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Entanglement entropy in pseudo-Hermitian models

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work in progress

Introduction & Motivation

- Holography relates quantum gravitational theories with non-gravitational theories

- Toward a quantum description of the Universe, we need the dS/CFT correspondence

  [Witten ’01, Strominger, ’01]

- Known examples of dS/CFT are the duality between “Vasiliev’s higher-spin theory on dS and Sp(N) vector model” and its extension

  [Anninos-Hartman-Strominger, ’11]

- Hamiltonian of the Sp(N) model is not Hermite [LeClair-Neubert, ’07]

  But, the Hamiltonian is pseudo-Hermite

I study entanglement entropy in pseudo-Hermitian models

Results

- Quantum Mechanics

Pseudo-Hermite

$$\eta H \eta^{\dagger} = H^{\dagger} \quad (H : \mathrm{Hamiltonian}\,, \quad \eta : \mathrm{unitary\ op.})$$

Using a modified density matrix $\rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi|\eta$, we obtain entanglement entropy similar to the usual one.

- The Sp(N) model

Vacuum state:

$$S_A \sim -\mathrm{const.}\frac{\mathrm{Area\ of\ }\partial A}{\varepsilon^{d-2}} + \cdots$$

Excited state:

$$\Delta S_A^{(n)} := S_{A,\mathrm{ex}}^{(n)} - S_{A,\mathrm{vacuum}}^{(n)} = \Delta S_A^{(n)}{}_{,\mathrm{scalar}}$$

Plan of my talk

1. Introduction
2. Pseudo-Hermitian property in Quantum Mechanics
3. Sp(N) model
4. Conclusion

Pseudo-Hermite in Quantum Mechanics

- Hamiltonian: $\eta H \eta^\dagger = H^\dagger \quad (\eta\eta^\dagger = 1 , \quad \eta = \eta^\dagger)$

PT-symmetric $\longrightarrow$ Eigenvalues of H are real, positive and discrete

$(\mathcal{PT} H (\mathcal{PT})^{-1} = H)$

not PT-symmetric $\longrightarrow$ Eigenvalues of H are complex and discrete

They contain E and its complex conjugate E^*

- Inner product

Usual inner product is not preserved:

$$\langle \psi_1(t) | \psi_2(t) \rangle = \langle \psi_1(0) | \mathrm{e}^{-iH^\dagger t} \mathrm{e}^{iHt} | \psi_2(0) \rangle \neq \langle \psi_1(0) | \psi_2(0) \rangle$$

New inner product is preserved:

$$\langle \psi_1(t) | \eta | \psi_2(t) \rangle = \langle \psi_1(0) | \eta | \psi_2(0) \rangle$$

insert the unitary op. η

Examples

- Hamiltonian

$$H = \begin{pmatrix} r\mathrm{e}^{i\theta} & s \\ s & r\mathrm{e}^{-i\theta} \end{pmatrix} \qquad \Big[\ r\,,s\,,\theta \ \text{ are real}\ \Big]$$

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \mathcal{P}H\mathcal{P} = H^\dagger$$

- Eigenvalues

$$E_\pm = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta}$$

- Eigenstates

$$|+\rangle = \frac{1}{\sqrt{2\cos\alpha}}\begin{pmatrix} \mathrm{e}^{i\alpha/2} \\ \mathrm{e}^{-i\alpha/2} \end{pmatrix} \quad \& \quad |-\rangle = \frac{i}{\sqrt{2\cos\alpha}}\begin{pmatrix} \mathrm{e}^{-i\alpha/2} \\ -\mathrm{e}^{i\alpha/2} \end{pmatrix}$$

$$\langle\pm|\mathcal{P}|\pm\rangle = \pm 1\,, \quad \langle\pm|\mathcal{P}|\mp\rangle = 0 \qquad \Big[\ \sin\alpha = \frac{r}{s}\sin\theta\ \Big]$$

