_\rho(t) \equiv \frac{a\sum_\vec{x}\langle \vec{\nabla}_0 A_0^a(\vec{x}, t) P^a(0, 0) \rangle}{\sum_\vec{x}\langle P^a(\vec{x}, t) P^a(0, 0) \rangle},$$

(5.1)

where $\vec{\nabla}_0$ denotes a symmetric lattice derivative. The pseudo-scalar density $P^a(x)$ must be the chirally improved one, $\bar{\psi}(\tau^a/2)\gamma_5(1 - \bar{a}D/2)\psi$, associated with the exact chiral symmetry on the lattice. For the on-shell matrix elements, such as that considered here, one can use the equation of motion to replace the $\bar{a}D$ term by $-am/(1 - \bar{a}m/2)$, which is negligible for our quark masses. We therefore use the local operator for $P^a(x)$.

The ratio (5.1) turns out to be insensitive to topology, as shown in Fig. 10. Fitting the average of $R_\rho(t)$ over all topological sectors with a constant in the range $7 \leq t \leq 13$, we obtain the results for

$$a\rho(ma) \equiv \frac{a\langle \vec{\nabla}_\mu A_\mu^a(x) P^a(0) \rangle}{\langle P^a(x) P^a(0) \rangle} = \frac{2ma}{Z_A} + O(a^2)$$

(5.2)

in the cases of the four quark masses $m = 0.0016, 0.0048, 0.008, 0.0160$, which are
shown in Fig. 11. With a quadratic fit we obtain

\[ a\rho(ma) = 0.00001(2) + 1.390(14)(ma) - 0.19(74)(ma)^2. \]  

(5.3)

The constant term is consistent with zero, and we can extract \( Z_A \) from the linear term as \( Z_A = 1.439(15) \), which is consistent with the value \( Z_A = 1.448(4) \) reported in Ref. 30, in which the same values of \( \beta \) and \( s \) were employed.

We now compare the renormalized axial-vector correlation function with the QChPT result,

\[
2Z_A^2 \sum_{\vec{x}} \langle A_0(\vec{x}, t) A_0(0, 0) \rangle^Q = 2 \left( \frac{F_\pi^2}{T} + 2m \Sigma_{|Q|}(\mu) T h_1(|t/T|) \right). \]  

(5.4)

From the constant term, we can determine \( F_\pi \), while \( \Sigma \) has to be extracted from the small \( t \) dependence. In Ref. 15) it is reported that the correlators exhibit large statistical fluctuations at \( |Q| = 0 \), and other topological sectors are insensitive to \( \Sigma \).
**Fig. 12.** Axial-vector current correlators in the cases $m = 0.0016$ (left) and 0.008 (right) for $|Q| = 0$ (top) and 1 (bottom). The dashed curves represent the results of simultaneous fitting the data for $Q = 0$ and 1 with $m = 0.0016$, 0.0048, 0.008 in the region $7 \leq t \leq 13$.

**Fig. 13.** Axial-vector current correlators in the cases $m = 0.0016$ (left) and 0.008 (right) for $|Q| = 2$. The dashed curves are the QChPT predictions with the parameter values determined in $|Q| = 0$ and 1 sectors.

We observed similarly large statistical noise, as shown in Fig. 12, but it turns out that the two-parameter ($F$ and $\Sigma$) fit is good when we treat the data obtained for different topologies and fermion masses simultaneously.

As Fig. 12 shows, our data for $m = 0.0016$, 0.0048, 0.008 in the $|Q| \leq 1$ sectors are described well by the QChPT formula (5.4). A simultaneous fit in the range $7 \leq t \leq 13$ yields $F_\pi = 98.3(8.3)$ MeV and $\Sigma^{1/3} = 259(50)$ MeV with $\chi^2$/dof =
0.19. These results are stable with respect to changes in the fit range: \((F_\pi, \Sigma^{1/3}) = (98.8(8.3) \text{ MeV}, 261(47) \text{ MeV})\) and \((99.2(8.3) \text{ MeV}, 261(45) \text{ MeV})\) for \(8 \leq t \leq 12\) and \(9 \leq t \leq 11\), respectively. The result for \(F_\pi\) is consistent with that of the previous work\(^{17}\) 102(4) MeV. The authors of Ref. 15) quoted a slightly larger value, \(\sim 130\) MeV.

