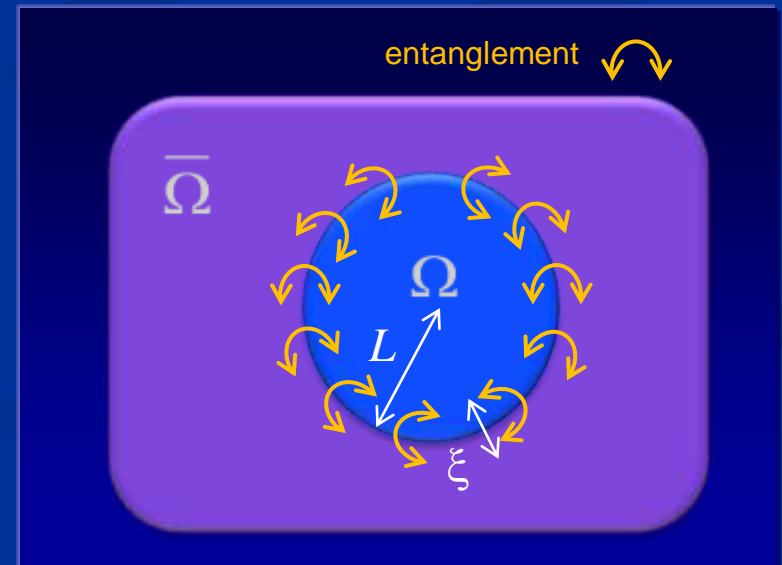
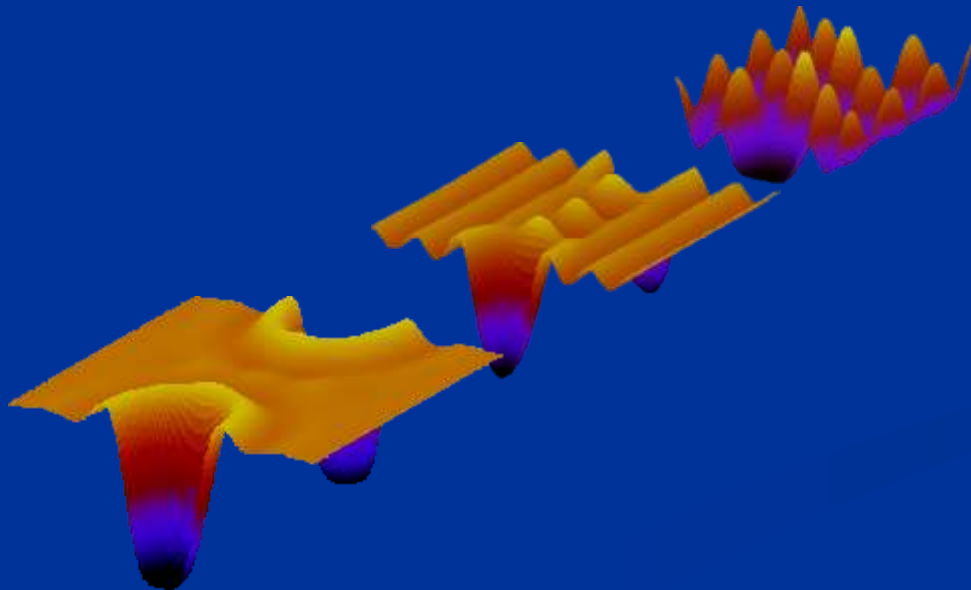


Quantum Hall States and Entanglement Entropy

Tohoku Univ. N. Shibata

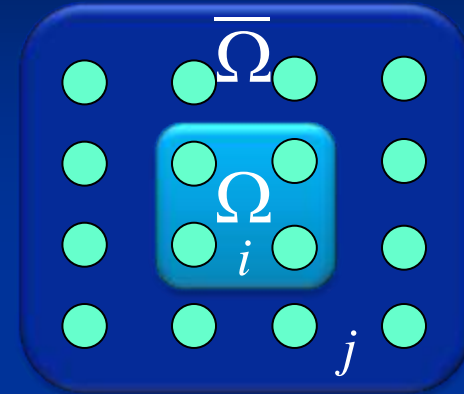
- Entanglement entropy
- Density matrix renormalization group
- Topological entanglement entropy of Quantum Hall states
- Relation between 2D quantum Hall systems and 1D systems



Entanglement entropy

Basis states : $|i j \rangle = |i \rangle_{\Omega} |j \rangle_{\bar{\Omega}}$

Wave function : $|\Psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle_{\Omega} |j\rangle_{\bar{\Omega}}$



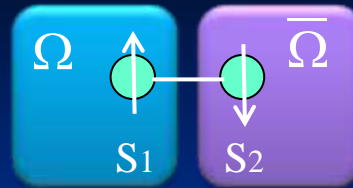
Reduced density matrix : $\rho_{\Omega} = Tr_{\bar{\Omega}} |\Psi\rangle\langle\Psi|$ $(\rho_{\Omega})_{ii'} = \sum_j \Psi_{ij}^* \Psi_{i'j}$

Entanglement entropy : $S_{\Omega} = -Tr \rho_{\Omega} \ln \rho_{\Omega}$

Measure of entanglement between two regions

Entanglement entropy

S=1/2 2-Spin system



S1 = ↑ or ↓

S2 = ↑ or ↓

Wave function

$$|\Psi\rangle = \sum_{S_1 S_2} \Psi_{S_1 S_2} |S_1\rangle |S_2\rangle$$

$$\Psi_{S_1 S_2} : \begin{pmatrix} \Psi_{\uparrow\uparrow} & \Psi_{\uparrow\downarrow} \\ \Psi_{\downarrow\uparrow} & \Psi_{\downarrow\downarrow} \end{pmatrix} = \Psi$$

Reduced density matrix

$$\rho_\Omega = \Psi^* \Psi^t$$

$$(\rho_\Omega)_{ii'} = \sum_j \Psi_{ij}^* \Psi_{i'j}$$

Entanglement entropy

$$S_\Omega = -\text{Tr} \rho_\Omega \ln \rho_\Omega$$

S1 is independent of S2

- Not correlated -

$$\frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle) \rightarrow \Psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} |\uparrow\rangle_{S_1} \otimes (|\uparrow\rangle + |\downarrow\rangle)_{S_2}$$

$$\rho_\Omega = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_\Omega = -\ln 1 = 0$$

disentangled

S1 depends on S2

- correlated -

$$\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \rightarrow \Psi = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Singlet state

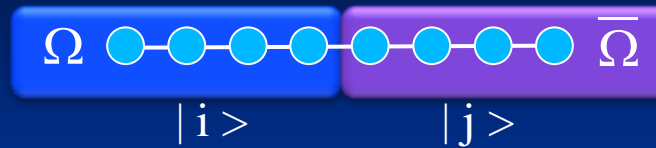
$$\rho_\Omega = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$S_\Omega = 2 \left(\frac{1}{2} \ln 2 \right) = \ln 2$$

entangled

Entanglement entropy

Many electron system



Basis states : $|i j\rangle = |i\rangle|j\rangle$ $i \rightarrow \{1,2,3,4, \dots m\}$ $|\Psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle|j\rangle$
 $j \rightarrow \{1,2,3,4, \dots m\}$

