Boundaries, Lattices and the Gravitational Anomaly

Domenico Orlando

Albert Einstein Center for Fundamental Physics – University of Bern

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Quantum Information in String Theory and Many-body Systems

Yukawa Institute for Theoretical Physics – Kyoto University

Collaboration with: **Simeon Hellerman (IPMU)**
Masataka Watanabe (IPMU)

Outline

Summary of the results

I will discuss two-dimensional conformal field theories with a non-vanishing gravitational anomaly $c_L \neq c_R$.

I will prove the following chains of implications:

lattice regulation ⇒ boundary CFT ⇒ $c_L = c_R$

and

Hilbert space factorization ⇒ boundary CFT ⇒ $c_L = c_R$

Thus proving the following facts for gravitationally-anomalouns theories:

1. they do not admit a lattice regularization [Nielsen–Ninomiya];
2. they do not admit a Hilbert space factorization, and hence
3. they do not admit any definition of entanglement.

A map of the talk

We will prove the following implications:

$$
\begin{tikzcd}
\text{lattice regularization} \arrow[d, dashed] \arrow[dr, Rightarrow] & & \\
\text{Hilbert space factorization} \arrow[r, Rightarrow] & \text{boundary CFT} \arrow[r, Longrightarrow] & c_L = c_R
\end{tikzcd}
$$

A map of the talk: contrapositive

And derive the following facts:

$$
\begin{tikzcd}
 & & \text{no lattice regularization} \\
c_L \neq c_R \arrow[r, Rightarrow] & \text{no boundary CFT} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & \text{no Hilbert space factorization} \arrow[d, Rightarrow] \\
 & & \text{no entanglement}
\end{tikzcd}
$$

Outline

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Lattice regularization

Consider a CFT $\mathcal{C}$ in $1+1$ dimensions.
Suppose that $\mathcal{C}$ has a UV completion described by a Hamiltonian system living on a lattice.

- lattice spacing ℓ
- sites labelled by integers n
- fundamental degrees of freedom Φ_n.

We also assume locality:

- canonical commutators in the lattice theory
- the lattice Hamiltonian only couples degrees of freedom separated by some bounded number $(\Delta n)_{\mathrm{max}}$ of lattice sites

Lattice regularization

We can write the lattice Hamiltonian as

$$H = \ell \sum_n \mathcal{H}_n ,$$

where the local Hamiltonian at n is a function of the sites in a neighborhood of n

$$\mathcal{H}_n = \mathcal{H}_n(\{ \Phi_{n'} \mid |n - n'| \leq (\Delta n)_{\max} \}).$$

We assume that the couplings are tuned so that the Hamiltonian flows in the IR to the two-dimensional CFT $\mathcal{C}$.

Lattice implies boundary in the discretum

We want to show that this regularization implies that $\mathcal{C}$ admits a consistent, unitary quantization on a space with boundary.

First, look at the lattice regularization.
Shift the local Hamiltonians by $(\Delta n)_{\max}$

$$\hat{\mathcal{H}}_n = \mathcal{H}_{n+(\Delta n)_{\max}}$$

and delete all the degrees of freedom with $n < 0$:

$$\hat{H} = \ell \sum_{n \geq 0} \hat{\mathcal{H}}_n \,.$$

This new Hamiltonian includes only interactions on the right of $n = 0$.

We do not need to impose any special value for the degrees of freedom at the boundary $n = 0$: the discrete system has free boundary conditions.

Lattice implies boundary in the continuum

Remember that $\mathcal{H}_n$ depends only on n' with $|n - n'| \leq (\Delta n)_{\max}$.

The deletion of the degrees of freedom at negative lattice sites does not affect the local dynamics of the theory away from the boundary; the Hamiltonian density is modified only within a distance $(\Delta n)_{\max} \times \ell$ of the boundary, which goes to zero in the continuum limit.

The system in the infrared is described by the conformal field theory $\mathcal{C}$ living on the half-line $\mathbb{R}_+$, with some boundary condition.

The boundary conditions

Unitarity and energy conservation are manifest in the lattice system on the half-line, and are inherited automatically by the continuum limit.

Conformal invariance of the theory with boundary is less obvious. We do not have *a priori* control over the boundary conditions of the continuum theory.
We need to use a conjecture of Friedan and Konechny on the monotonicity of the Affleck-Ludwig g-function under renormalization group flow
The infrared limit of the lattice system on the positive half-lattice is the CFT $\mathcal{C}$ quantized on $\mathbb{R}_+$ with an energy-conserving, unitary, and conformally invariant boundary condition.

Equivalently, we can think of a boundary state $|B\rangle$.