The correlators for \(|Q| = 2\) are not consistent with the above fit parameters, as shown in Fig. 13. As discussed in 
\[\text{§2}\] this may indicate that the topological sector \(|Q| = 2\) is already too large to apply the QChPT in the \(\epsilon\)-regime.

5.2. \(\Sigma, \Sigma_{\text{eff}}\) and \(\alpha\) from connected \(S\) and \(PS\) correlators

As discussed in the previous section, the scalar and pseudo-scalar connected correlators are approximated rather accurately with only the lowest 200+ \(|Q|\) eigen-modes for small quark masses \((m = 2.6–13\) MeV\). Noting this, we define the scalar (pseudo-scalar) correlators
\[
\langle S(t) \rangle^Q \equiv -2 \sum_x (1 + s)^2 \langle S^3(x) S^3(0) \rangle_{\text{low-modes}},
\]
\[
\langle P(t) \rangle^Q \equiv 2 \sum_x (1 + s)^2 \langle P^3(x) P^3(0) \rangle_{\text{low-modes}},
\]
for the topological sectors \(1 \leq |Q| \leq 3\) \((0 \leq |Q| \leq 3)\), in the cases \(m = 0.0016, 0.0032, 0.0048, 0.0064,\) and 0.008, in which the error resulting from higher modes is estimated to be \(\lesssim 1\%\), which can be ignored in comparison with statistical errors. We take the average of the source point over \((L/2)^3 \times (T/2)\) lattice sites.

To fit our data with the one-loop QChPT formulae (2.26) and (2.27), we have to determine the five parameters in these formulae: \(F_\pi, \Sigma, \Sigma_{\text{eff}}, m_0^2\) and \(\alpha\). Because the sensitivity to \(F_\pi\) is weak with these correlators, we use the jackknife samples of \(F_\pi\) obtained with the axial-vector current correlator. These yield \(F_\pi = 98.3(8.3)\) MeV. Unfortunately, there are still too many parameters to fit with the QChPT expressions. Therefore, to determine \(m_0^2\) we use the relation (2.6) and input the value of topological susceptibility \(\chi \equiv \langle Q^2 \rangle / V\), using \(r_0^4 \chi = 0.059(3)\) obtained in a recent work\(^{31}\). This gives \(m_0 = 940(80)(23)\) MeV, where the second uncertainty represents the uncertainty on \(r_0^4 \chi\).

Using these inputs, we fit the correlators (5.5) and (5.6) in the range \(7 \leq t \leq 13\) with different \(Q\) and \(m\) simultaneously. Figure 14 shows the correlators with fit curves. For \(|Q| \leq 1\), the data for all available quark masses, \(m = 0.0016, 0.0032, 0.0048, 0.0064,\) and 0.008, are fit well, and we actually obtain \(\chi^2/\text{dof} \sim 0.7\). (Note that the correlations between different values of \(t, m\) and channels \((PS\) and \(S)\) are not taken into account.) Our fit results are \(\Sigma^{1/3} = 257 \pm 14 \pm 0\) MeV, which is consistent with the results given in Ref. 32), \(\Sigma_{\text{eff}}^{1/3} = 271 \pm 12 \pm 0\) MeV, and \(\alpha = -4.5 \pm 1.2 \pm 0.2\), where the first uncertainty is the statistical uncertainty, and the second one is from the uncertainty on \(\langle Q^2 \rangle\). They are insensitive to the choice of the fit range, the central values vary only slightly (for instance, \(\pm 1\) MeV for \(\Sigma^{1/3}\) and \(\Sigma_{\text{eff}}^{1/3}\)) when we choose shorter fitting ranges \(8 \leq t \leq 12\) or \(9 \leq t \leq 11\).

