Chaos from Information Metric

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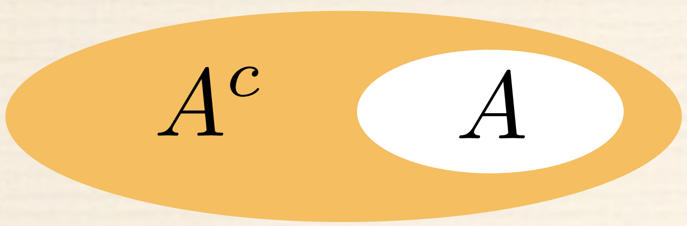
Scrambling

Local perturbation is added to thermal equilibrium.

Perturbation is scrambled. $=$ Reduced density matrices of large subregions are thermal.



$$\mathrm{Tr}[\rho_A(t)\mathcal{O}_A] \approx \langle \mathcal{O}_A \rangle_{\text{thermal}}$$

<u>Fast scrambling conjecture</u> [Sekino, Susskind]

Most rapid scramblers have scrambling time

$$t_w \sim log\ n$$

n : Number of degrees of freedom

BHs are fast scramblers.

⇨ Dual CFT of Einstein gravity is fastest scrambler among large N theories.

[Shenker, Stanford, Maldacena]

<u>Black hole information paradox</u> [Hayden, Preskill]

Scrambling

Add local perturbation W to thermal system at $t = -t_w$.

If $W(-t_w)$ is scrambled, $W(-t_w)$ is typical operator.

$$W(-t_w) = \sum_{\mathcal{O}} c_{\mathcal{O}}(-t_w)\mathcal{O}$$

For almost all local V, $\left[W(-t_w),\ V\right] \neq 0$ should hold.

Characterization of Scrambling [Shenker, Stanford]

For almost all local V with $[W,\ V] = 0$,

$$C(t) := \langle [W(-t),\ V][W(-t),\ V]^\dagger \rangle_\beta$$

are nonzero and order 1.

Scrambling time

Lyapunov exponent

$$C(t) \sim \frac{1}{A} \cdot e^{\lambda_L t} \quad \Rightarrow \quad t_w \sim \frac{1}{\lambda_L} \mathrm{Log} A$$

Fidelity, Information metric, and Loschmidt echo

[Peres, Jalabert, Pastawski, Jacquod, Silvestrov, Beenakker, Cerruti, Wisniacki, Cucchietti, Gorin, Prosen, Zurek, Seligman, etc…]

Inner product between two states with identical Hamiltonian will be conserved.

No butterfly effect?

Inner product between two states with different Hamiltonians H^{λ} and $H^{\lambda+\delta\lambda}$ decays rapidly in chaotic systems.

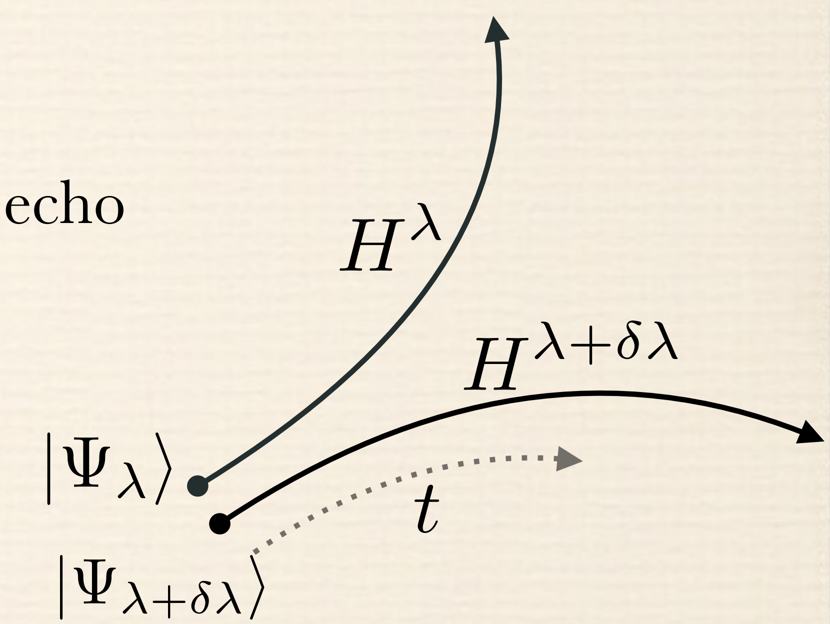
Butterfly effect !