Outline

Introduction

Lattice regularization implies boundary CFT

Boundary consistency implies $c_L = c_R$

Consequences: a generalized Nielsen–Ninomiya theorem

Tensor factorization implies boundary CFT

Consequences: no entanglement

Boundary states and central charge

A boundary state representing a energy-conserving conformal boundary condition, must satisfy [Cardy]

$$\left(L_n - \tilde{L}_n\right) |B\rangle = 0 \qquad \forall n$$

Now, act with $\left(-L_{-n} + \tilde{L}_n\right)$ on the right and subtract the same equation with $n \to -n$. Use the commutation relations of the Virasoro algebra and the condition becomes

$$\left[2n\left(L_0 - \tilde{L}_0\right) + \frac{n^3 - n}{12}(c_L - c_R)\right] |B\rangle = 0.$$

But $\left(L_0 - \tilde{L}_0\right) |B\rangle = 0$, so we find that

$$(c_L - c_R) |B\rangle = 0$$

Non-trivial boundary conditions $|B\rangle$ are only possible if $c_L = c_R$.

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A generalized Nielsen–Ninomiya theorem

We have proven two implications:

lattice regulation ⇒ boundary CFT

and

boundary CFT ⇒ $c_L = c_R$

Now we can string them together and take the contrapositive:

$c_L \neq c_R$ ⇒ no bCFT ⇒ no lattice regulation

This is the generalized version of the renowned Nielsen–Ninomiya theorem.

Outline

The relevance for Quantum information theory

Why is our result relevant for quantum information?
To see that, we need to refine the first implication: lattice $\Rightarrow$ bCFT.

Let Σ be the spatial slice of our system and let $\Sigma = A \cup B$. We want to show that if the Hilbert space factorizes $\mathfrak{H}_{A\cup B} = \mathfrak{H}_A \otimes \mathfrak{H}_B$, then we can define a bCFT.

In other words, we want to show that if the Hilbert space is factorizable, starting from the CFT Hamiltonian H_Σ we can define two Hamiltonians H_A and H_B on the two Hilbert spaces associated the bounded regions A and B. These Hamiltonians are locally the same as the initial Hamiltonian.

Factorizaton of the operator algebra

The typical situation is the inverse. Given an operator $\mathcal{O}_A$ on $\mathfrak{H}_A$ we can associate it to and operator $\mathcal{O}_\Sigma \in \mathfrak{A}(\mathfrak{H}_\Sigma)$ as follows:

$$\mathcal{O}_\Sigma = \mathcal{O}_A \otimes \mathbb{1}_B$$

which defines the embedding of $\mathfrak{A}(\mathfrak{H}_A)$ in $\mathfrak{A}(\mathfrak{H}_\Sigma)$.

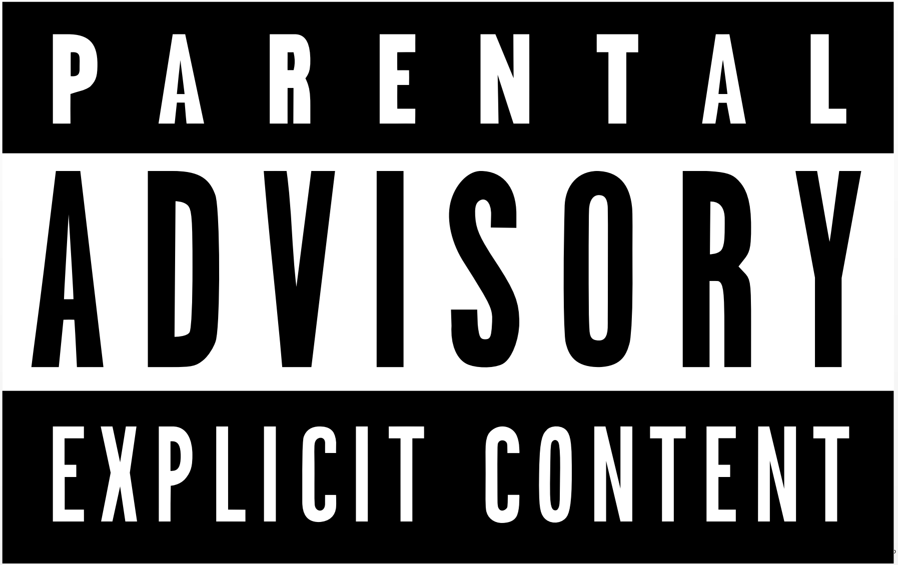
What we need is to define a projection that goes the other way round:

$$\mathfrak{A}(\mathfrak{H}_\Sigma) \to \mathfrak{A}(\mathfrak{H}_A)$$

which, roughly speaking, is obtained by taking the partial trace of the operator

$$\mathcal{O}_\Sigma \mapsto \mathcal{O}_A \propto \mathrm{Tr}_B[\mathcal{O}_\Sigma]$$

Parental advisory



Regularization

In general $\mathcal{O}$ is not trace-class, so we need to regularize.
The natural choice is to use the data of the CFT and introduce a heat-kernel regularization using the full Hamiltonian H_Σ:

$$\mathcal{O}_A \propto \mathrm{Tr}'_B[\mathcal{O}_\Sigma] \equiv \lim_{\varepsilon \to 0} \mathrm{Tr}_B[\mathcal{O}_\Sigma e^{-\varepsilon H_\Sigma}].$$

In this way we can define the regularized Hamiltonian on the sector A using Stone's theorem:

$$H_A \propto \mathrm{Tr}'_B[H] = -\lim_{\varepsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}\,\varepsilon} \mathrm{Tr}_B[e^{-\varepsilon H_\Sigma}]$$

Regularization

Let me comment on the operator we have defined:

$$\mathrm{Tr}_B[e^{-\varepsilon H_\Sigma}]$$

This is the exponential of the von Neumann entropies of the reduced density matrices of the thermal ensemble at temperature $1/\varepsilon$.