From the ratio \(\Sigma_{\text{eff}} / \Sigma\), we can identify the size of the NLO correction in the \(\epsilon\)
Fig. 14. Scalar (top) and pseudo-scalar (bottom) correlators for $m = 0.0032$ (left) and 0.0064 (right). The dotted curves are the fit results obtained using all available values of the mass parameters, $m = 0.0016, 0.0032, 0.0064$ and 0.008, for topological charges 0 and 1.

Expansion. To the one-loop level, it is written

$$\frac{\Sigma_{\text{eff}}}{\Sigma} = 1 + \frac{1}{N_c F_\pi} \left( m_0^2 \bar{G}(0) + \alpha \bar{\Delta}(0) \right),$$

where the parameters $\bar{G}(0)$ and $\bar{\Delta}(0)$ are regularization dependent. For this ratio we obtain 1.163(59), which indicates that the $\epsilon$ expansion is actually converging.

We obtain a large negative value for $\alpha$, which is also reported in Refs. 33) and 34). These results contradict a previous precise calculation, 35) which obtained a small value $\alpha = 0.03(3)$. If we instead assume $\alpha = 0$, and fit $F_\pi$ as a free parameter, we obtain $F_\pi = 136.9(5.3)$ MeV, while $\Sigma$ and $\Sigma_{\text{eff}}$ are almost unchanged. (Details are listed in Table III.) Therefore, there is an apparent inconsistency in the determination of $F_\pi$ between the axial-vector and (pseudo-)scalar correlators if $\alpha \sim 0$ is assumed. A possible cause is that $|Q| = 1$ is not small enough to derive the partition function Eq. (2.5) (see the Appendix). Equation (2.6) may also contain a systematic error due to the use of finite $V$ as well as finite $a$.

The data for larger topological charge, $|Q| = 2$, are also plotted in Fig. 14. They do not quite agree with the predictions of the QChPT, represented by the dashed curves in the plots. A simultaneous fit with all the data, including $|Q| = 0, 1$ and 2, yields a very large $\chi^2$/dof ($\simeq 12$). This problem encountered for larger topological charge also exists for the axial-vector correlator, as discussed in the previous subsection.
5.3. Scalar condensate

The free parameter in the scalar condensate is $\Sigma_{\text{eff}}$, as seen in (2.8) and (2.9). To avoid the problem of ultraviolet divergence we compare the difference $-\langle \bar{\psi}\psi \rangle^Q - \langle \bar{\psi}\psi \rangle^0$ with the QChPT result, $\Sigma_Q(\mu') - \Sigma_{Q=0}(\mu')$. We use the low-mode approximation with 200+|$Q$| eigenmodes, and the low-mode averaging is done in the same way as for the meson correlators. Figure 15 plots the difference as a function of the quark mass for $|Q| = 1, 2$ and 3. We find that the lattice data agree remarkably well with the QChPT prediction with $\Sigma_{\text{eff}} = 271(12)$ MeV, as determined from the (pseudo-)scalar connected correlator, even in higher topological sectors. In fact, if we fit the scalar condensate with $\Sigma_{\text{eff}}$ as a free parameter, we obtain 256(14) MeV, which is consistent with the above result.

5.4. Disconnected PS correlators

We also computed the disconnected pseudo-scalar correlators, again using the low-mode approximation. The convergence obtained with 200+|$Q$| lowest-lying modes is quite good for the pseudo-scalar channel, as discussed in §4.3.