Fidelity or Loschmidt echo

$$|\langle\Psi_{\lambda+\delta\lambda}|e^{iH^{\lambda+\delta\lambda}t}e^{-iH^{\lambda}t}|\Psi_{\lambda}\rangle|$$

$$= 1 - G_{\lambda\lambda}(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3)$$

Information metric



- Detect sensitivity of the states to external environment or internal imperfection

Measure of decoherence

- Irreversibility

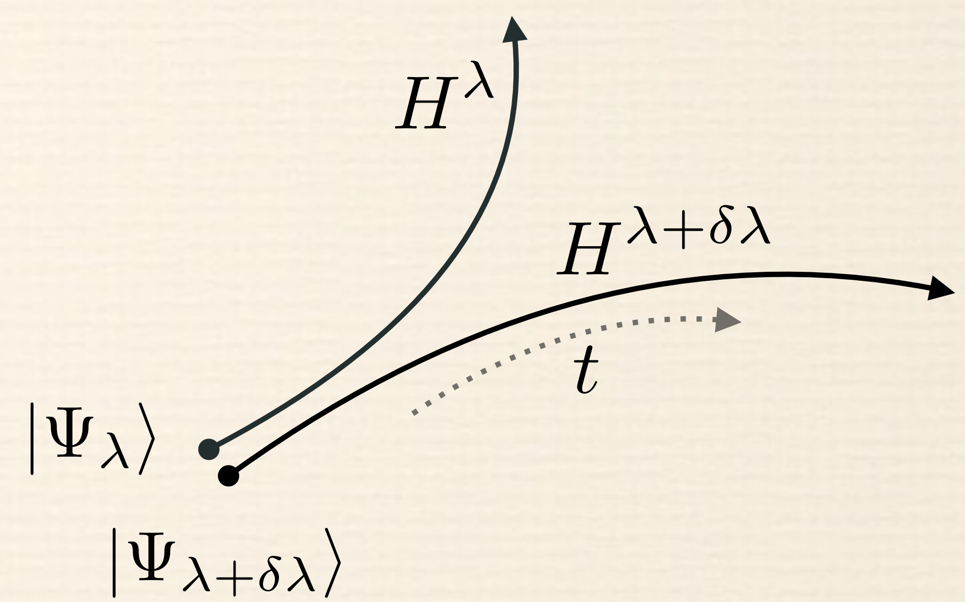
Information metric of TFD state

We perturb TFD state of the theory with external coupling $H_{CFT} + \lambda \cdot V$

at $t = -t_w$ by local operator W on $\mathcal{H}_L$. TFD state at t=0 is

$$|\Psi_{TFD}(\beta, \lambda, t_w)_W\rangle = e^{-iH^\lambda t_w} W e^{iH^\lambda t_w} |\Psi_{TFD}(\beta, \lambda, t_w)\rangle$$

Information metric is



$$|\langle\Psi_{TFD}(\beta, \lambda + \delta\lambda, t_w)_W|\Psi_{TFD}(\beta, \lambda, t_w)_W\rangle|$$

$$= 1 - G^W_{\lambda\lambda}(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3)$$

$$= \frac{1}{\sqrt{Z_W(\beta, \lambda) Z_W(\beta, \lambda + \delta\lambda)}} |\mathrm{Tr}[e^{-\frac{\beta}{2} H_L^{\lambda+\delta\lambda}} W^{\lambda+\delta\lambda}(-t_w) W^\lambda(-t_w) e^{-\frac{\beta}{2} H_L^\lambda}]|$$