Ω is independent of $\bar{\Omega}$

$$\frac{1}{\sqrt{m}} (|1 1\rangle + |1 2\rangle + |1 3\rangle + \dots + |1 m\rangle) \rightarrow \Psi = \begin{pmatrix} \frac{1}{\sqrt{m}} & \dots & \frac{1}{\sqrt{m}} \\ 0 & \dots & 0 \\ \vdots & \mathbf{0} & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{m}} |1\rangle_i \otimes (|1\rangle + |2\rangle + \dots + |m\rangle)_j$$

$$\rho_{ii'} = \sum_j \Psi_{ij}^* \Psi_{i'j}$$

$$\rho = \begin{pmatrix} \frac{1}{\sqrt{m}} & \dots & \frac{1}{\sqrt{m}} \\ 0 & \dots & 0 \\ \vdots & \mathbf{0} & \vdots \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{m}} & \dots & 0 \\ \frac{1}{\sqrt{m}} & \dots & 0 \\ \vdots & \mathbf{0} & \vdots \\ \frac{1}{\sqrt{m}} & \dots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \rightarrow$$

Entanglement entropy

$$S = -\text{Tr} \rho \ln \rho$$

$$= -\ln 1 = 0$$

disentangled

Entanglement entropy

Basis states :



$$|i j\rangle = |i\rangle |j\rangle$$

$$|i\rangle$$

$$|j\rangle$$

Ω and $\bar{\Omega}$ are correlated

$$\frac{1}{\sqrt{m}} (|11\rangle + |22\rangle + |33\rangle + \dots + |mm\rangle) \rightarrow \Psi = \begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m}} & \\ 0 & & \ddots \\ & & & \frac{1}{\sqrt{m}} \end{pmatrix}$$

$$\rho_{ii'} = \sum_j \Psi_{ij}^* \Psi_{i'j}$$

$$\rho = \begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m}} & \\ 0 & & \ddots \\ & & & \frac{1}{\sqrt{m}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{m}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{m}} & \\ 0 & & \ddots \\ & & & \frac{1}{\sqrt{m}} \end{pmatrix} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & \\ 0 & & \ddots \\ & & & \frac{1}{m} \end{pmatrix}$$

Entanglement entropy

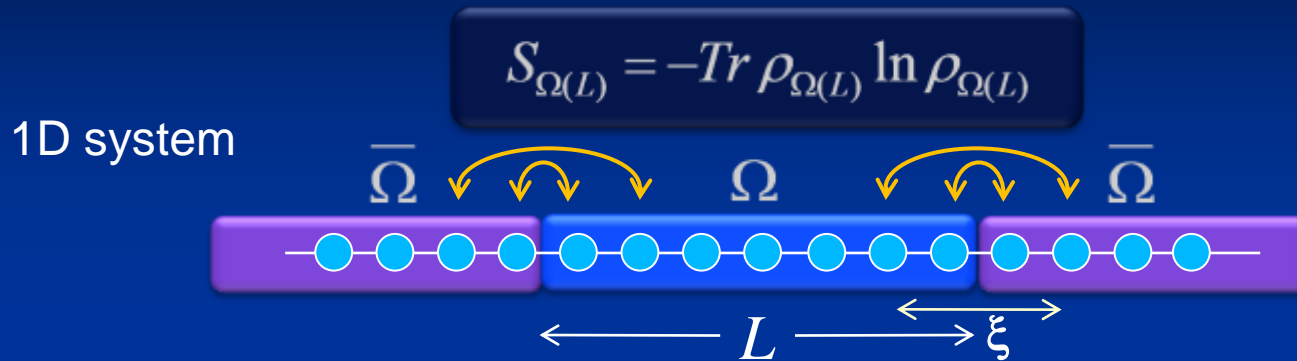
$$S = -\text{Tr} \rho \ln \rho$$

$$= m \left(\frac{1}{m} \ln m \right) = \ln m$$

Maximally entangled

Scaling of entanglement entropy

Relation between correlations and scaling of entanglement entropy



Short range correlation :

$$S_{\Omega(L)} \approx \text{const.}$$

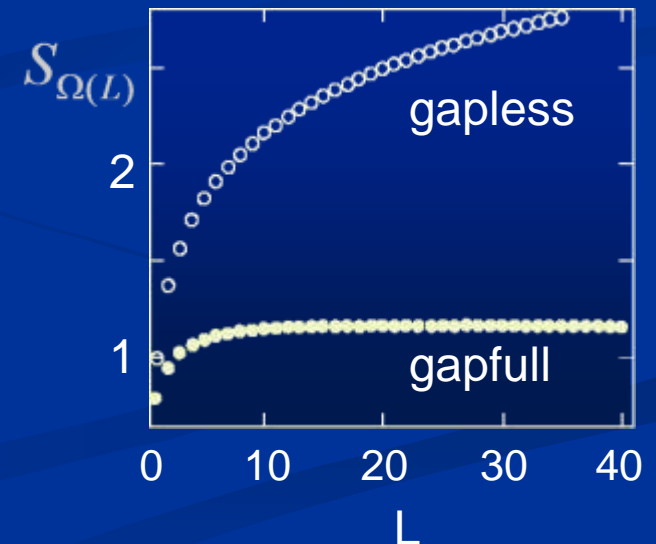
($L \gg$ correlation length ξ)

Power law correlation : (1D critical system)

$$S_{\Omega(L)} \approx \frac{c}{6} \ln L + s_0$$

c: central charge

1D-XXZ model



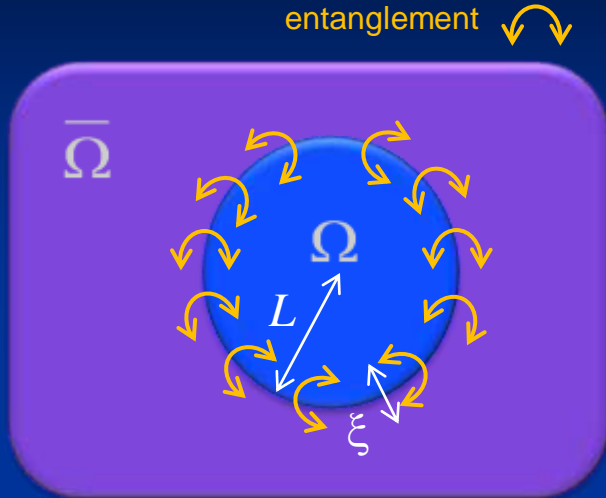
Scaling of entanglement entropy

Short range correlation ($L \gg \xi$)

D-dimensional system

Area law $S_{\Omega(L)} \propto L^{D-1}$

L^{D-1} : boundary size (length)



Srednicki, PRL, 1993 Wolf, Verstraete, Hastings, Cirac, PRL, 2008

Topological order in 2D

$$S_{\Omega(L)} = \underbrace{\alpha L}_{\text{Boundary term}} - \underbrace{\ln D}_{\text{Topological term}}$$

