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# Entanglement entropy in 1D critical systems with open boundaries

## and Sinusoidal deformation

Toshiya Hikihara (Hokkaido Univ.)

## ◆ Outline

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### 1. Boundary oscillation in Entanglement entropy

- ◆ Oscillating part of EE is proportional to oscillating part of nearest-neighbor spin correlations

[ N. Laflorencie et al. (2006)]

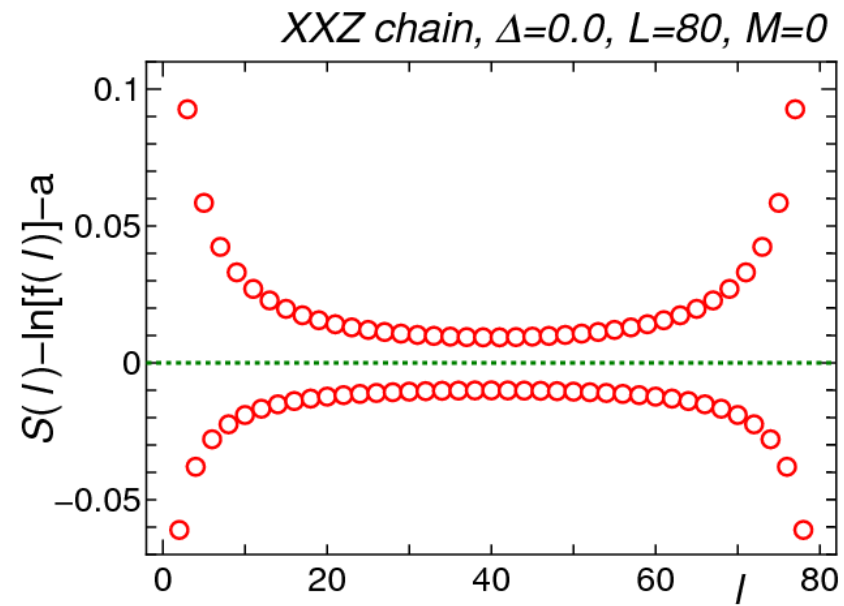
### 2. Elimination of the boundary oscillation in EE by Sinusoidal deformation

[Collab. with T. Nishino (Kobe)]

- ◆ Sinusoidal deformation realizes completely “periodic” ground state
- ◆ Numerical evidence :  
Entanglement entropy, spin correlation,  
and wave-function overlap



# 1. Boundary oscillation in Entanglement entropy



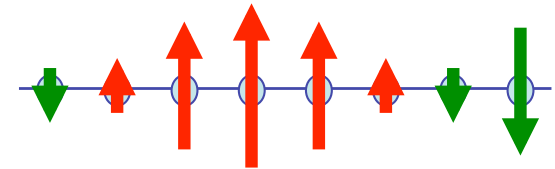
## ◆ 1D quantum critical system

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➤ 1D quantum critical system at low temperature ( $T=0$ )

**(typically) Tomonaga-Luttinger liquid (TLL)**

- ◆ Gapless excitations of collective modes
- ◆ Divergent correlation length :  $\xi \rightarrow \infty$
- ◆ Algebraically decaying correlation functions



➤ Important parameters to characterize the system

- ◆ central charge  $c$
- ◆ TLL parameter  $K$
- ◆ charge/spin velocity  $v$
- ◆ Fermi momentum  $k_F$

# ◆ Entanglement Entropy in 1D critical systems

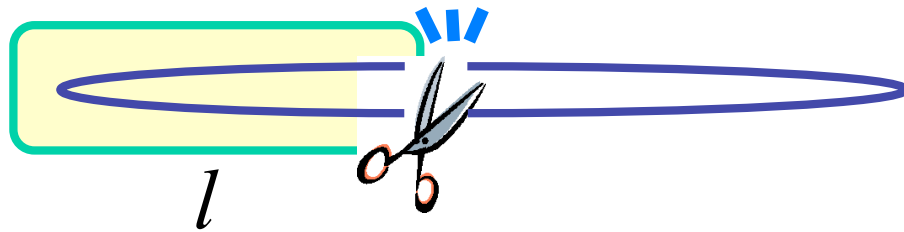
- Entanglement entropy for block  $\Omega$

$$S_{\Omega} = -\text{Tr}_{\Omega}(\rho_{\Omega} \ln \rho_{\Omega})$$

$$(\rho_{\Omega} = \text{Tr}_{\bar{\Omega}}|0\rangle\langle 0|$$

: reduced DM)

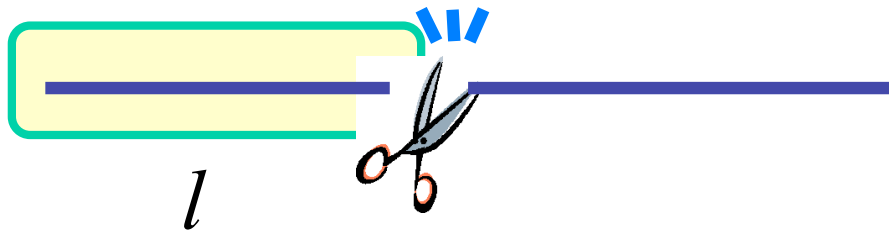
- ◆ Periodic b.c.



$$S(l) = \frac{c}{3} \ln[f(l)] + a$$

$$\left[ f(l) = \frac{L}{\pi} \sin\left(\frac{\pi}{L}l\right) \approx l \right]$$

- ◆ Open b.c.



$$S(l) = \frac{c}{6} \ln[f(l)] + a$$

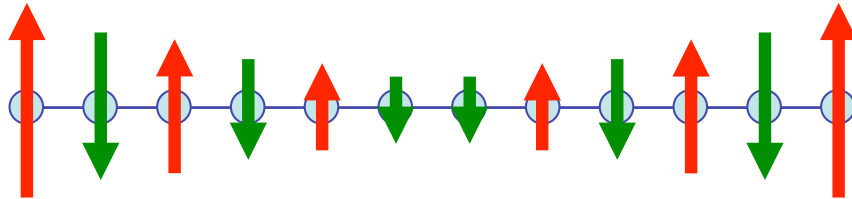
Calabrese-Cardy (2004)

**Slope of EE- $\ln l$  plot gives central charge  $c$**

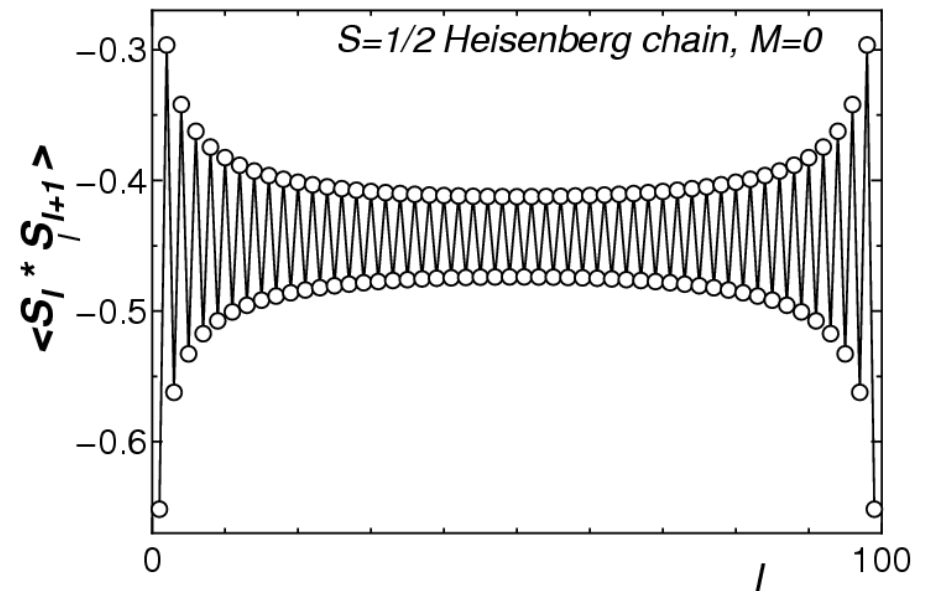
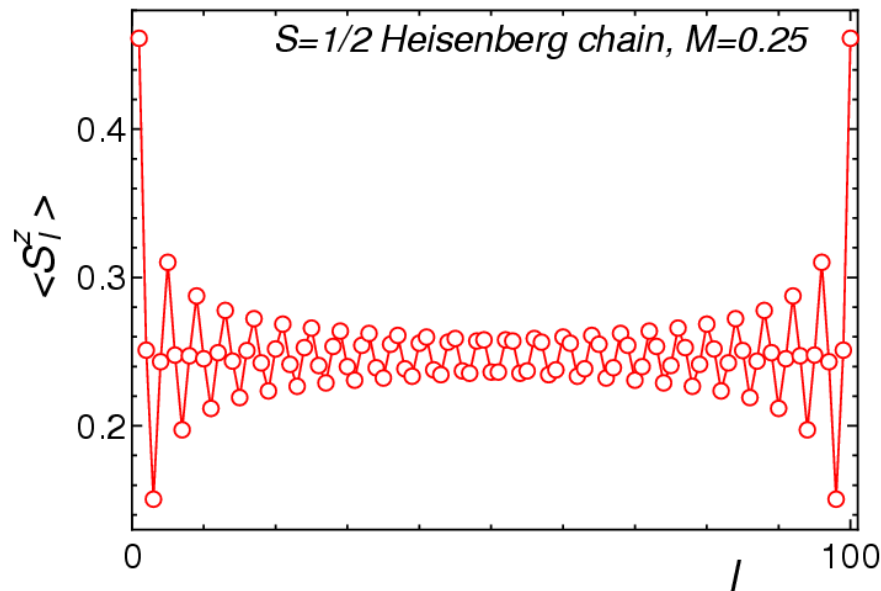
(provided that topology of state identified)