Information metric

For general quantum systems, information metric can be divided into

$$G^{W}_{\lambda\lambda} = G^{(0)}_{\lambda\lambda} + G^{W:c}_{\lambda\lambda}$$

The first part is

$$G^{(0)}_{\lambda\lambda} = \frac{1}{2}\int_{0}^{\beta} \underset{t^1>\frac{\beta}{2}>t^2}{dt^1 dt^2}\, \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)} e^{Ht^1} V e^{-Ht^1} e^{Ht^2} V e^{-Ht^2} W(-t_w)^2\Big]$$

$$-\frac{3}{8}\Big(\int_0^\beta dt \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)} e^{Ht} V e^{-Ht} W(-t_w)^2\Big]\Big)^2$$

$$\langle V(t)V\rangle_\beta \sim \mathcal{O}(e^{-\frac{t}{t_r}})$$

For large N theory, when scrambling time is much larger than relaxation time,

$$G^{(0)}_{\lambda\lambda} \approx \frac{1}{2}\int_{0}^{\beta} \underset{t^1>\frac{\beta}{2}>t^2}{dt^1 dt^2}\, \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z(\beta,\lambda)} e^{Ht^1} V e^{-Ht^1} e^{Ht^2} V e^{-Ht^2}\Big]$$

This is equal to unperturbed information metric. In particular, it's time independent.

Information metric

The second part is

$$
\begin{aligned}
G^{W:c}_{\lambda\lambda} = \; & Re\Big[\, \frac{1}{2}\int_0^{t_w} dt^1 dt^2 \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)}\Big[W(-t_w),\; V(-t^1)\Big]\cdot\Big[W(-t_w),\; V(-t^2)\Big]^{\dagger}\Big] \\
& +\frac{i}{2}\int_0^{\frac{\beta}{2}} dt_E \int_0^{t_w} dt \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)} e^{Ht_E} V e^{-Ht_E}\Big[\,\Big[W(-t_w),\; V(-t)\Big],\; W(-t_w)\Big]\Big]\;\Big] \\
& -\frac{1}{2}\Big(\int_0^{t_w} dt \mathrm{Tr}\Big[\frac{e^{-\beta H}}{Z_W(\beta,\lambda)}\Big[W(-t_w),\; V(-t)\Big]\; W(-t_w)\Big]\Big)^2
\end{aligned}
$$

All of the terms are proportional to commutator $\Big[W(-t_w),\; V(-t)\Big]$

- Rapid growth of the second part of information metric means scrambling.
- Expected to behave $\sim e^{\lambda_L t_w}$
- For large N theory with hierarchy, rapid growth of information metric itself is enough to ensure scrambling.

Holographic proposal for Information Metric

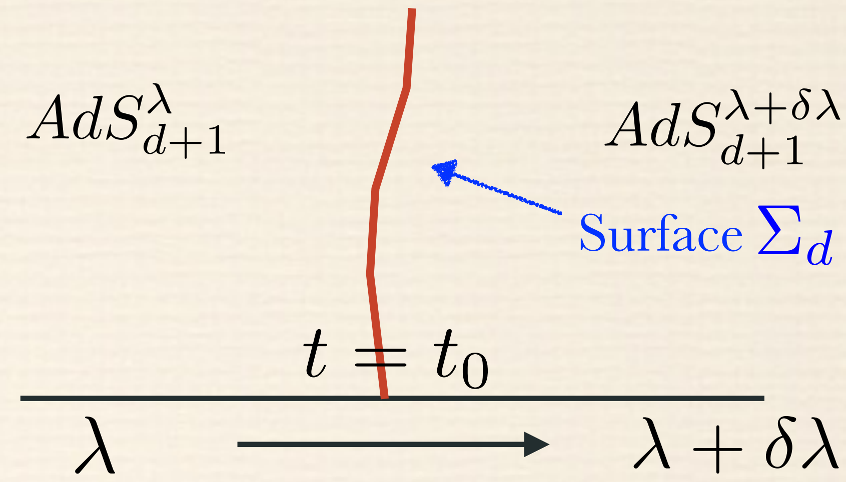
[M.M, Numasawa, Shiba, Takayanagi, Watanabe]

Consider time dependent states

$|\Psi_\lambda\rangle$ and $|\Psi_{\lambda+\delta\lambda}\rangle$ governed by Hamiltonians

$H_{CFT} + \lambda \cdot V$ and $H_{CFT} + (\lambda + \delta\lambda) \cdot V$

with marginal deformation V, whose dual spacetimes are the same, except for scalar field configurations.