Non-trivial
universal correction
 $\ln D$
Topological term

Fractional quantum Hall state

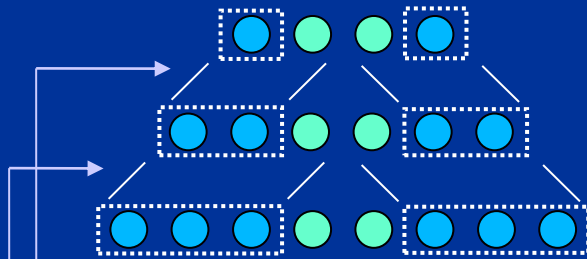
$$\nu = 1/m \text{ Laughlin state : } D = m^{1/2}$$

Kitaev & Preskill; Levin & Wen, PRL, 2006

Application of DMRG to quantum Hall systems

To confirm the prediction of boundary term and topological term

Real space renormalization

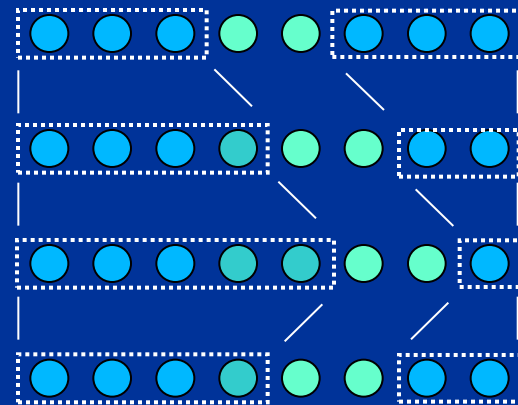


Large size of systems

Extend the blocks
with the restriction of basis states

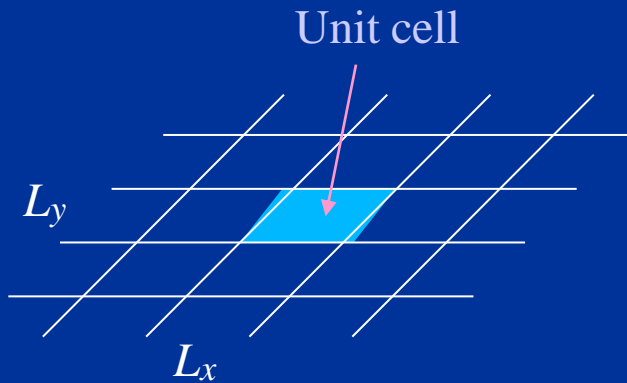
Steven White (1992)

Variational method



Controlled accuracy

Application of DMRG



Periodic boundary conditions
for both x and y directions

$$k_y = 2\pi n / L_y = X_n / l^2$$

Initial basis states (Landau gauge)

$$\varphi_{XN}(\mathbf{r}) = \exp \left[i \frac{X_n y}{l^2} - \frac{(x - X_n)^2}{2l^2} \right] H_N \left(\frac{x - X_n}{l} \right)$$

H_N : Hermite polynomials

One particle states are uniquely
specified by X_n and N

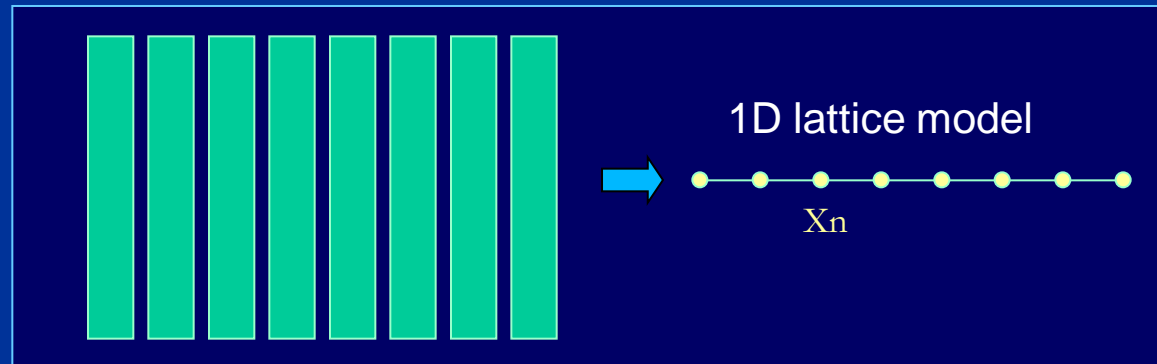


X_n : guiding center
 N : Landau level index

Mapping on to effective 1D lattice model

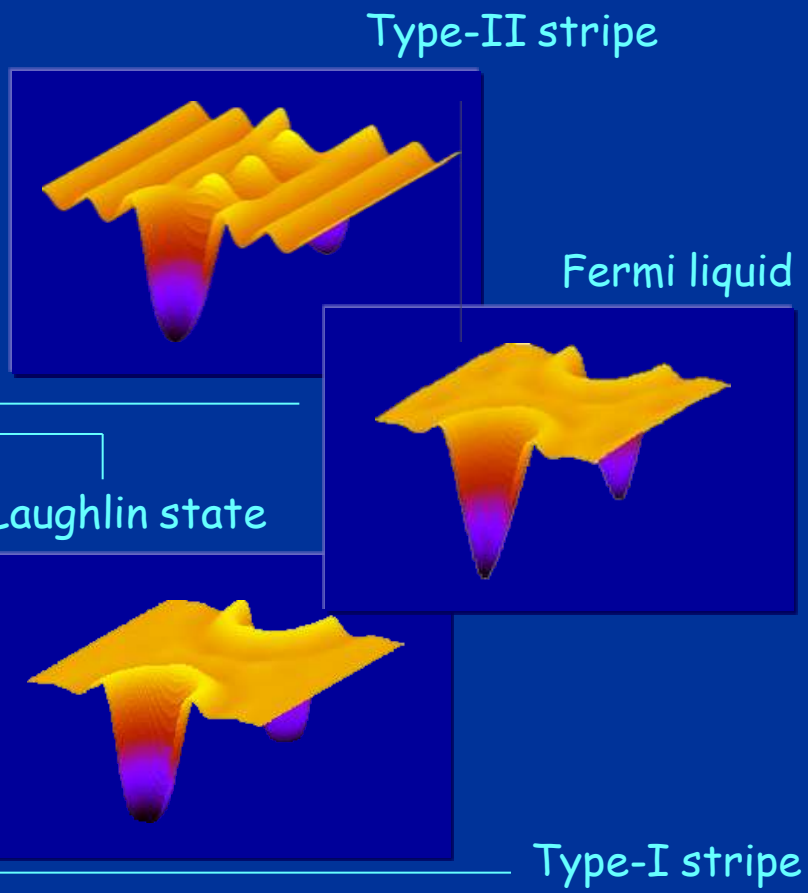
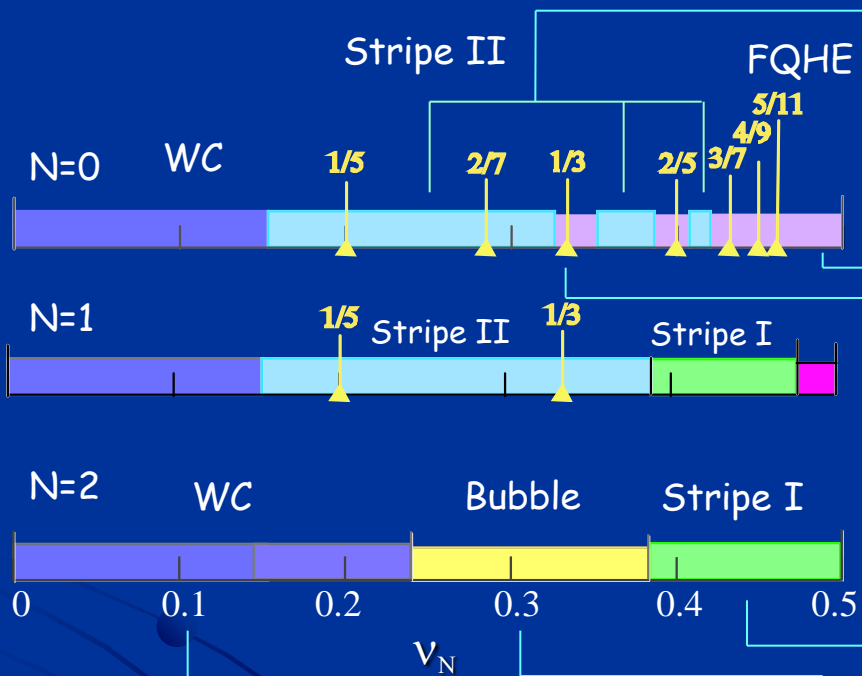
Ground state energy (Ne=10)