# ◆ Boundary oscillation in TLL

Friedel oscillation



Staggered pattern of strong/weak bonds



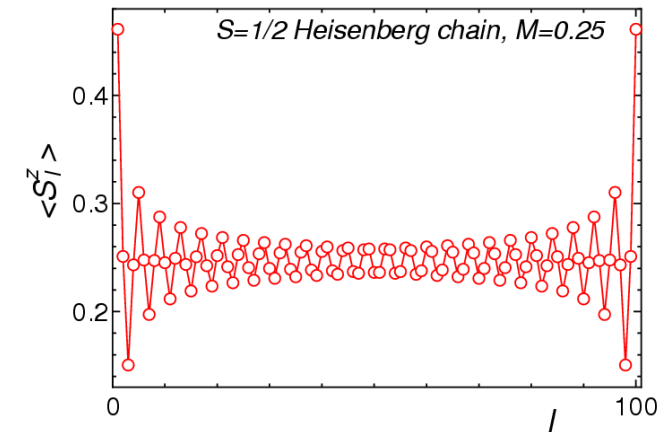
**$2k_F$**  - oscillation induced at open boundaries

## ◆ Boundary modulation in TLL

### ▶ Bosonization for spin-1/2 XXZ chain

$$\tilde{H} = \frac{v}{2} \int dx \left[ \frac{1}{K} (\partial_x \theta)^2 + K (\partial_x \phi)^2 \right]$$

[  $\phi(x), \theta(x)$  : boson field ]



$$\langle S_l^z \rangle = M + a \frac{\sin(2k_F l)}{[g(2l+1)]^K} + \dots \rightarrow M + \tilde{a} \sin(2k_F l) l^{-K}$$

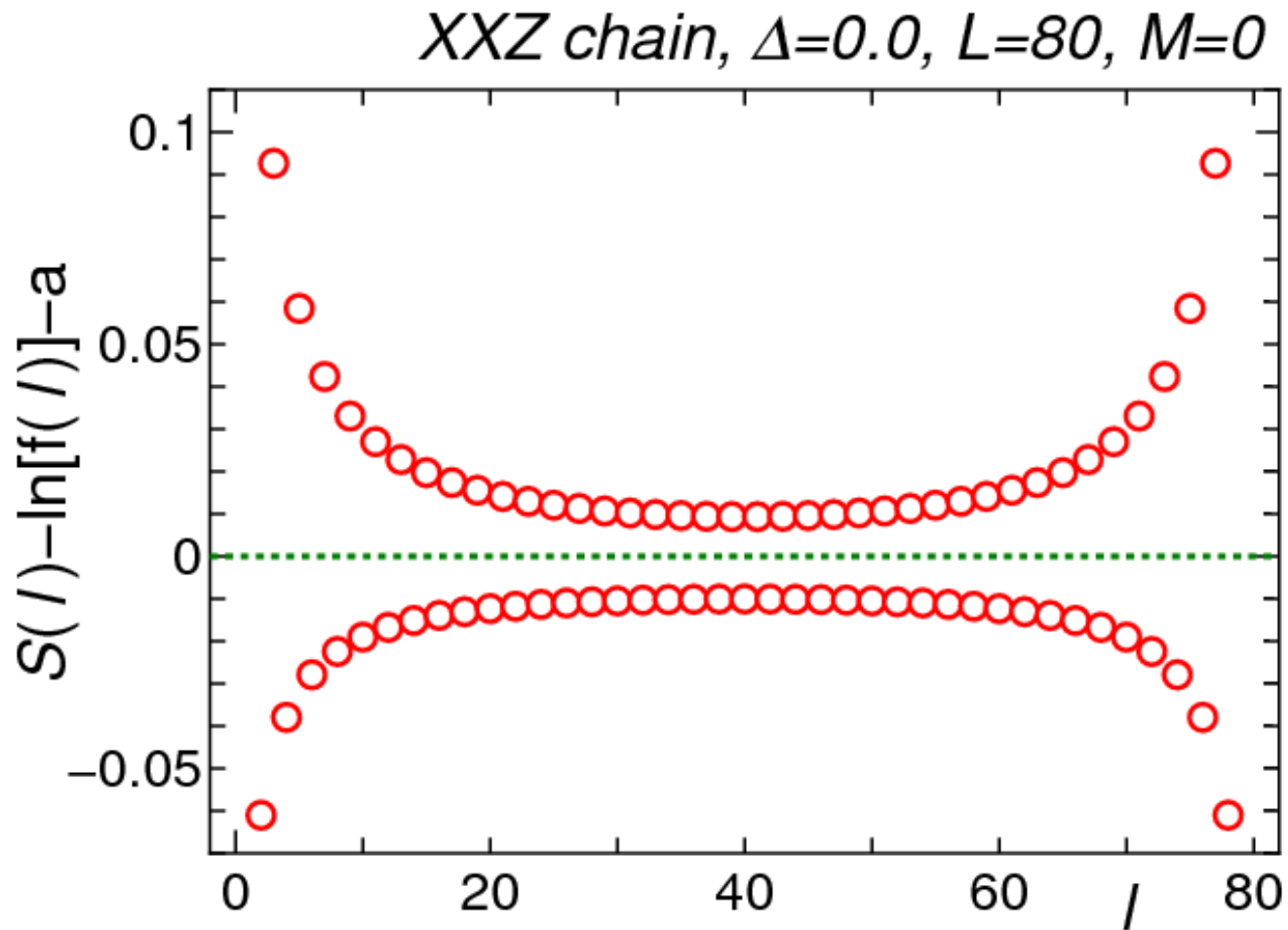
$$\langle S_l^\alpha S_{l+1}^\alpha \rangle = c_0^\alpha + c_1^\alpha \frac{\cos(2k_F l)}{[g(2l+1)]^{1/K}} + \dots \rightarrow c_0^\alpha + \tilde{c}_1^\alpha \cos(2k_F l) l^{-1/K}$$

$$\left[ g(x) = \frac{2(L+1)}{\pi} \sin\left(\frac{\pi x}{2(L+1)}\right) \right]$$

Boundary oscillation decays **algebraically**

with exponent related to  **$K$**

## ◆ Boundary oscillation in Entanglement Entropy



Entanglement Entropy also includes  
oscillating components from the boundaries

## ◆ Oscillating part in EE and spin correlations

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▶ N. Laflorencie et al., PRL 96, 100603 (2006)

◆ Spin-1/2 XXZ chain

$$H = J \sum_l \left( S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta S_l^z S_{l+1}^z \right) \quad (J > 0)$$