Entanglement entropy

- Density matrix

$$\rho_{\rm tot} = |\Psi\rangle\langle\Psi| \quad \Rightarrow \quad \rho_{\rm tot} = |\Psi\rangle\langle\Psi|\eta$$

to make $\rho_{\rm tot} \to \mathrm{e}^{iHt}\rho_{\rm tot}\mathrm{e}^{-iHt}$ under time evolution

In previous example, inner products are not positive-definite

Eigenvalues of ρ_A are negative

$\Rightarrow$ EE becomes negative

- Comment

1. When $\eta = S^\dagger S$, inner products are positive definite, so EE is positive
2. If there is “charge conjugation op.” $\mathcal{C}$ ($\mathcal{C}|\pm\rangle = \pm|\pm\rangle$) ,

   $\langle\psi|\mathcal{PC}|\phi\rangle$ becomes positive definite

Sp(N) model

- Sp(N) model is the holographic dual of Vasiliev's higher-spin theory

[Anninos-Hartman-Strominger, '11]

- Action (d-dim, free) [LeClair-Neubert, '07]

$$I = -\frac{1}{2}\int \mathrm{d}^d x\, \Omega_{ab}\partial_\mu \chi^a \partial^\mu \chi^b \quad \text{where} \quad \Omega_{ab} = \begin{pmatrix} 0 & 1_{N/2\times N/2} \\ -1_{N/2\times N/2} & 0 \end{pmatrix}$$

anti-commuting

- Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^{d-1} x\, \Omega_{ab}(\pi^a \pi^b + \partial_i \chi^a \partial_i \chi^b)$$

$$\Rightarrow \quad CHC^\dagger = H^\dagger \quad \left(\begin{array}{l} CC^\dagger = 1\,, \quad C = C^\dagger \\ C\chi^a C = \chi^{a+N/2} \end{array} \right)$$

Density matrix

- Modification of density matrix

$$\rho_{\rm tot} = |\Psi\rangle\langle\Psi| \quad \Rightarrow \quad \rho_{\rm tot} = |\Psi\rangle\langle\Psi|C$$

to make $\rho_{\rm tot} \to \mathrm{e}^{iHt}\rho_{\rm tot}\mathrm{e}^{-iHt}$ under time evolution

- Path-integral representation of wave functions

$$\langle\chi_-|\Psi\rangle = \frac{1}{\sqrt{Z}}\int\prod_{-\infty<x_0<0}\prod_{\boldsymbol{x}}\mathcal{D}\chi\,\mathrm{e}^{-S[\chi]}\delta[\chi(0,\boldsymbol{x}) - \chi_-(\boldsymbol{x})]$$

$$\langle\Psi|C|\chi_+\rangle = \frac{1}{\sqrt{Z}}\int\prod_{0<x_0<\infty}\prod_{\boldsymbol{x}}\mathcal{D}\chi\,\mathrm{e}^{-S[\chi]}\delta[\chi(0,\boldsymbol{x}) - \chi_+(\boldsymbol{x})]$$

$$\boxed{[\rho_{\rm tot}]_{\chi_-\chi_+} = \frac{1}{Z}\int\mathcal{D}\chi\,\mathrm{e}^{-S[\chi]}\delta[\chi(-0,\boldsymbol{x}) - \chi_-(\boldsymbol{x})]\delta[\chi(+0,\boldsymbol{x}) - \chi_+(\boldsymbol{x})]}$$

EE of vacuum state

- EE can be obtained as an analytical continuation of Renyi entropy:

$$S_A = \lim_{n\to 1} S_A^{(n)}$$
$$S_A^{(n)} := \frac{1}{1-n}\log \mathrm{tr}\rho_A^n$$

$$\boxed{\mathrm{tr}\rho_A^n = \frac{1}{Z^n}\int \prod_{x\in\Sigma_n} \mathrm{d}\phi\, \mathrm{e}^{-S[\phi]}}$$