DMRG	m=100	-3.239340
	m=200	-3.239686
	m=300	-3.239981
	m=400	-3.239993
N=2		
v=1/2	Exact	-3.239995



Ground state of Quantum Hall systems

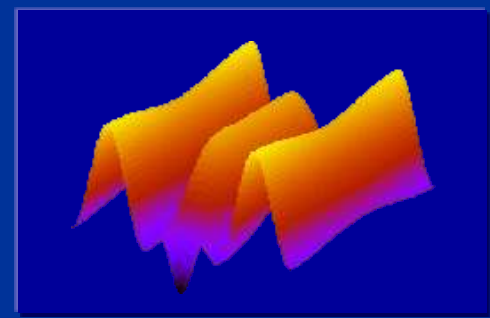
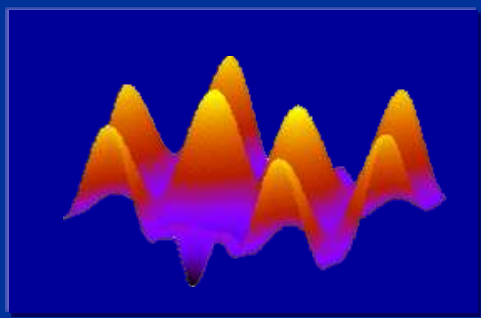
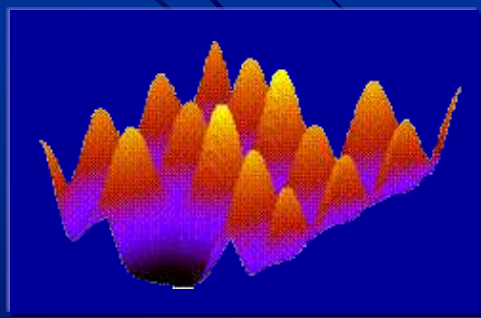
DMRG