◆ Numerically found

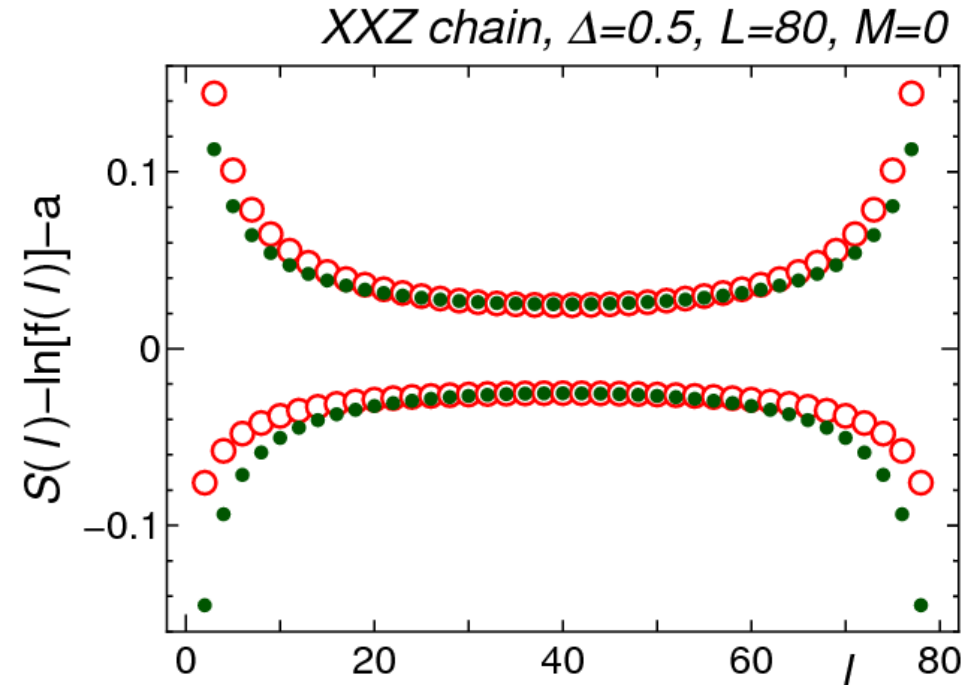
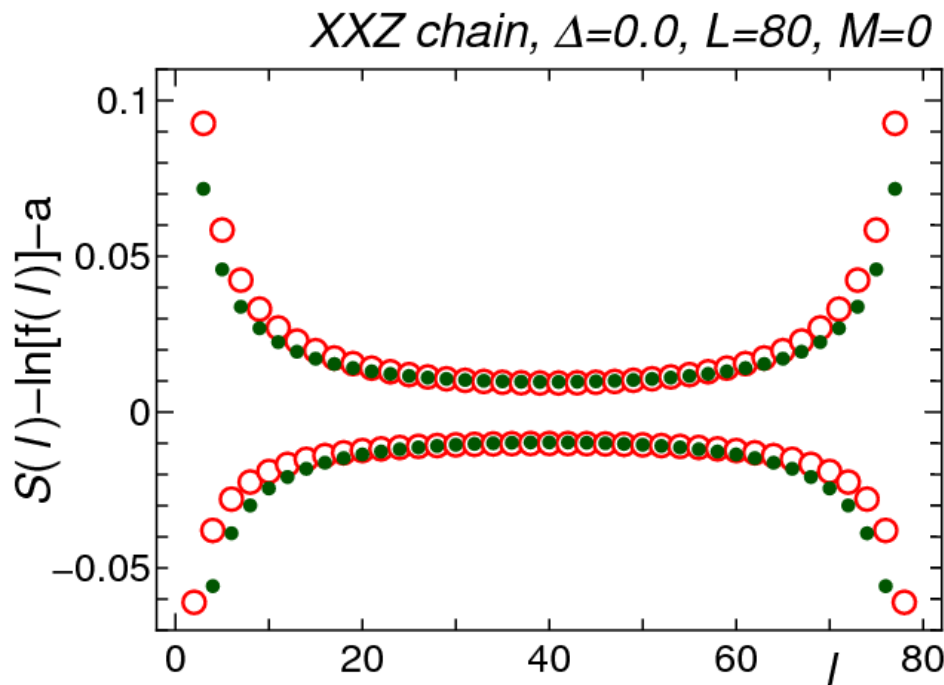
$$\begin{aligned} [S(l)]_{stag} &= - \left( \frac{\pi}{2\nu} \right) \left\langle S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta S_l^z S_{l+1}^z \right\rangle_{stag} \\ &= - \left( \frac{\pi}{2\nu} \right) c_1 \frac{(-1)^l}{[g(2l+1)]^{1/K}} \end{aligned}$$

Oscillating part of EE gives

information about TLL parameter  **$K$**

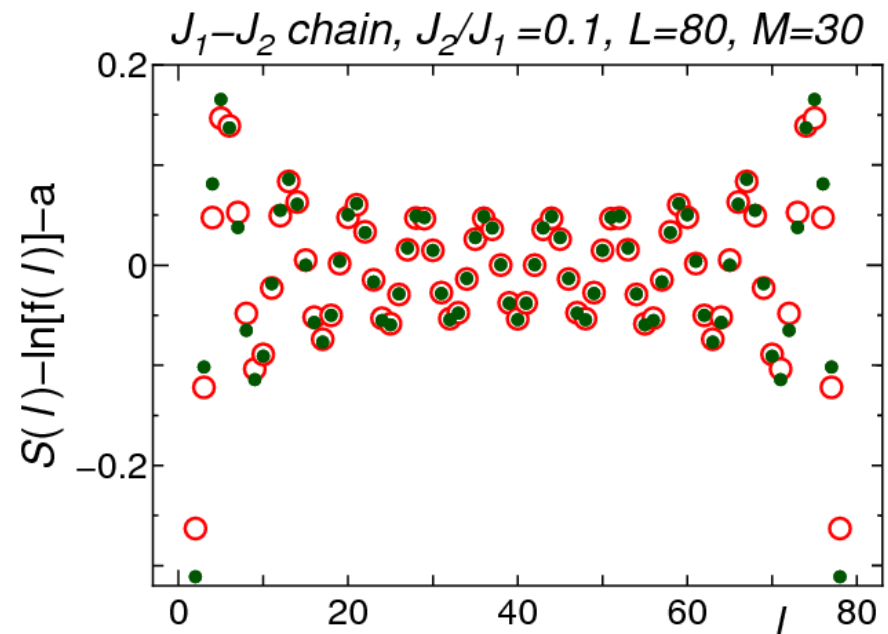
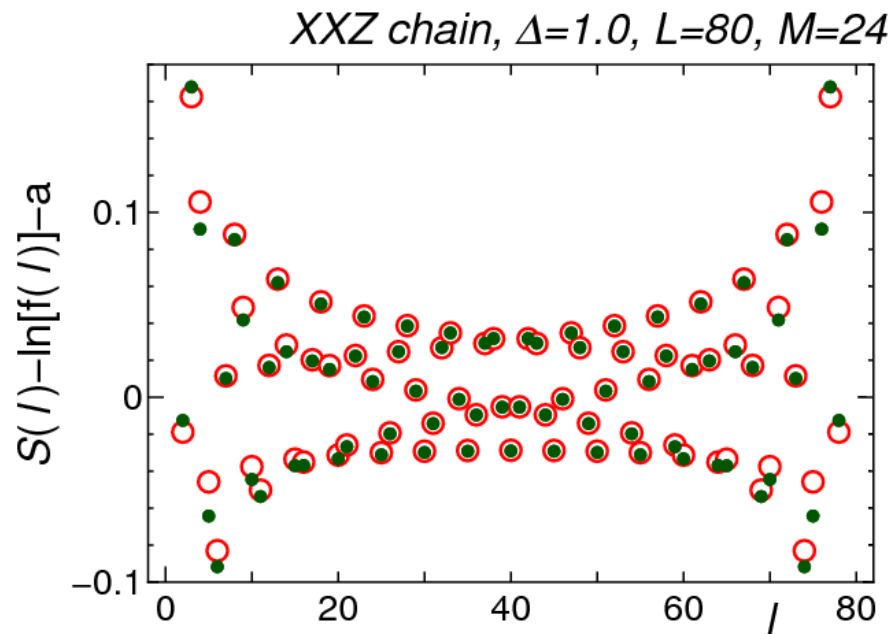
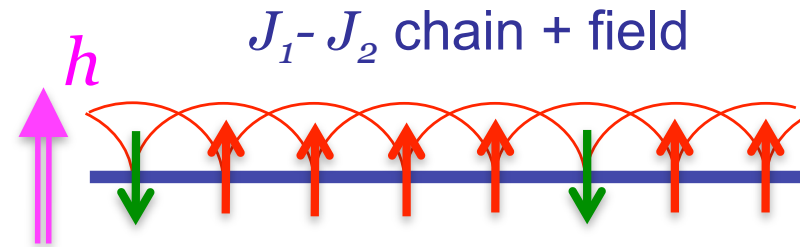
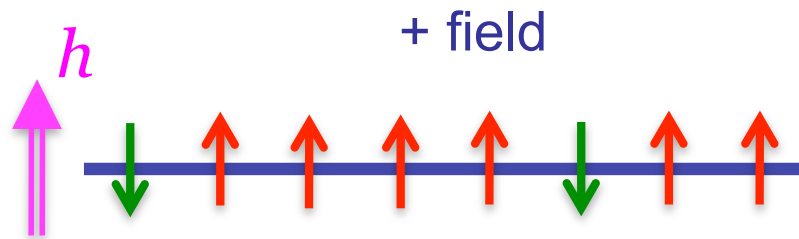
# ◆ Oscillating part in EE and spin correlations