Proposal

Consider a extremal volume codimension 1 surface Σ_d which ends at $t = t_0$ at the boundary of spacetime.

$$|\langle \Psi_{\lambda+\delta\lambda}(t_0)|\Psi_\lambda(t_0)\rangle| = 1 - G_{\lambda\lambda}(\delta\lambda)^2 + \mathcal{O}((\delta\lambda)^3)$$

$$\boxed{G_{\lambda\lambda} = n_d \int_{\Sigma_d} \sqrt{g}}$$

Volume of extremal volume surface is proportional to information metric.

$n_d > 0$

$n_d \sim \mathcal{O}(1)$

- The proposal is consistent with the result of pure AdS case and two sided BH case.

* "Complexity = Maximal volume" was proposed in [Stanfords, Susskind]

Shock wave geometry

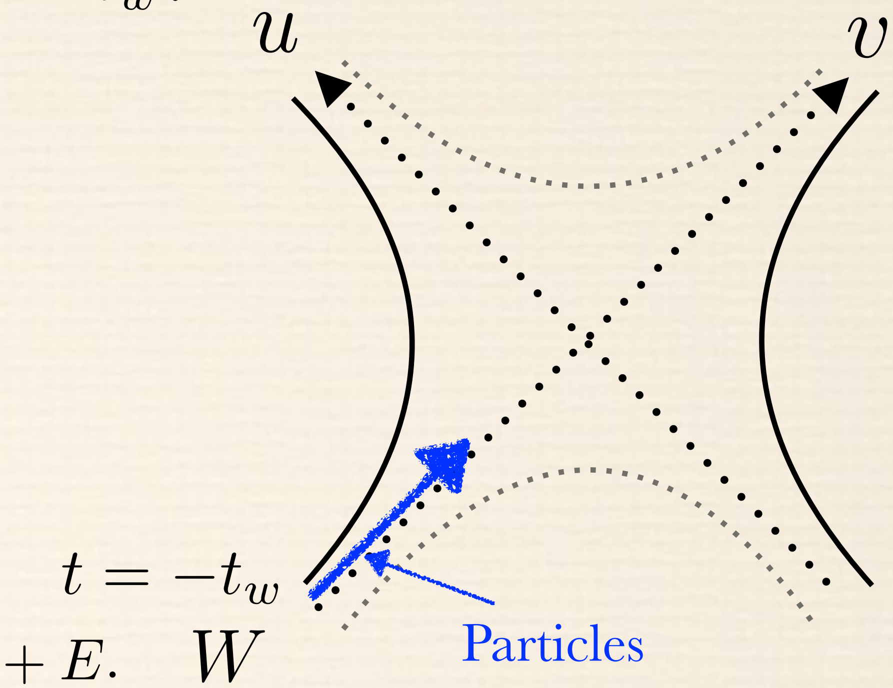
[Shenker, Stanford]

A few particles thrown into BH from left boundary at $t = -t_w$.
Proper energy at $t = 0$ is

$$E_p \approx \frac{E}{R} e^{\frac{2\pi}{\beta} t_w}$$

Back reaction can't be neglected for $t_w >> \beta$. The geometry can be approximated by shock wave geometry.