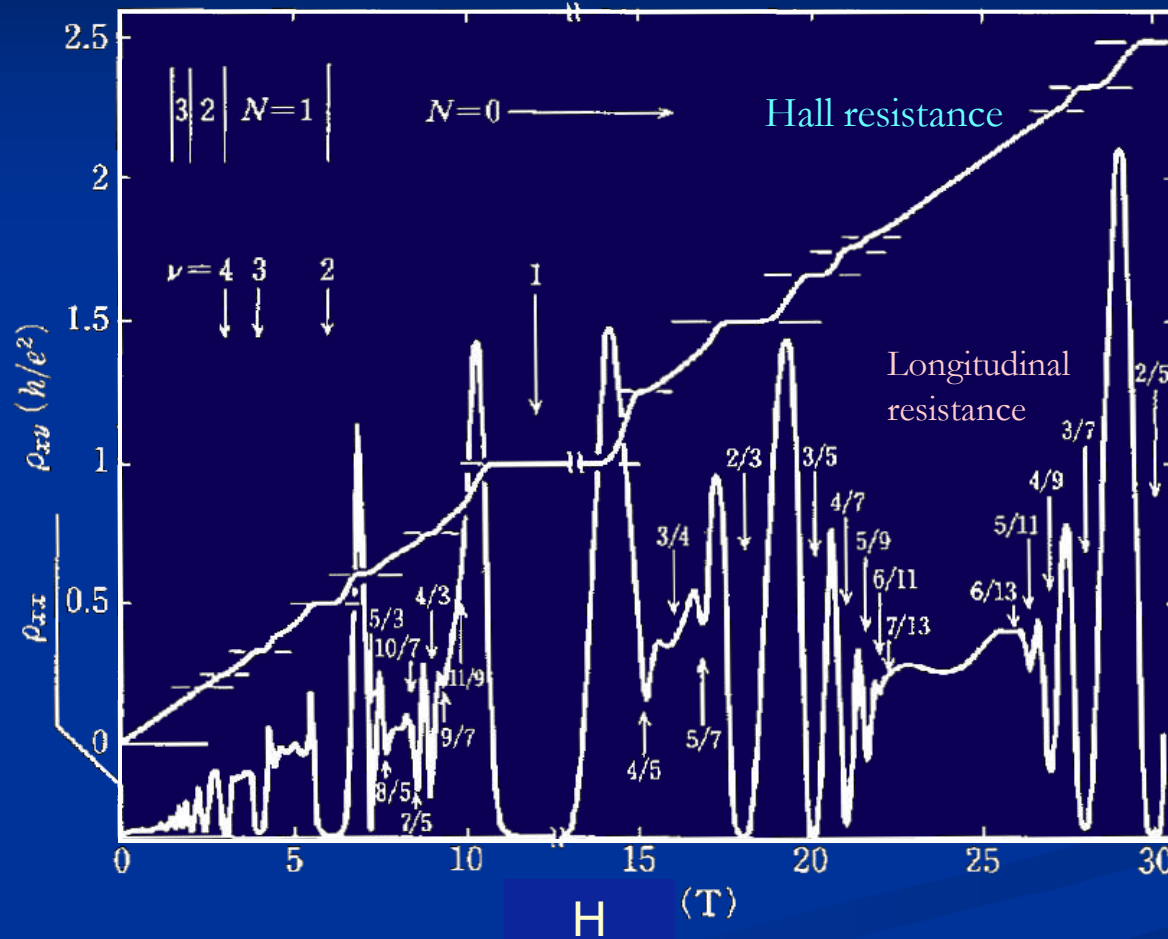
Wigner crystal

Bubble

Type-I stripe

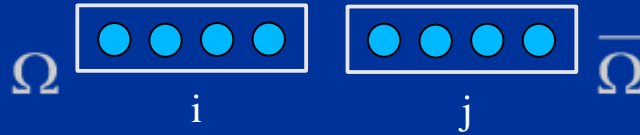


Fractional quantum Hall effect



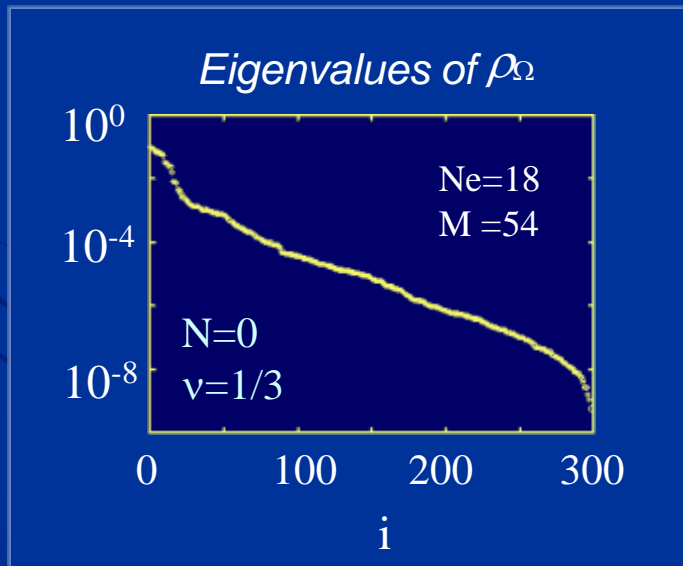
R. Willett *et al* (1987)

DMRG and Entanglement entropy



Ground state $|\Psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle |j\rangle$

Density matrix $(\rho_{\Omega})_{ii'} = \sum_j \Psi_{ij} \Psi_{ij'}$



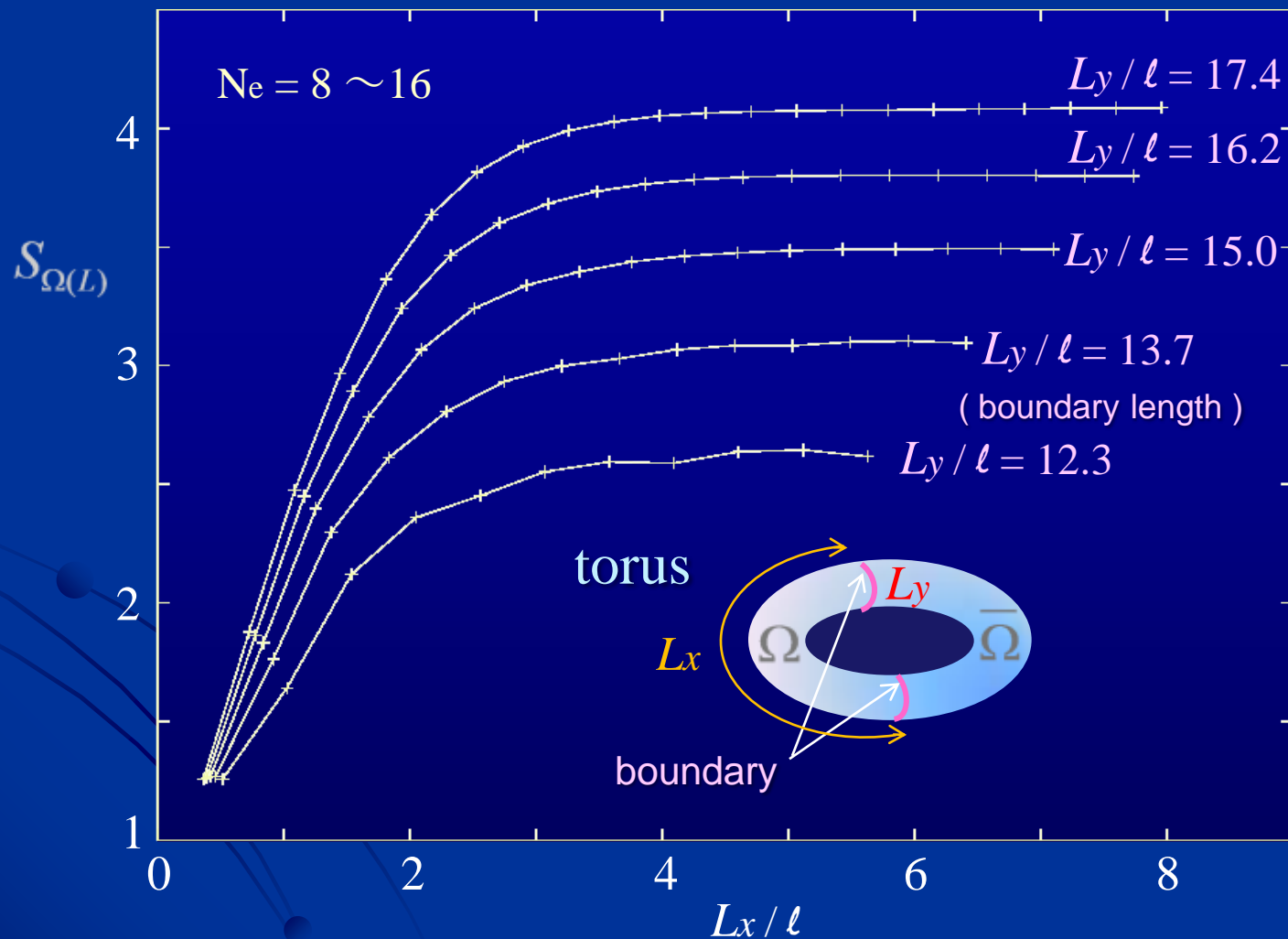
$$S_{\Omega} = -\text{Tr} \rho_{\Omega} \ln \rho_{\Omega}$$

Entanglement entropy is calculated from eigenvalues of ρ_{Ω}

Entanglement entropy

Torus geometry

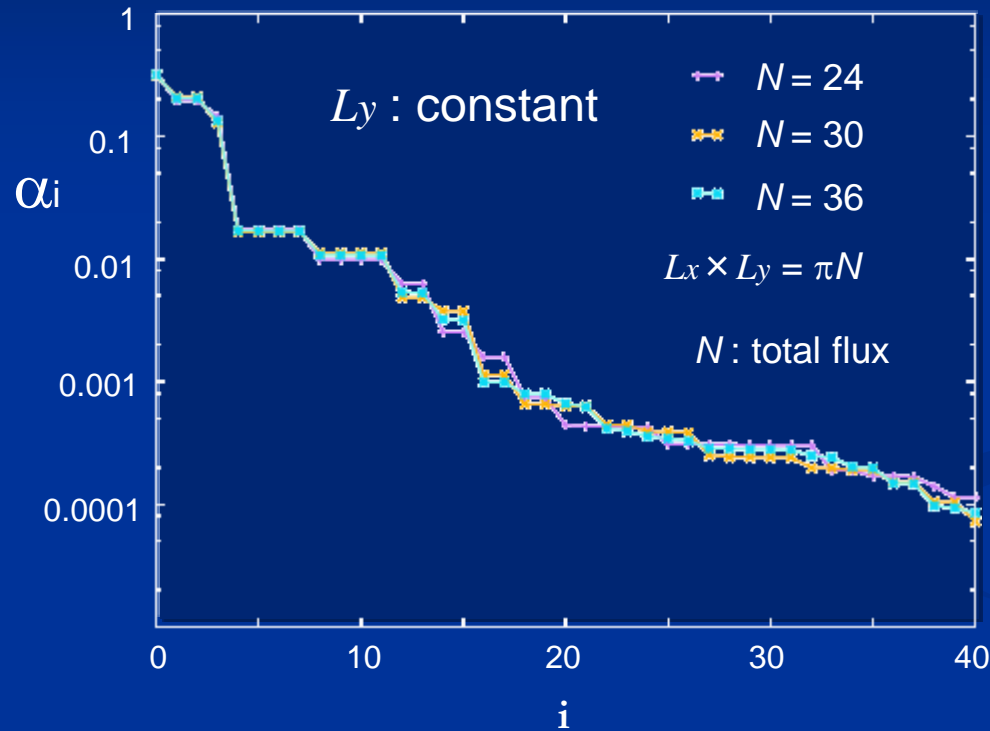
Fractional quantum Hall state $\nu = 1/3$