Spin-1/2 XXZ chain



$$S(l) = \left( a + \frac{1}{6} \ln[f(l)] \right) - \left( \frac{\pi}{2\nu} \right) c_1 \frac{(-1)^l}{[g(2l+1)]^{1/K}} + \dots$$

# ◆ Oscillating part in EE and spin correlations

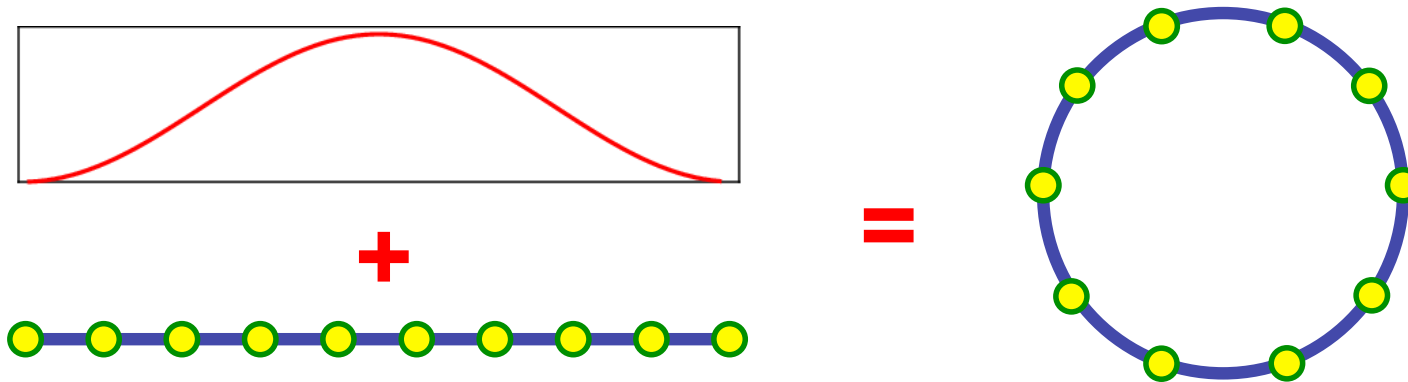


$$S(l) = \left( A + \frac{1}{6} \ln[f(l)] \right) - B \langle \mathbf{S}_l \cdot \mathbf{S}_{l+1} \rangle + \dots \quad (\text{in TLL1 phase})$$

**Analytic derivation of the relation is highly desirable...**



## 2. Elimination of the boundary oscillation in EE by Sinusoidal deformation



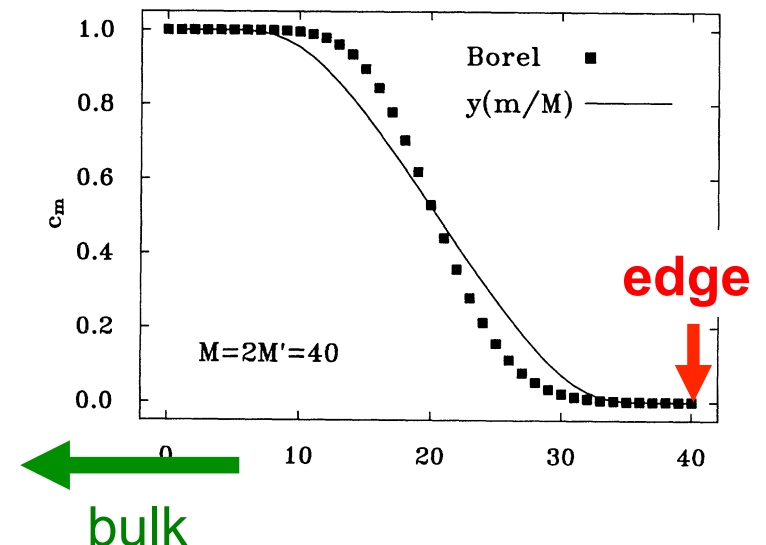
## ◆ Smooth boundary condition

- ▶ Boundary oscillation :  
(sometimes) an obstacle for studying thermodynamic limit

- ◆ Smooth boundary condition [Vekic-White 1993,1996]

"smoothly turn off (set to zero) the parameters of the Hamiltonian near the edges of the system."

Succeed (to some extent) to  
suppress boundary oscillation  
get smooth dependence  
on chemical potential