In the QChPT, the disconnected pseudo-scalar correlator is given by

$$
\langle P^d(t) \rangle^Q = \int d^3x (1 + s)^2 \langle 2 P^3(x) P^3(0) - P^0(x) P^0(0) \rangle^Q
= \int d^3x (1 + s)^2 \left[ C^d_p - \frac{\Sigma^2}{2F^2} \left( \frac{d_+}{Nc} (m_0^2 G(x) + \alpha \bar{\Delta}(x)) - e_+ \bar{\Delta}(x) \right) \right],
$$

where

$$
C^d_p = \frac{Q^2}{m^2 V^2},
$$

$$
d_+ = -4 \left( 1 + \frac{Q^2}{\mu^2} \right),
$$
In Fig. 16 lattice data for the topological sectors $|Q| = 0$–3 are displayed for two representative quark masses, $m = 0.0032$ and 0.0064. The QChPT predictions are plotted with the parameters determined through the axial-vector and (pseudo-)scalar connected correlators: $\Sigma^{1/3} = 257$ MeV, $F_\pi = 98.3$ MeV, $m_0 = 940$ MeV, and $\alpha = -4.5$. We observe that the agreement is marginal, though the correlator’s magnitude and shape are described well qualitatively. Instead, if we fit the disconnected correlator with $\Sigma$ and $\alpha$ as free parameters while fixing $F_\pi$ and $m_0$ to the same value, we obtain $\Sigma^{1/3} = 227(32)$ MeV and $\alpha = -3.5(1.2)$, which are statistically consistent with the input values. Therefore, we conclude that both the connected and disconnected correlators are consistently described by the QChPT in the $\epsilon$-regime. Details of the fit results are listed in Table III.

§6. Conclusions

In the $\epsilon$-regime of ChPT, the meson correlators are strongly affected by the fermion zero-mode, and thus by the topological charge of the background gauge field. This prediction of the effective field theory is explicitly confirmed by a first-principles calculation of lattice QCD using the overlap Dirac operator, with which
Table III. Summary of the fitting results. The first column lists the topological sectors used in the fit. The values in \( \cdots \) are input parameters. The first uncertainty is statistical. The second and third uncertainties reflect the uncertainties in the input parameters, \( \langle Q^2 \rangle \) and \( F_\pi \), respectively.

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<th>( \Sigma^{1/3} ) (MeV)</th>
<th>( \Sigma^{1/3}_{\text{eff}} ) (MeV)</th>
<th>( \alpha )</th>
<th>( m_0 ) (MeV)</th>
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<td>\leq 1)</td>
<td>98.3(8.3)</td>
<td>257(14)(00)</td>
<td>271(12)(00)</td>
<td>-4.5(1.2)(0.2)</td>
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</tr>
<tr>
<td>scalar condensate ((\bar{\psi}\psi)^n - \langle \bar{\psi}\psi \rangle^n) (0 \leq</td>
<td>Q</td>
<td>\leq 3)</td>
<td>139.3(4.1)(1.4)</td>
<td>244(32)(00)</td>
<td></td>
<td>-3.6(0.1)(0.2)(0.8)</td>
</tr>
<tr>
<td>(1 \leq</td>
<td>Q</td>
<td>\leq 3)</td>
<td></td>
<td></td>
<td></td>
<td>256(14)</td>
</tr>
</tbody>
</table>

we can preserve the exact chiral symmetry at finite lattice spacing.