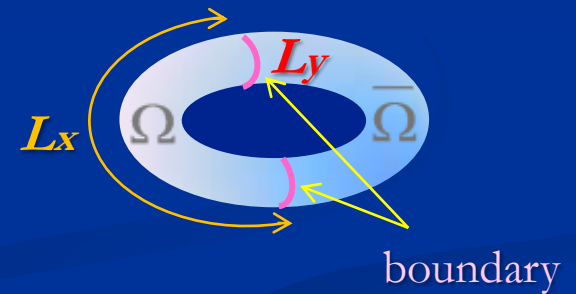
Fixed boundary length

Fractional quantum Hall state $\nu = 1/3$

density matrix eigenvalues α_i



Torus geometry



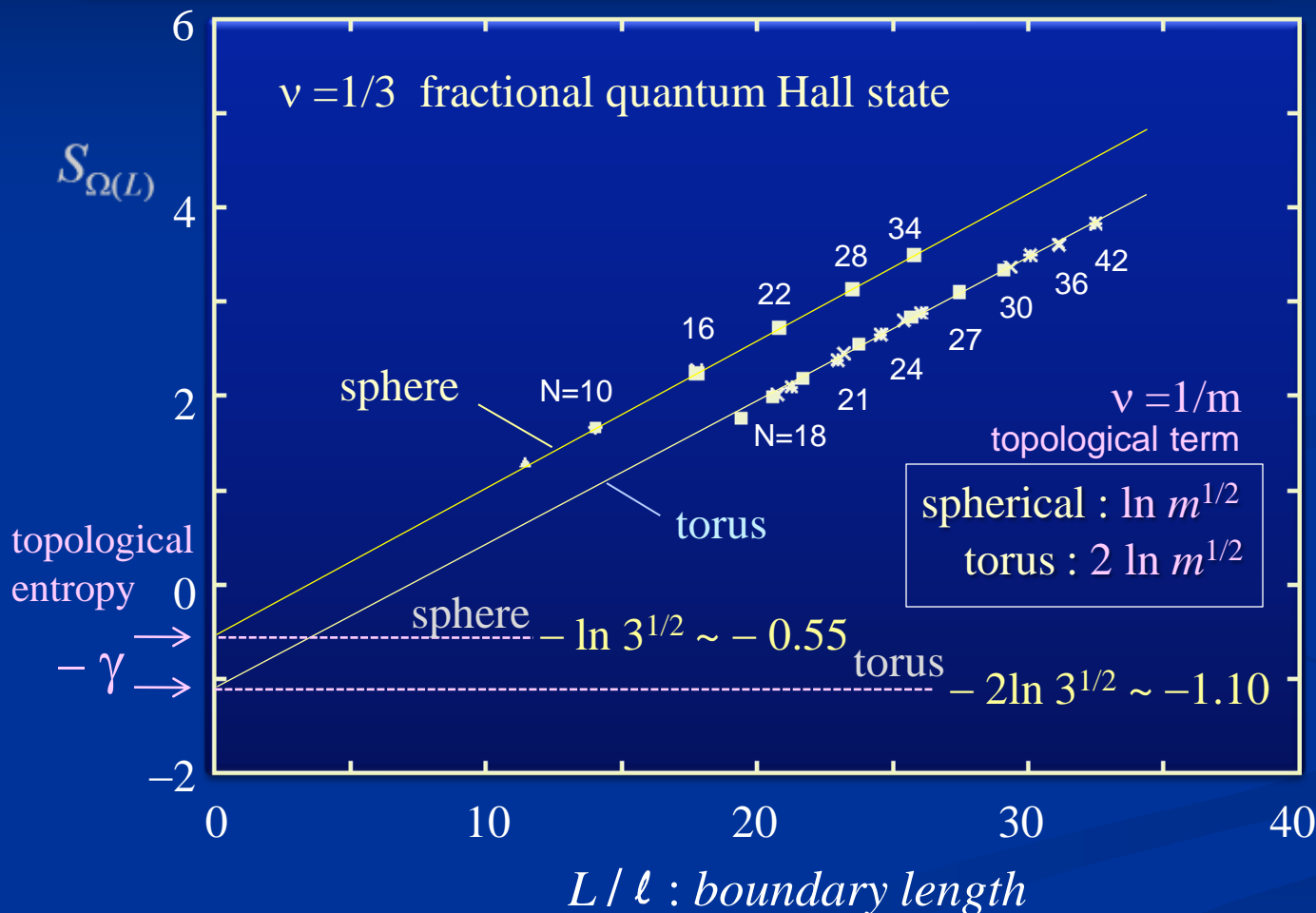
α_i is almost the same when the boundary length is the same



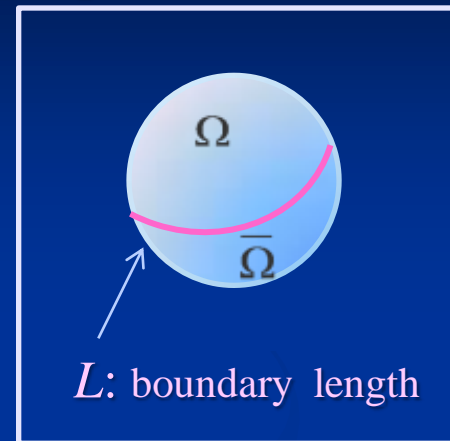
Area law

Entanglement entropy

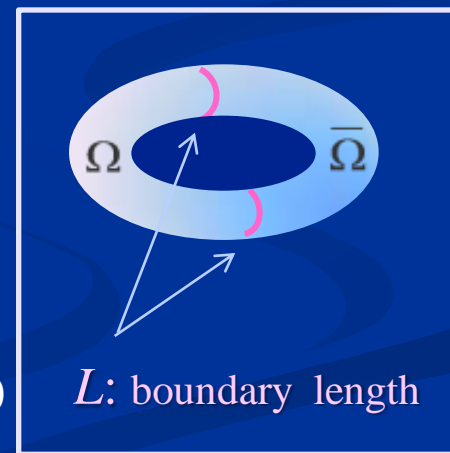
entanglement entropy $S_{\Omega(L)} = \frac{\alpha L}{\ell} - \underline{\gamma} \rightarrow$ topological term
 boundary term