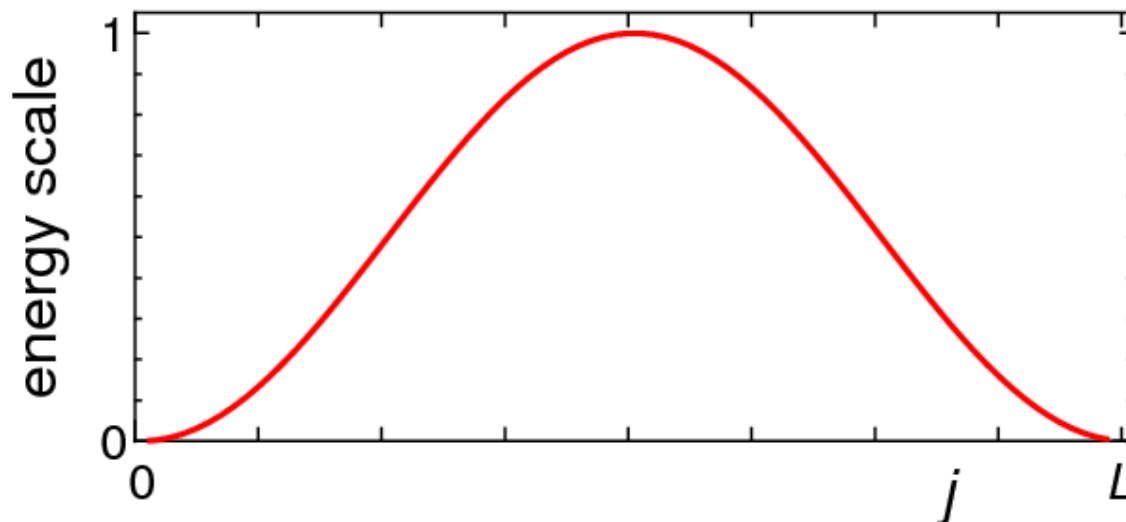
## ◆ Sinusoidal deformation

- ▶ Gendiar et al. (2009) : Sinusoidal (spherical) deformation

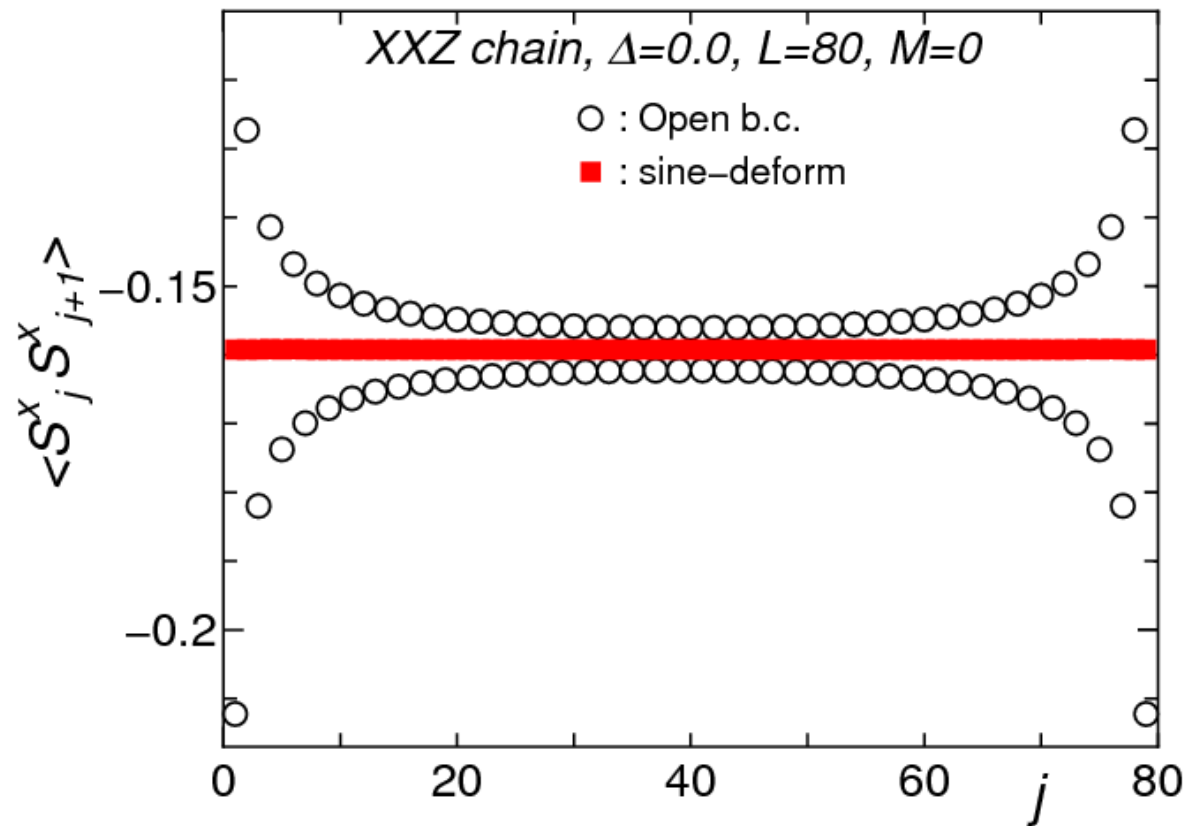
$$H = -t \sum_{j=1}^{L-1} \sin^2\left(\frac{\pi}{L} j\right) (c_j^+ c_{j+1} + c_{j+1}^+ c_j) - \mu \sum_{j=1}^L \sin^2\left(\frac{\pi}{L} \left(j - \frac{1}{2}\right)\right) c_j^+ c_j$$

Free fermions in open chain

with energy-scale deformation  $\sim \sin^2\left(\frac{\pi}{L} \left(x - \frac{1}{2}\right)\right)$



## ◆ Sinusoidal deformation



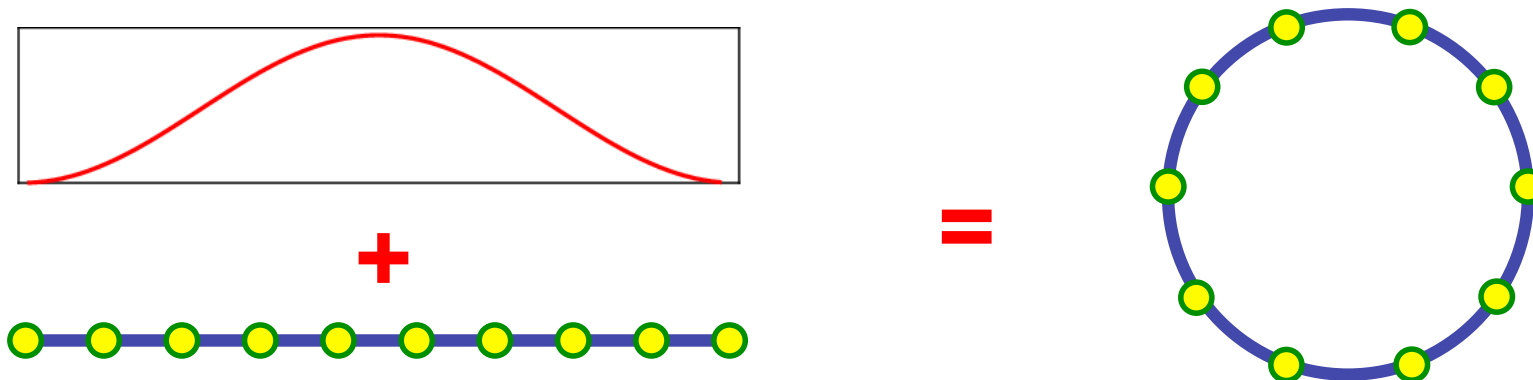
- ◆ Nearest-neighbor correlation becomes **position independent** almost completely
  - ◆ Also reported correction of g.s. energy :  $E_{gs} \sim \frac{1}{L^2}$
- ➔ Sin.-deformation eliminates boundary effects efficiently

## ◆ Sinusoidal deformation

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**Q.** What's happening in the system with sinusoidal deformation?

**A.** Sinusoidal deformation is not only an efficient smooth boundary condition. It changes the topology of the ground state and realizes the uniform periodic state.



# ◆ Entanglement Entropy in 1D critical systems

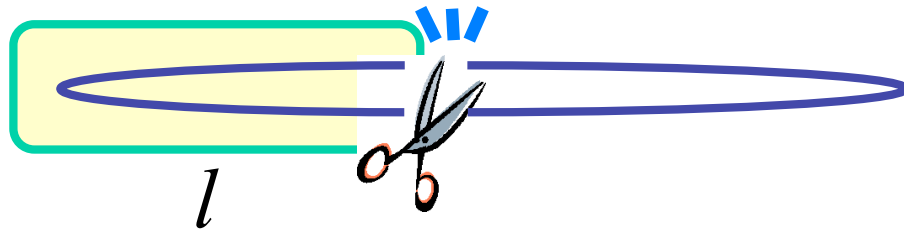
➤ Entanglement entropy for block  $\Omega$

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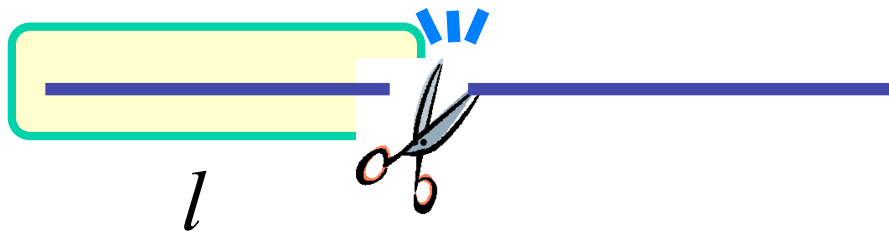
◆ Periodic b.c.



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◆ Open b.c.



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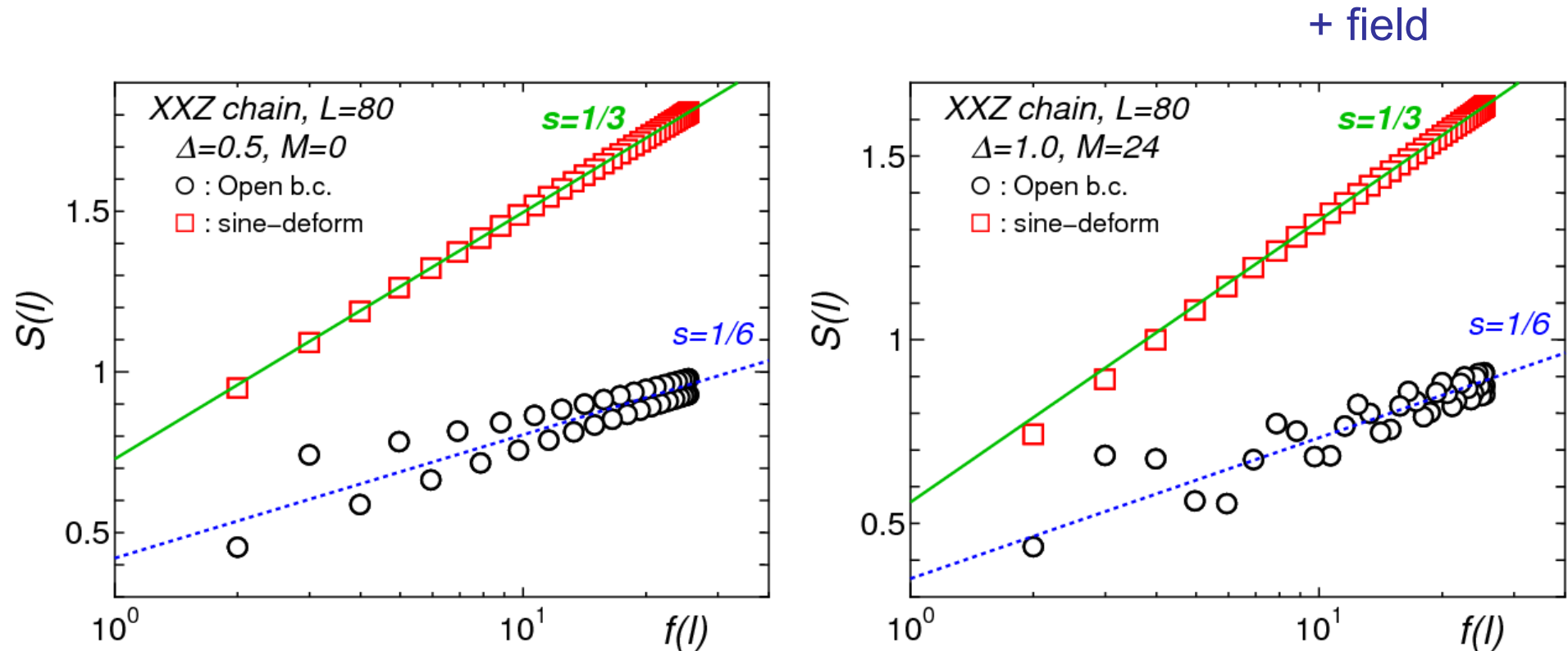
Calabrese-Cardy (2004)