- EE of a half plane is evaluated as

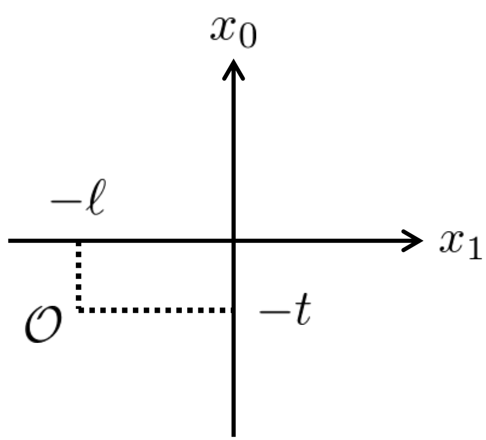
$$\boxed{S_A = -\frac{NV_{d-2}}{6(d-2)(4\pi)^{(d-2)/2}}\cdot\frac{1}{\varepsilon^{d-2}} + \mathcal{O}(\varepsilon^{-(d-4)})}$$

The most divergent term satisfies area law but the coefficient is negative

This result is similar to EE of ghost fields [Donnelly-Wall '12]

EE of locally excited state

[Nozaki-Numasawa-Takayanagi, '14, Nozaki '14]



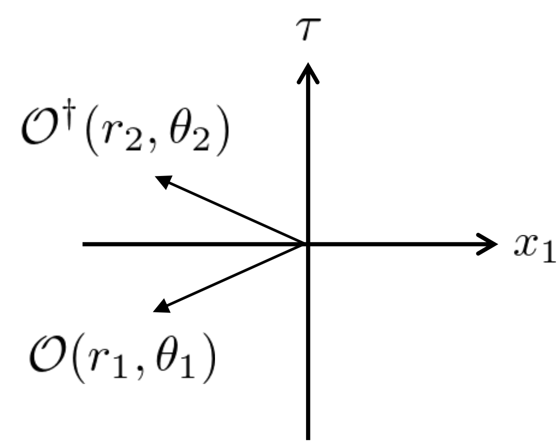
- Consider a locally excited state:

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(-t,-\ell,\boldsymbol{x})|0\rangle$$

$$\Rightarrow \quad \rho_{\rm tot} = \mathcal{N}^{-2}\mathcal{O}(-t,-\ell,\boldsymbol{x})|0\rangle\langle 0|C\mathcal{O}_C^\dagger(-t,-\ell,\boldsymbol{x}) \qquad \left(\mathcal{O}_C := C\mathcal{O}C\right)$$

- Density matrix in path-integral representation

$$[\rho_{\rm tot}]_{\chi_-\chi_+} = \frac{1}{Z}\int \mathcal{D}\chi\,\mathcal{O}_C^\dagger(r_2,\theta_2)\mathcal{O}(r_1,\theta_1)\,\mathrm{e}^{-S[\chi]}$$
$$\times\delta[\chi(-0,\boldsymbol{x})-\chi_-(\boldsymbol{x})]\delta[\chi(+0,\boldsymbol{x})-\chi_+(\boldsymbol{x})]$$



EE of locally excited state

- Difference of Renyi entropy between the excited state and the vacuum state of a half plane:

$$\Delta S_A^{(n)} := S_{A,\mathrm{ex}}^{(n)} - S_{A,\mathrm{vacuum}}^{(n)}$$

replica trick

$$\Rightarrow \quad \Delta S_A^{(n)} = \frac{1}{1-n} \log \left(\frac{\langle \prod_k \mathcal{O}_C^\dagger(r_2, \theta_{2,k})\, \mathcal{O}(r_1, \theta_{1,k}) \rangle_{\Sigma_n}}{\langle \mathcal{O}_C^\dagger(r_2, \theta_2)\, \mathcal{O}(r_1, \theta_1) \rangle_{\Sigma_1}^n} \right)$$