sphere



torus

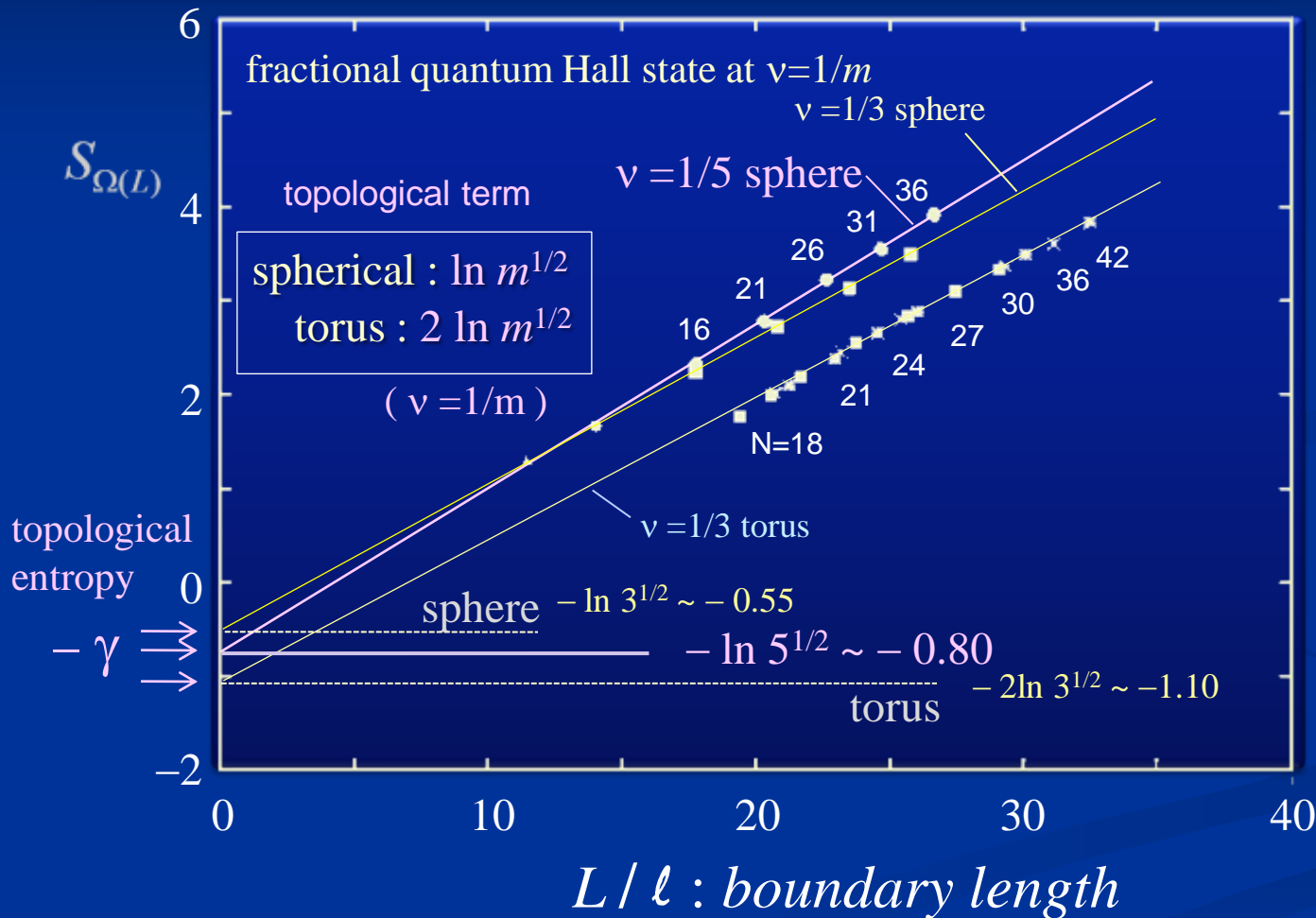


Disk geometry : Haque, Zozulya, and Schoutens, PRL, 2007

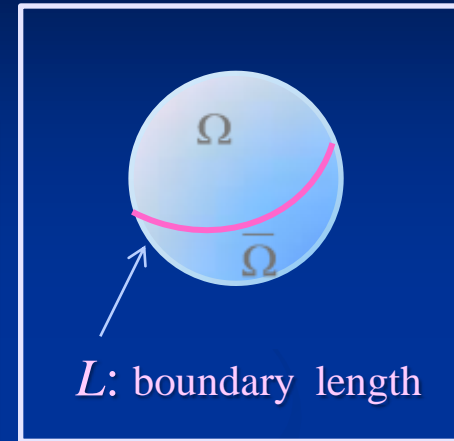
Torus geometry : Friedman and Levine, PRB, 2008

Entanglement entropy

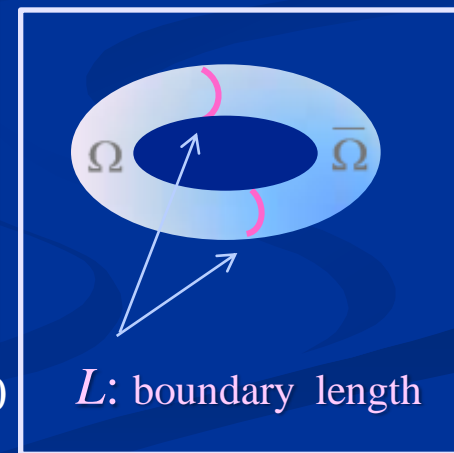
entanglement entropy $S_{\Omega(L)} = \frac{\alpha L}{\ell} - \underline{\gamma} \rightarrow$ topological term
 boundary term