Two sided BH



The geometry is given by attaching two BHs with mass M and $M + E$.
For large t_w and $E << M$,

$$v_L = v_R + \alpha$$

$$\alpha = \frac{E}{u_w} \frac{d}{dM} [-e^{f'(R) r_*(r)}]|_{r=R} \propto \frac{1}{N^2} e^{\frac{2\pi}{\beta} t_w}$$

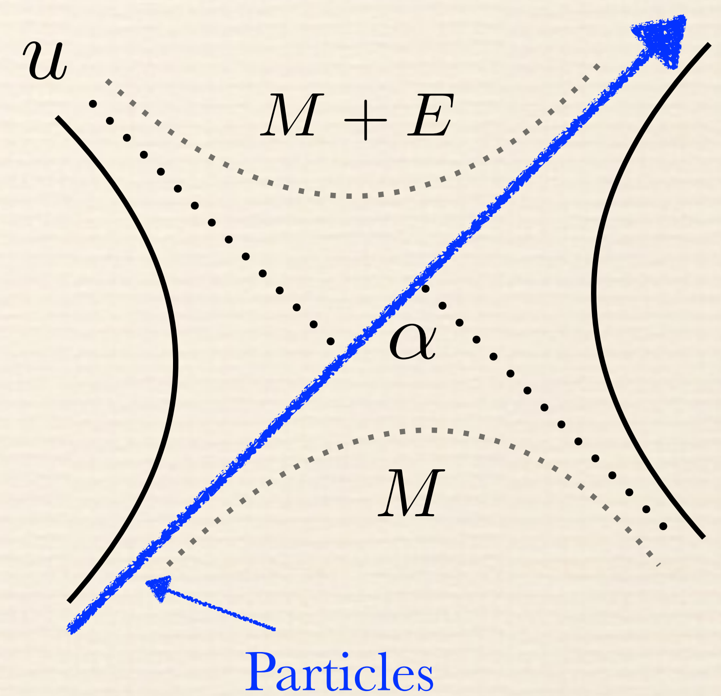
$t = 0$ slice of this spacetime is dual to

$$|\Psi_{TFD}(\beta, \lambda, t_w)_W\rangle = e^{-iH^\lambda t_w} W e^{iH^\lambda t_w} |\Psi_{TFD}(\beta, \lambda, t_w)\rangle$$

By computation,

$$\lambda_L = \frac{2\pi}{\beta} \qquad t_w = \frac{\beta}{2\pi} \log N^2 + \cdots$$

Shock wave geometry



Information metric from holography

Holographically, $|\langle \Psi_{TFD}(\beta, \lambda + \delta\lambda, t_w)_W | \Psi_{TFD}(\beta, \lambda, t_w)_W \rangle|$

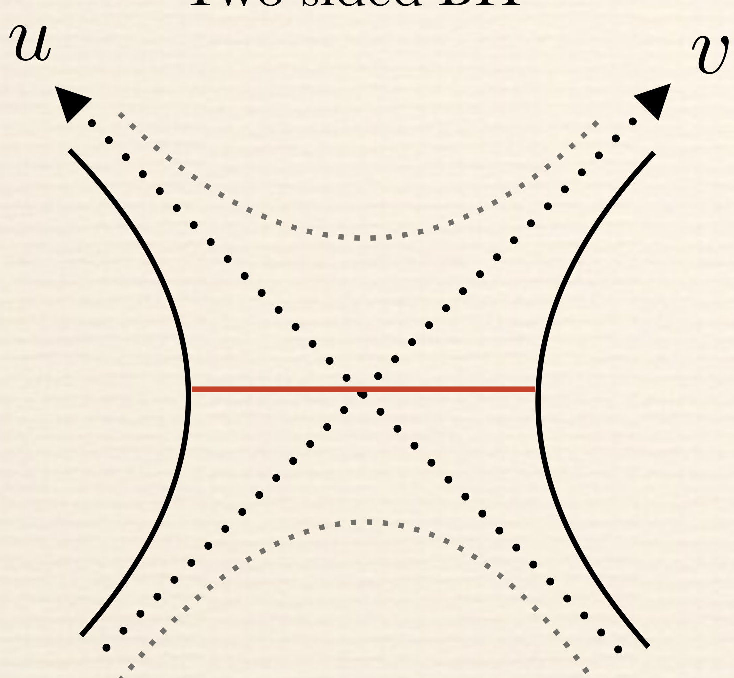
$$= e^{-n_d (\delta\lambda)^2 \int_{\Sigma_d} \sqrt{g}}$$

$$= 1 - n_d (\delta\lambda)^2 \int_{\Sigma_d} \sqrt{g} + \mathcal{O}((\delta\lambda)^4)$$

Σ_d is the maximum volume surface.