Physically we can think of it as the effective dimension of the Hilbert space $\mathfrak{H}_A$ accessible at energy $1/\varepsilon$.
In this sense, the scaling of ε describes an RG flow.

The boundary

The construction is well defined in the bulk (away from the boudary between A and B), where we are assured that all the corrections to the unregulated Hamiltonian vanish for $\varepsilon \to 0$.

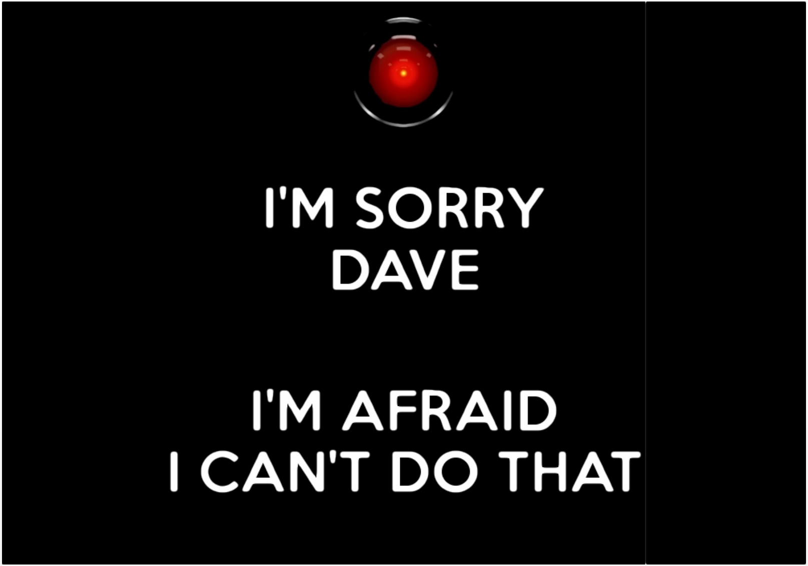
At the boundary there might be operators that appear with negative powers of ε (*e.g.* a cosmological constant would appear as ε^{-1}).

Again we must appeal to the Konechny–Friedan hypothesis. Scaling $\varepsilon \to 0$ is a boundary renormalization group flow for a CFT, and leads to a bCFT (if g is bounded).

To recap: we have shown that if the Hilbert space is factorizable $\mathfrak{H}_{A\cup B} = \mathfrak{H}_A \oplus \mathfrak{H}_B$, then we can define a boundary conformal field theory on the two halves, with Hamiltonians H_A and H_B.

Outline

I'm sorry Dave, I'm afraid I can't do that



Consequences

We have now the following chain of implications:

Hilbert space factorization ⇒ boundary CFT

and

boundary CFT ⇒ $c_L = c_R$

Once more we take the contrapositive:

$c_L \neq c_R$ ⇒ no bCFT ⇒ no Hilbert space factorization

In a gravitationally anomalous theory there is no tensor factorization of the Hilbert space into Hilbert spaces supported in complementary regions of a spatial slice.

It follows that no definition of entanglement is possible.

Thank you
for your attention

A map of the talk

$$
\begin{array}{ccccc}
 & & & & \text{no lattice regularization} \\
 & & & \nearrow & \\
c_L \neq c_R & \Longrightarrow & \text{no boundary CFT} & & \\
 & & & \searrow & \\
 & & & & \text{no Hilbert space factorization} \\
 & & & & \Downarrow \\
 & & & & \text{no entanglement}
\end{array}
$$

What I am not saying

In any separable Hilbert space you can always write a basis and tensor factorize that basis into factors.
Our result is that those tensor factors can't be thought of as representing Hilbert spaces living in complementary regions of a spatial slice.
If the spatial slice is $\Sigma = A \cup B$ then we don't have in general $\mathfrak{H}_\Sigma = \mathfrak{H}_A \otimes \mathfrak{H}_B$.

The Konechny–Friedan hypothesis

In the lattice description we have obtained a theory with free boundary conditions. Does this flow to a conformal boundary?

The intuition is the following. Go to the scale Λ_{cft} where the bulk Hamiltonian flows to the CFT. Either the bCFT is conformal, or there still are non-conformal operators with large coefficients.

- either they are irrelevant and flow to zero at long distances
- or they are relevant and can be integrated out, leaving us with a boundary condition with fewer degrees of freedom.

So eventually, one expects, the process should terminate when we reach a fixed point, or at the latest when there are no boundary degrees of freedom left, and we should get to a conformal boundary condition.

The Konechny–Friedan hypothesis

To make this intuition more precise, there is actually a functional on boundary theories in two-dimensional CFT, the Affleck-Ludwig g-function.

This is monotonic along RG flows and stationary if and only if the RG flow is at a fixed point, essentially like Zamolodchikov c-function.

Problem: the g-function is only conjectured to be bounded below.

Friedan and Konechny have provided a great deal of evidence for the hypothesis.