To reach the \( \epsilon \)-regime we need small quark masses to satisfy the condition \( M_\pi L \lesssim 1 \). This can be achieved by using the eigenmode decomposition of the fermion propagator. Then, the connected scalar and pseudo-scalar correlators are precisely reproduced by using only the 200 lowest-lying eigenmodes. This number would be unchanged even if we decreased the lattice spacing, as long as the physical volume is kept fixed at \( \sim (1.2 \text{ fm})^3 \times (2.4 \text{ fm}) \), since the small eigenvalue distribution depends only on the combination \( \lambda \Sigma V \) in the \( \epsilon \)-regime. The \( a \to 0 \) limit only affects higher modes, which are irrelevant to the low energy dynamics. For these connected meson correlators, the small quark mass regimes are reached without extra computational cost, and we can also employ the low-mode averaging technique to substantially reduce the statistical noise due to near-zero mode contributions. For the axial-vector current correlator, on the other hand, the approximation obtained with only low-lying modes is much worse, and we had to treat them exactly, using the (costly) CG solver. We also investigated the disconnected pseudo-scalar correlator using the low-lying mode approximation. Computing the disconnected diagrams is usually very costly, as they require many fermion inversions, but with this approximation they are obtained without extra cost. We confirm that the disconnected pseudo-scalar correlator is approximated well by the 200 lowest-lying eigenmodes for our lattice.

Strong \( Q \) and \( \mu \) dependences of the quenched ChPT are accurately reproduced by our lattice calculation with \( \beta = 5.85 \) on a \( 10^3 \times 20 \) lattice. Thus we are able to extract some of the low energy constants: \( F_\pi \), \( \Sigma \), \( m_0^2 \) and \( \alpha \). The last two are the artifacts of the quenched approximation. Fitting our data for meson correlators simultaneously
with different quark masses and topological charges, we obtain $F_\pi = 98.3(8.3) \text{ MeV}$, $\Sigma^{1/3} = 257(14)(00) \text{ MeV}$ ($\Sigma_{\text{eff}}^{1/3} = 271(12)(00) \text{ MeV}$), $m_0 = 940(80)(23) \text{ MeV}$, and $\alpha = -4.5(1.2)(0.2)$, from the connected correlators with $|Q| \leq 1$. In these numerical results, the second uncertainty reflects the uncertainty in $\langle Q^2 \rangle$. We also obtain consistent results for disconnected pseudo-scalar correlator and the chiral condensate.

Despite such noteworthy success of QChPT, we also find problems. First, the correlators in $|Q| \geq 2$ sectors are not well fitted with the parameters determined in $|Q| \leq 1$. This may indicate a problem of the $\epsilon$-expansion in QChPT, which results from the fact that the partition function with a fixed topology is valid only for $|Q| \ll \langle Q^2 \rangle$. Because $\langle Q^2 \rangle$ is proportional to $V$, this condition is relaxed on larger lattices. It would therefore be interesting to extend our study to larger volumes. Second, the numerical result for $\alpha$ is a rather large negative value. Such a large value may raise a question regarding the validity of the partition function (2.5). Previous lattice calculations (e.g., see Ref. 35), indicate a value consistent with zero, and our result is clearly inconsistent with them. If we assume $\alpha = 0$, then our data favor a larger value of $F_\pi \simeq$, specifically $130 \text{ MeV}$, which contradicts the result for the axial-vector correlator. Our calculations are, of course, not free from other systematic errors due to higher-order terms in QChPT, finite lattice spacing, etc. However, because of the exact chiral symmetry, the finite lattice spacing error does not destroy the consistency with (Q)ChPT, although the extracted parameters are altered.

Once these problems are solved, the lattice simulation in the $\epsilon$-regime could become a useful alternative to the conventional large volume (or large quark mass) simulations. A clear advantage of this method is that sufficiently small quark masses can be realized, and there is practically no question concerning the applicability of ChPT. Obviously, dynamical simulations in the $\epsilon$-regime are most desirable. With the overlap fermion, we do not foresee any fundamental problem, because the overlap-Dirac operator is well-conditioned, even in the massless limit, though it is numerically too costly with the current generation computers.