sphere

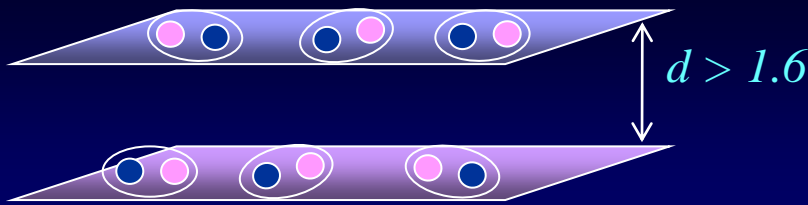


torus



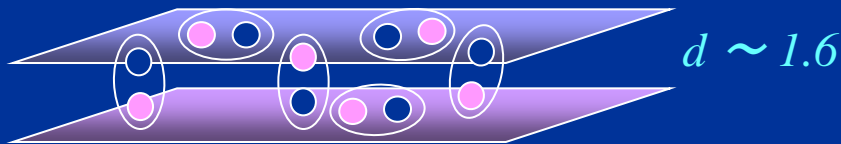
Bilayer quantum Hall system at $\nu=1$

↑ Magnetic field



Composite fermion liquid

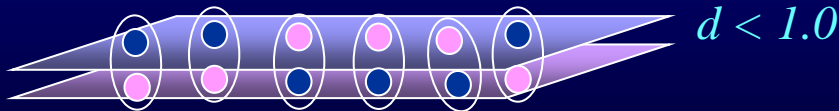
Gapless Fermi liquid



**Break down of
composite fermion**

Continuous
transition

↑ Magnetic field



Excitonic state

Macroscopic coherence of excitons

Finite charge gap

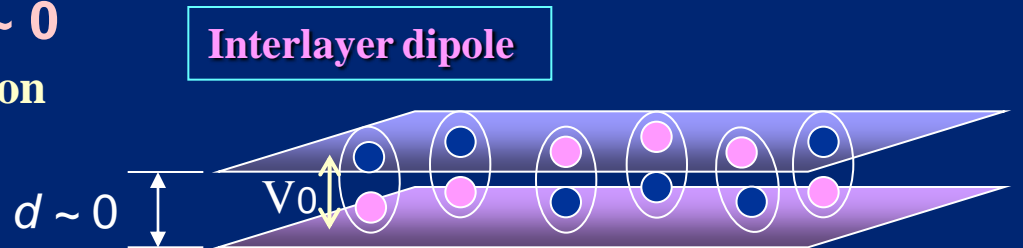
Zero pseudo-spin gap

V_0 - V_1 bilayer system at $\nu=1$

small interlayer distance $d \sim 0$
strong inter-layer Coulomb repulsion



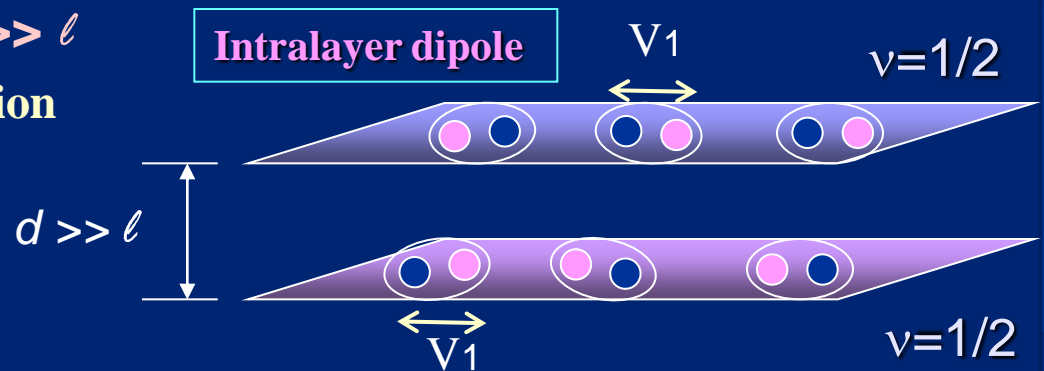
Inter-layer interaction V_0



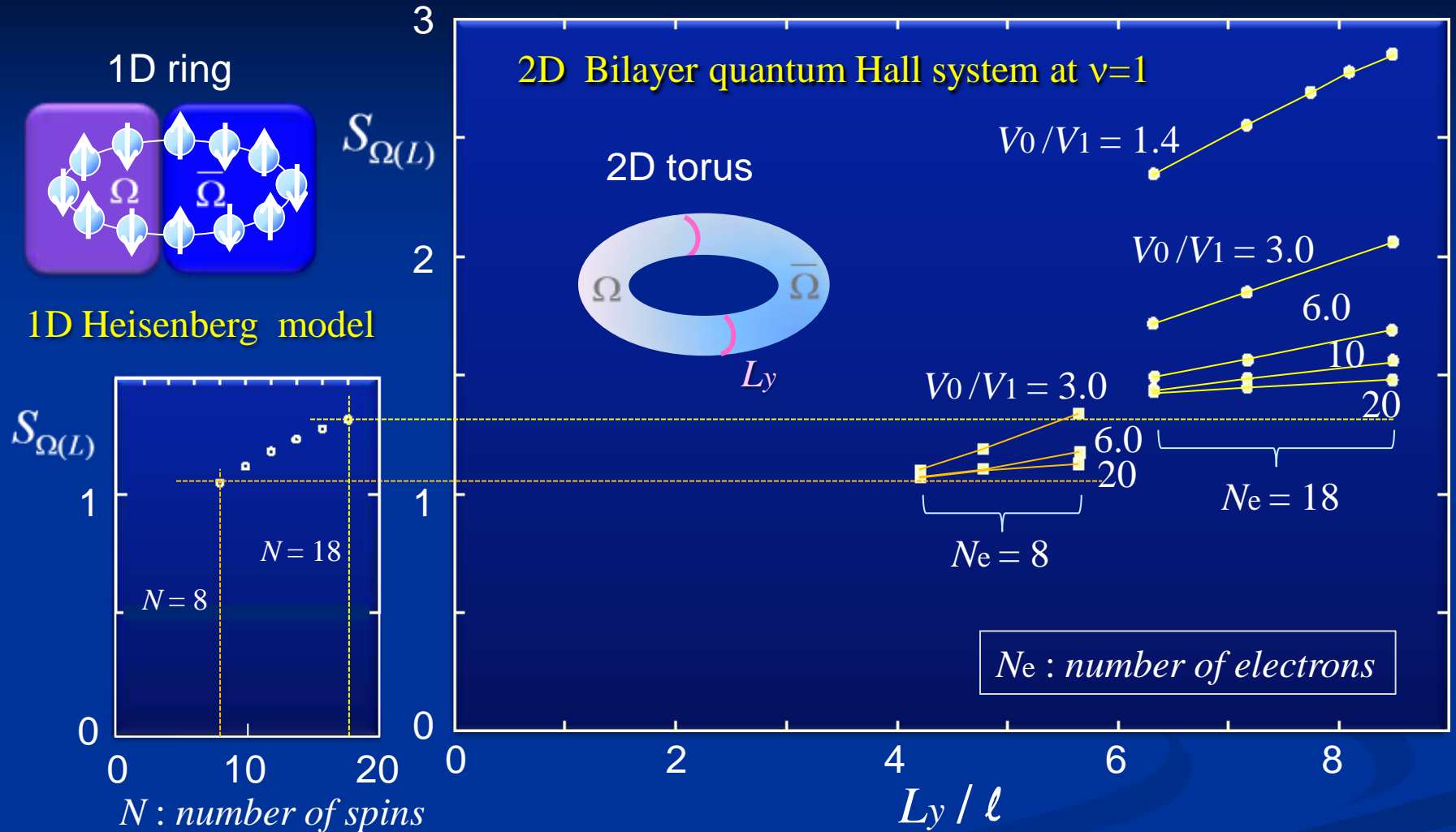
large interlayer distance $d \gg \ell$
strong intra-layer Coulomb repulsion



Intra-layer interaction V_1



Relation between 2D systems and 1D systems



2D topological charge entropy = $2 \ln m^{1/2} = 0$ ($\nu = 1/m = 1$)

1D spin entropy = 2D pseudo-spin entropy

Summary

DMRG is applied to quantum Hall systems

Entanglement entropy

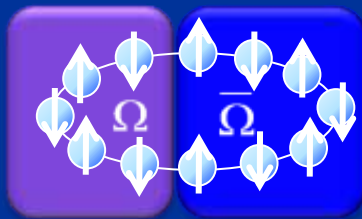
Measure of

Entanglement between two regions

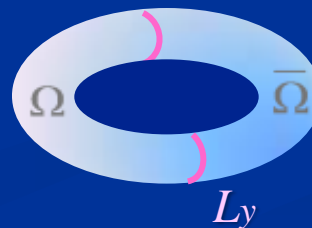
Correlation and topological order

Similarity between different models

1D ring



2D torus



sphere