- When $\mathcal{O} = \chi^a$ ($\mathcal{O}_C^\dagger = \chi^{a+N/2}$), $\Delta S_A^{(n)}$ is expressed by Green functions

$$\langle \chi^{a+N/2}(r, \theta, \boldsymbol{x}) \chi^b(s, \theta', \boldsymbol{x}') \rangle = \underline{G(r, s, \theta, \theta', \boldsymbol{x}, \boldsymbol{x}')}\delta^{ab}$$

Green function

$$G(r, s, \theta + 2\pi n, \theta', \boldsymbol{x}, \boldsymbol{x}') = G(r, s, \theta, \theta', \boldsymbol{x}, \boldsymbol{x}')$$

EE of locally excited state

- Green function

$$d_0 = 2, d_{l\geq 1} = 1$$

$$G(r,s,\theta,\theta',\boldsymbol{x},\boldsymbol{x}') = \frac{1}{4\pi n(2\pi rs)^{\frac{d-1}{2}}}\sum_{l=0}^{\infty} d_l \cos\left(\frac{\theta-\theta'}{n}l\right)\int_0^\infty \mathrm{d}v\, v^{\frac{d-3}{2}} \mathrm{e}^{-\frac{1+a^2}{2a}v} I_{l/n}(v)$$

modified Bessel func.

$$\left(\quad \frac{a}{1+a^2} = \frac{rs}{|\boldsymbol{x}-\boldsymbol{x}'|^2 + r^2 + s^2}\quad\right)$$

This propagator is the same as that of standard scalar field theory

$$\Rightarrow \quad \Delta S_A^{(n)}{}_{,Sp(N)} = \Delta S_A^{(n)}{}_{,\mathrm{scalar}}$$

Positive quantity!!

- Example (4-dim, n=2)

$$G(r,s,\theta,\theta',\boldsymbol{x},\boldsymbol{x}') = \frac{1}{4n\pi^2 rs(a-a^{-1})}\cdot\frac{a^{\frac{1}{n}} - a^{-\frac{1}{n}}}{a^{\frac{1}{n}} + a^{-\frac{1}{n}} - 2\cos\left(\frac{\theta-\theta'}{n}\right)}$$

$$\Delta S_A^{(2)} = 0 \quad (t<\ell)\,, \qquad \Delta S_A^{(2)} = \log\left(\frac{2t^2}{\ell^2+t^2}\right) \quad (t\geq\ell)$$

Conclusion

- In pseudo-Hermitian quantum mechanics, I modify the density matrix:

Modified density matrix

$$\rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi|\eta$$

$$\eta H \eta^{\dagger} = H^{\dagger} \quad (H : \mathrm{Hamiltonian}\,, \quad \eta : \mathrm{unitary\ op.})$$

- Entanglement entropy

1. When the inner product $\langle\psi|\eta|\phi\rangle$ is positive definite, EE is also positive
2. When $\langle\psi|\eta|\phi\rangle$ isn't positive definite, EE becomes negative

   However, the further modification of the density matrix resolves this problem

   (I don't consider this possibility in Sp(N) model because
   I don't know a counterpart of this operator)

Conclusion

- Sp(N) model:

Modified density matrix

$$\rho_{\rm tot} = |\Psi\rangle\langle\Psi|C$$

$$CHC^\dagger = H^\dagger\,, \quad C\chi^a C^\dagger = \chi^{a+N/2}$$

$$(H:\text{Hamiltonian}\,, \quad \eta:\text{unitary op.})$$

- Entanglement entropy

Vacuum state

$$S_A \sim -\text{const.}\frac{\text{Area of }\partial A}{\varepsilon^{d-2}} + \cdots \propto -S_{A,\text{scalar}}$$

Excited state

$$\Delta S_A^{(n)} := S_{A,\text{ex}}^{(n)} - S_{A,\text{vacuum}}^{(n)} = \Delta S_A^{(n)}{}_{,\text{scalar}}$$

Thank you for
your attention!!