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**Appendix**

In this appendix we derive the partition function with a fixed topological charge, (2.5). It is obtained from the partition function (2.1) by Fourier transforming with
Low-Lying Mode Contribution to the Quenched Meson Correlators

\[ Z_Q(M) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{i\theta Q} Z(\theta, M) \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int dU_0' d\xi (S\det U_0')^Q \exp \left[ -\int d^4x \left( \mathcal{L} + i\sqrt{2}Q \Phi_0 \right) \right] \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int dU_0' d\xi (S\det U_0')^Q \exp \left[ -\frac{V m_0^2}{2N_c} \left( \Phi_0 \frac{F_\pi\theta}{\sqrt{2}} \right)^2 \right. \]

\[ \left. - \frac{\sqrt{2}iQ}{F_\pi} \left( \Phi_0' - F_\pi\theta \right) \right] \times \exp \left[ \frac{m\Sigma V}{2} \text{Str}(U_0' + U_0'^{-1}) \right] \]

\[ + \int d^4x \left( -\frac{1}{2} \text{Str}(\partial_\mu \xi \partial_\mu \xi) - \frac{m_0^2}{2N_c} (\text{Str} \xi)^2 - \frac{\alpha}{2N_c} (\partial_\mu \text{Str} \xi)^2 \right) \]

\[ = \frac{1}{\sqrt{2\pi\langle Q^2 \rangle}} e^{-Q^2/2\langle Q^2 \rangle} \int dU_0' d\xi (S\det U_0')^Q \exp \left[ \frac{m\Sigma V}{2} \text{Str}(U_0' + U_0'^{-1}) \right] \]

\[ + \int d^4x \left( -\frac{1}{2} \text{Str}(\partial_\mu \xi \partial_\mu \xi) - \frac{m_0^2}{2N_c} (\text{Str} \xi)^2 - \frac{\alpha}{2N_c} (\partial_\mu \text{Str} \xi)^2 \right) + O(\epsilon^4) \]  

(A.1)

Here we have used

\[ U = U_0 e^{i\sqrt{2}\xi/F_\pi}, \quad \Phi_0 \equiv \frac{F_\pi}{\sqrt{2}} \text{Str}(-i\ln U_0), \quad U_0' = U_0 U_0, \]

\[ e^{iQ\theta} = (S\det U_0')^Q \exp \left( -\int d^4x \frac{\sqrt{2}iQ}{F_\pi V} \Phi_0 \right), \]

\[ \Phi_0' \equiv \frac{F_\pi}{\sqrt{2}} \text{Str}(-i\ln U_0') = \Phi_0 + \frac{F_\pi \theta}{\sqrt{2}}. \]

(A.2)  

(A.3)  

(A.4)  

(A.5)  

(A.6)

In the last line of (A.1), we carried out the integration over \( \theta \) as the Gaussian

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \exp \left[ -\frac{V m_0^2 F_\pi^2}{4N_c} \left( \theta - \frac{\sqrt{2}}{F_\pi} \Phi_0 \right)^2 \right. \]

\[ \left. + iQ \left( \theta - \frac{\sqrt{2}}{F_\pi} \Phi_0 \right) \right] \]

\[ = \exp \left( -\frac{Q^2}{2\langle Q^2 \rangle} \right) \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta' \exp \left[ -\frac{\langle Q^2 \rangle}{2} \left( \theta' - \frac{iQ}{\langle Q^2 \rangle} \right)^2 \right] \]

\[ \sim \frac{1}{\sqrt{2\pi\langle Q^2 \rangle}} \exp \left( -\frac{Q^2}{2\langle Q^2 \rangle} \right), \]

(A.7)

where \( \langle Q^2 \rangle = V m_0^2 F_\pi^2/2N_c \) and \( \theta' = \theta - \sqrt{2}\Phi_0/F_\pi \). To justify this Gaussian integral, we need the condition \(|Q|/\langle Q^2 \rangle \ll 1\); otherwise the integral (A.7) should depend on
\( \Phi'_0 \), which implies that \( \Phi'_0 \) and \( \theta \) cannot be treated independently, and the partition function Eq. (2.5) is not valid. On our lattice with \( \langle Q^2 \rangle = 4.34(22) \), it is not clear if this condition is satisfied for \( |Q| \neq 0 \).

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