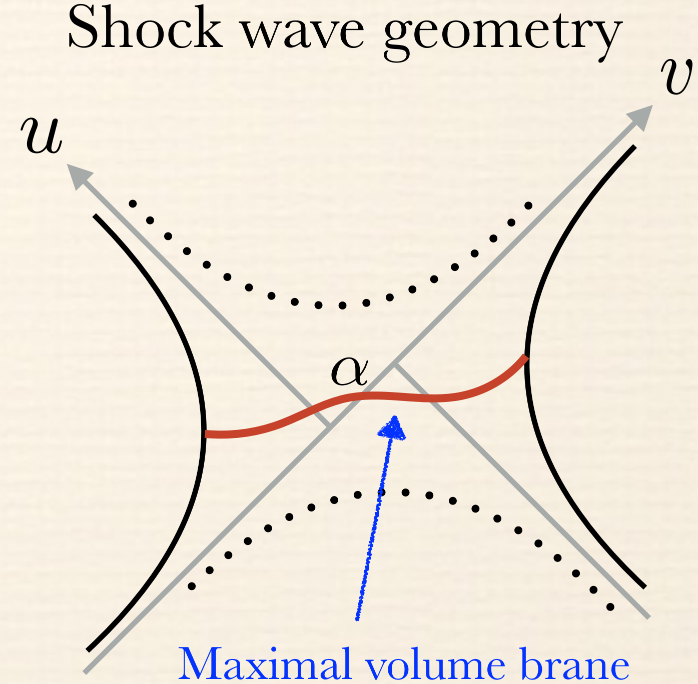
$$G^W_{\lambda\lambda} = G^{(0)}_{\lambda\lambda} + G^{W:c}_{\lambda\lambda}$$