**EE can identify the topology of the state**

(provided that central charge  $c$  is known)

# ◆ EE in sinusoidally-deformed systems

## ➤ EE in spin-1/2 XXZ chain

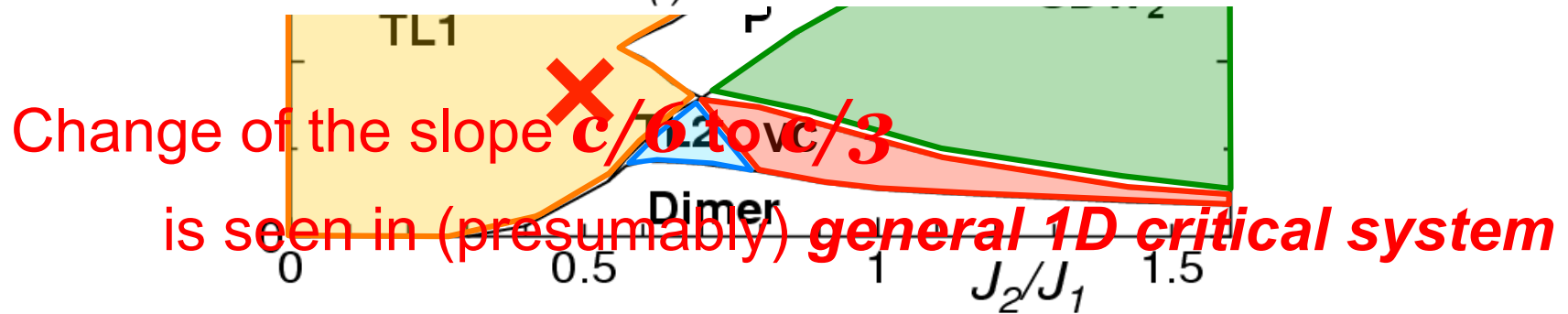
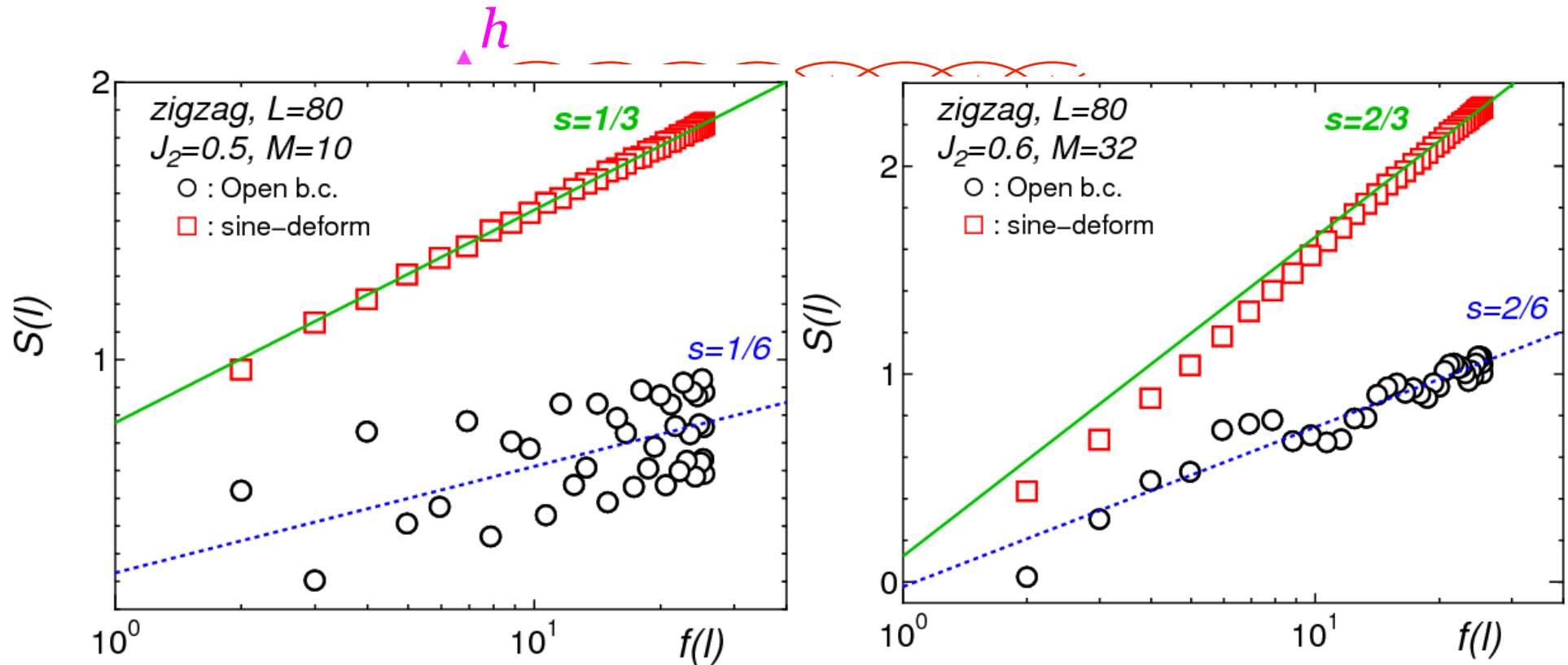


Sinusoidal deformation eliminates boundary modulation

**and changes the slope  $c/6$  to  $c/3$**

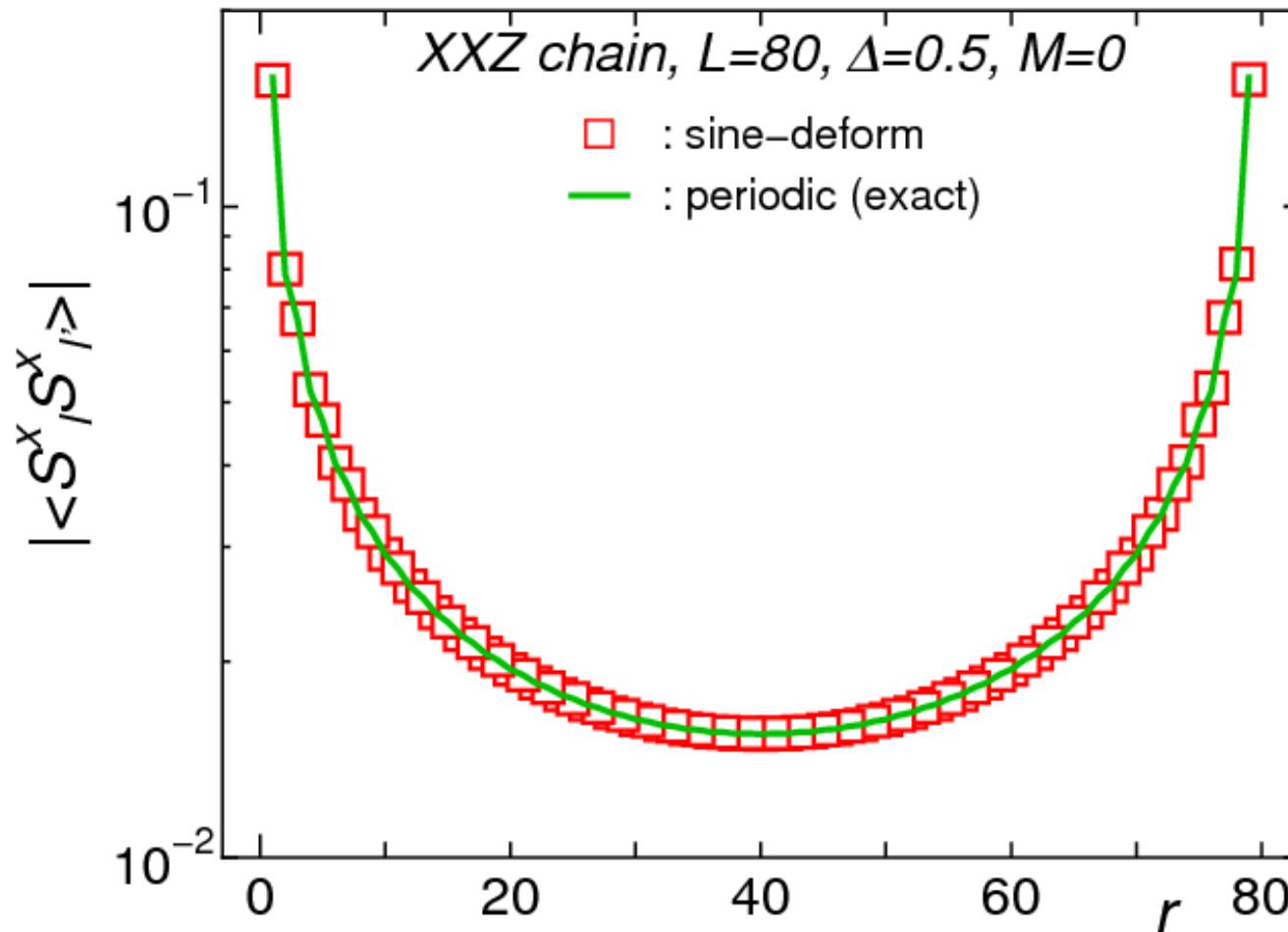
# ◆ EE in sinusoidally-deformed systems

- EE in spin-1/2  $J_1$ - $J_2$  chain under magnetic field



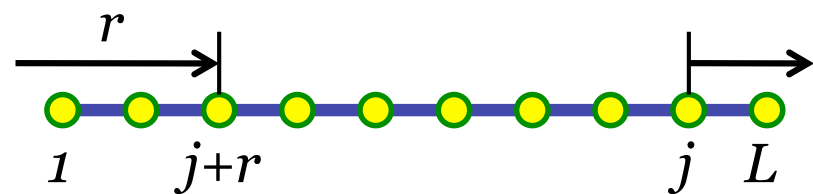
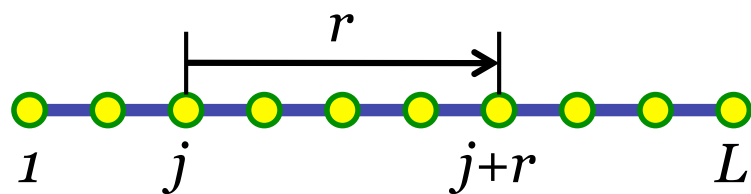
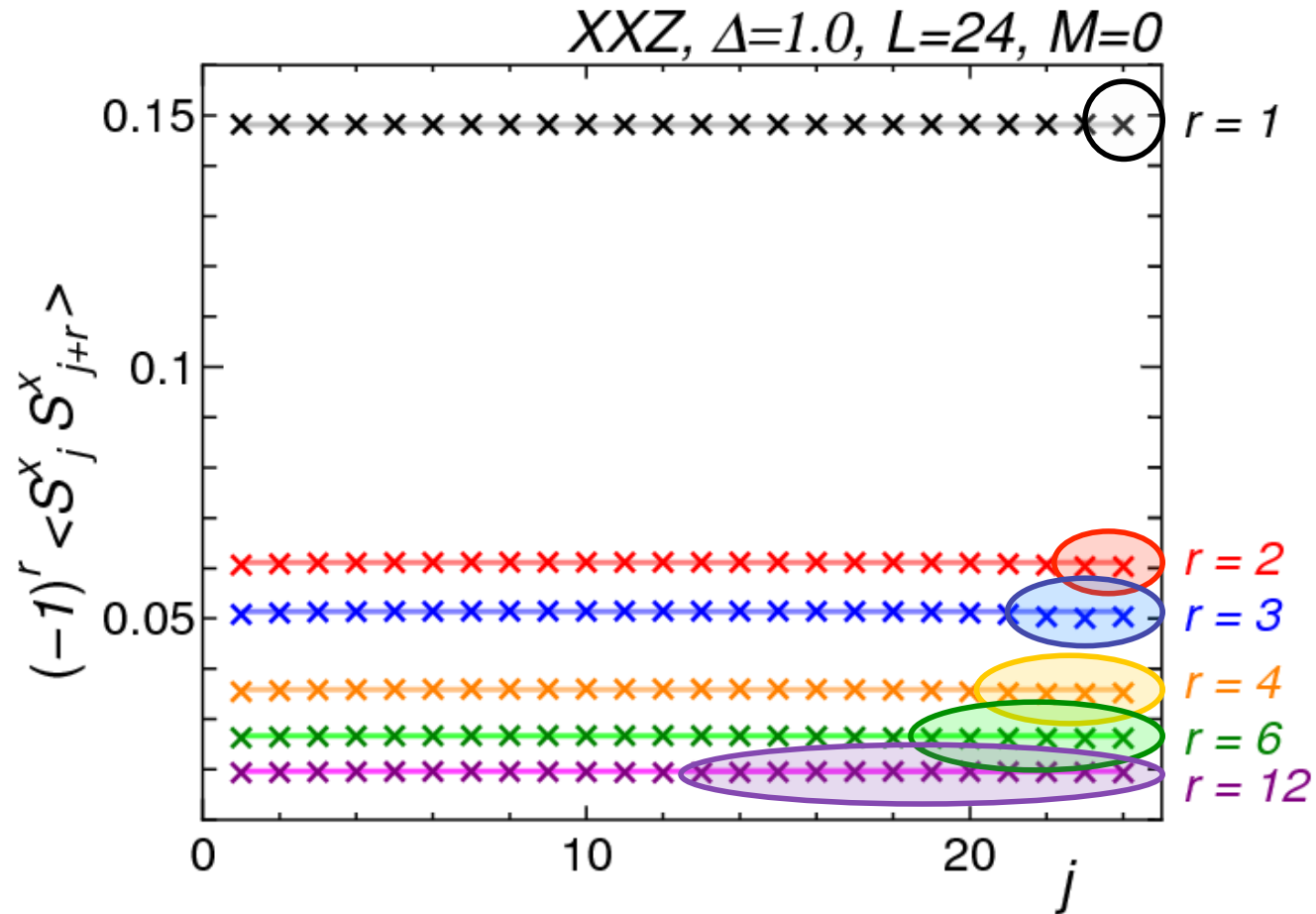
# ◆ Correlations in sinusoidally-deformed systems