Two sided BH



$G^{(0)}_{\lambda\lambda} = n_d \mathrm{Vol}$

Shock wave geometry



$G^{(W:c)}_{\lambda\lambda} = n_d \Delta \mathrm{Vol}$

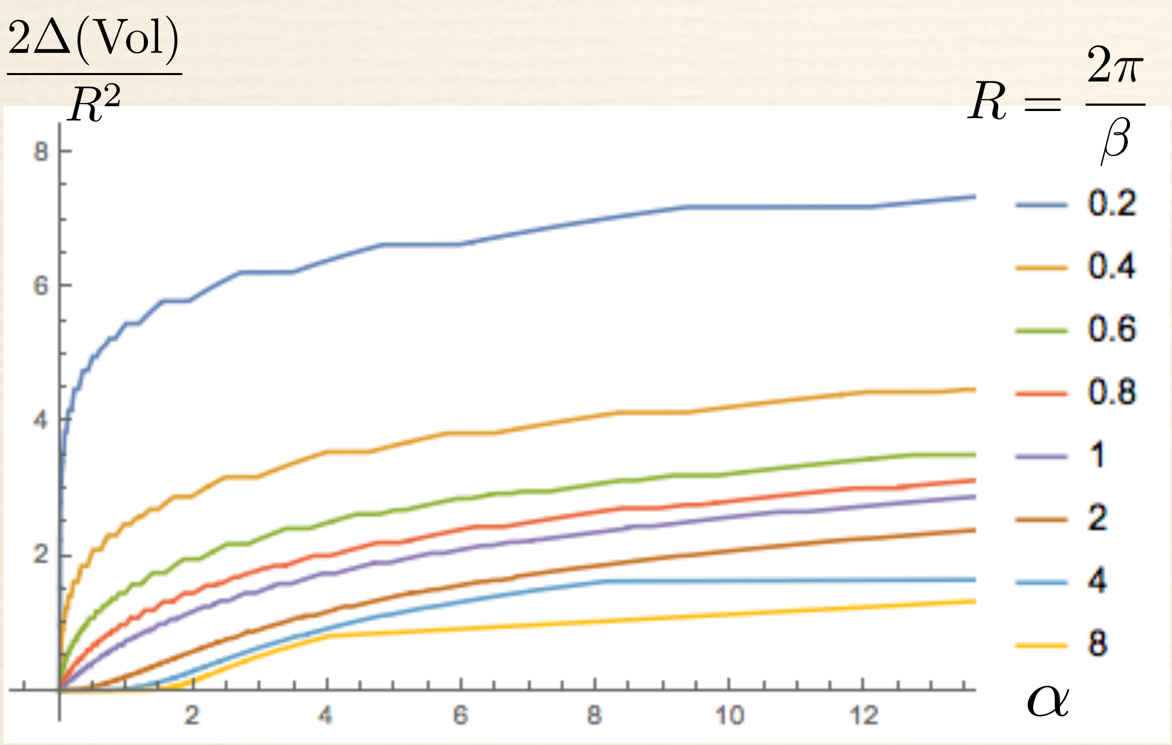
Information metric from holography

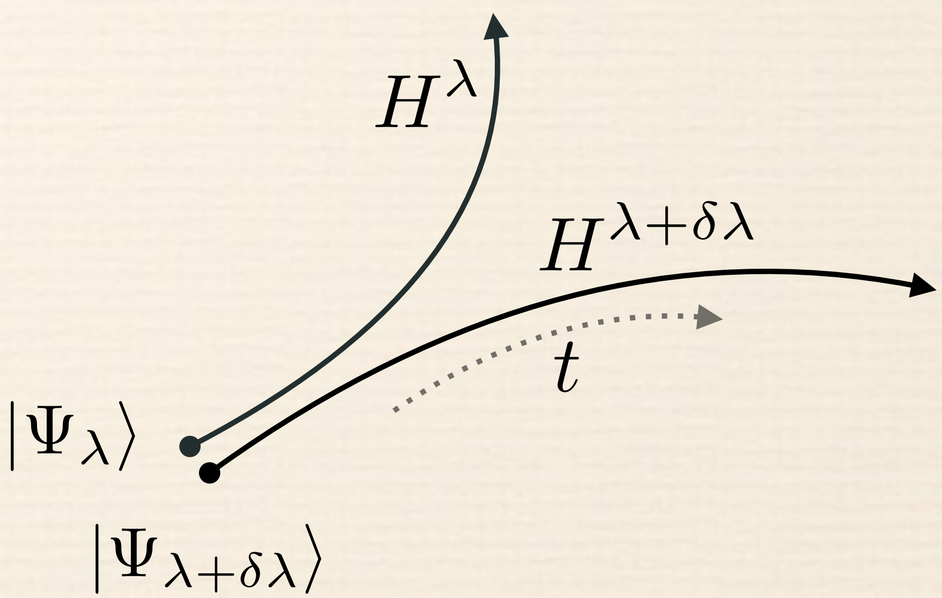
For d=2, numerical result shows $G^{W:c}_{\lambda\lambda}$ grows proportional to $\alpha = \frac{E}{4M} e^{\frac{2\pi}{\beta} t_w}$

$$G^{W:c}_{\lambda\lambda} = f_0 \alpha + \mathcal{O}(\alpha^2) \qquad f_0 > 0 \quad f_0 \sim \mathcal{O}(N^0)$$

The second part of information metric indeed grows exponentially rapidly in holographic CFT.

→ Consistent with intuitive picture of butterfly effect.





Conclusion

Rapid growth information metric of TFD state for external environment or internal imperfections signifies scrambling, at least for large N theories.

Using holographic proposal for information metric, we confirmed information metric indeed grows exponentially fast in holographic theory.

$\Rightarrow$ Butterfly effect !

Future problems

Analytical and numerical calculations.

Various regimes of Loschmidt echo: e.g. Strong perturbation.

Confirm equilibration of reduced density matrices = growth of commutators?