## ➤ Spin correlation in spin-1/2 XXZ chain



# ◆ Correlations in sinusoidally-deformed systems

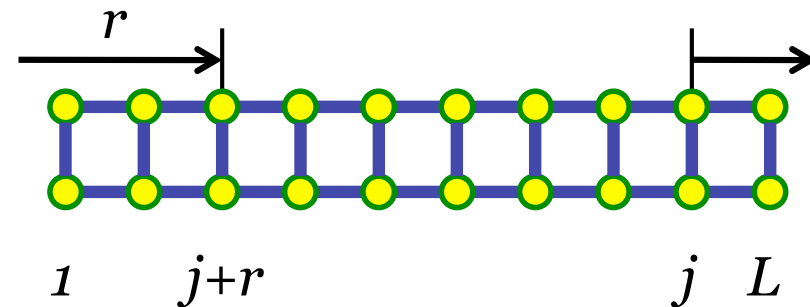
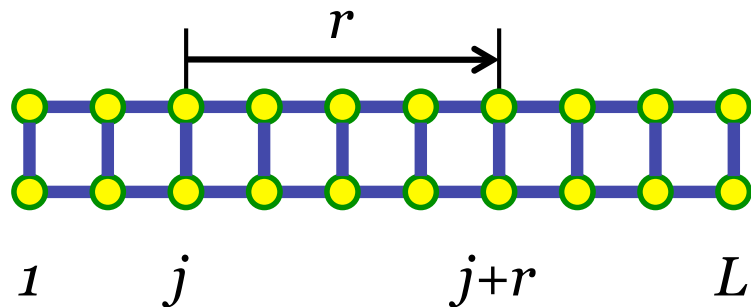
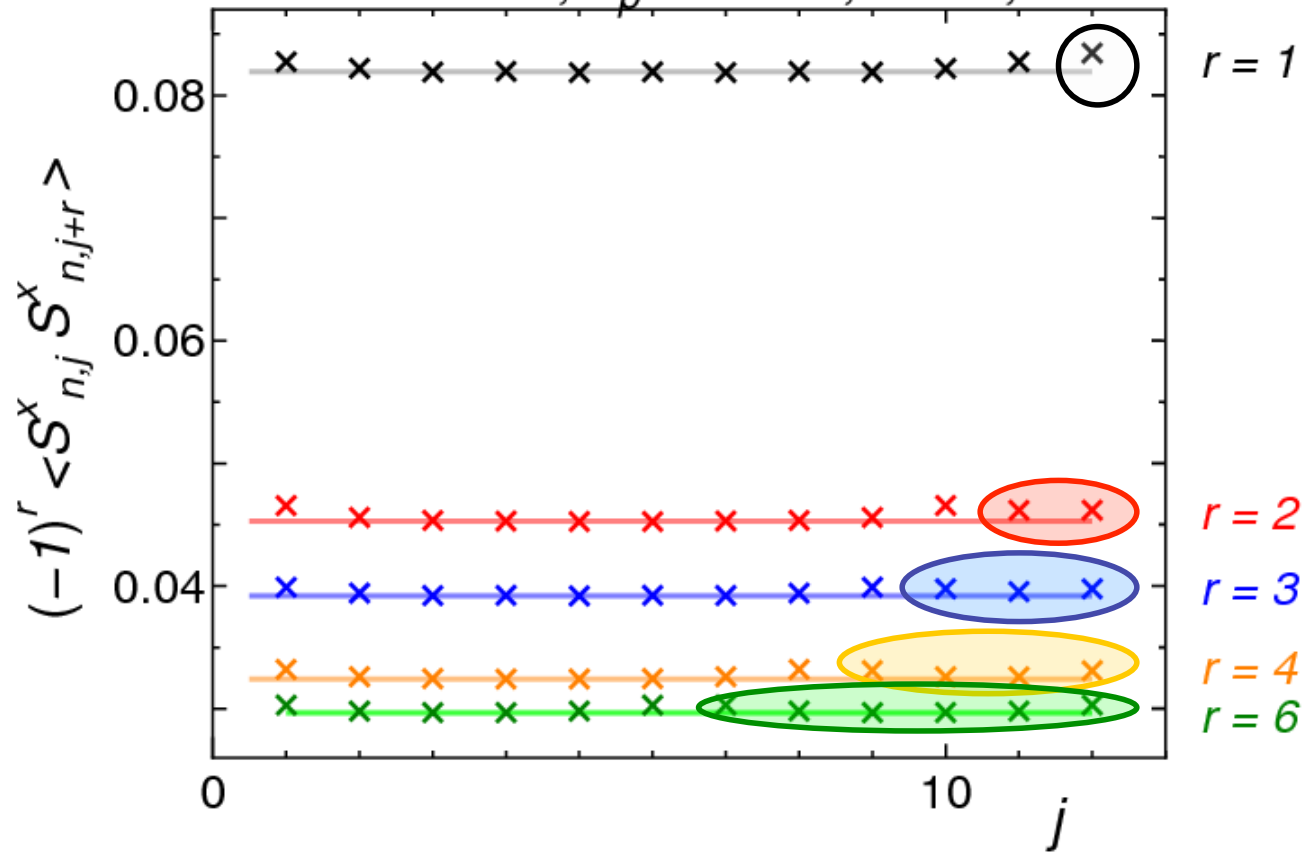
## ➤ Spin correlation in spin-1/2 XXZ chain



# ◆ Correlations in sinusoidally-deformed systems

## ➤ Spin correlation in two-leg spin ladder

*ladder,  $J_p/J=10.0$ ,  $L=12$ ,  $M=6$*



## ◆ Wave functions

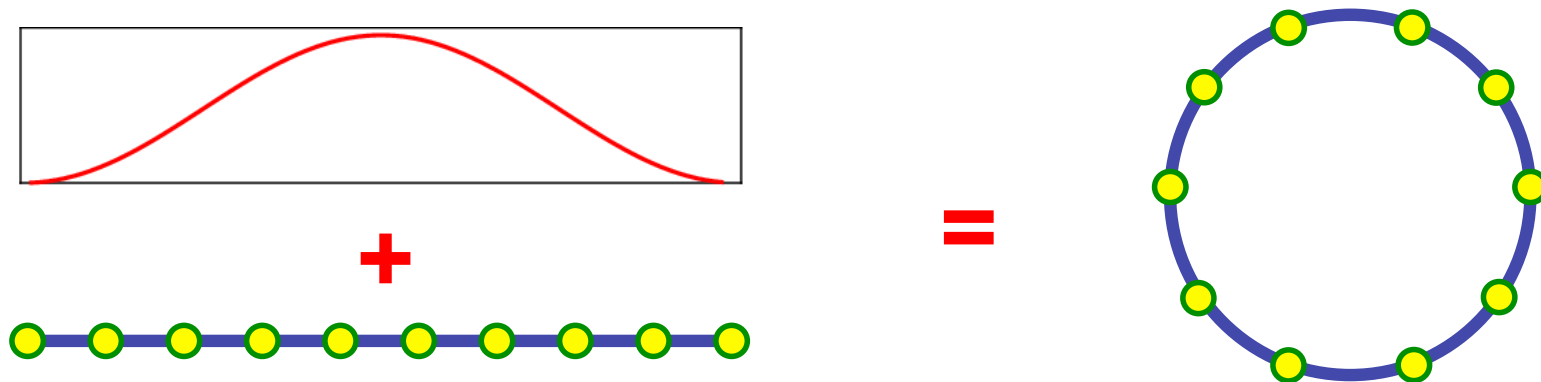
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- Overlap between wave functions of uniform-periodic and sinusoidally-deformed systems

$$\left| 1 - \langle \mathbf{v}_{\text{Sin}} | \mathbf{v}_{\text{PBC}} \rangle \right| < 10^{-4}$$

found numerically by exact diagonalization

**Sinusoidal deformation realizes the ground state of uniform periodic system at level of wave function**



## ◆ Wave functions

### ➤ Equivalence of the wave functions

- Not trivial even in the XX (free fermion) case

$$H = -t \sum_{j=1}^{L-1} \sin^2 \left( \frac{\pi}{L} j \right) (c_j^+ c_{j+1} + c_{j+1}^+ c_j) - \mu \sum_{j=1}^L \sin^2 \left( \frac{\pi}{L} \left( j - \frac{1}{2} \right) \right) c_j^+ c_j$$

- ◆ One-particle eigen w.f.  $\varphi_n$  is not the eigen w.f.  $\phi_k \sim e^{ikj}$  of the uniform periodic chain

→ When (and only when) the particles are filled up to the Fermi energy, the Slater determinants become equivalent

$$\prod_{\varepsilon_n < \varepsilon_F} \Psi_n^+ |0\rangle = \prod_{\varepsilon_k < \varepsilon_F} \Phi_k^+ |0\rangle$$

## ◆ Summary

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- Boundary oscillation in Entanglement entropy
  - ◆ Oscillating part of EE : proportional to NN spin correlations
    - information of TLL parameter  $K$
- Sinusoidal deformation
  - ◆ Change the slope of Entanglement Entropy from  $c/6$  to  $c/3$
  - ◆ spin correlations exhibit "translational" symmetry
  - ◆ wave-function overlap

**Sinusoidal deformation changes  
the topology of the system  
→ realize “periodic” ground state**