YITP long term workshop 2016
Quantum Information in String Theory and Many-body Systems
9 June 2016

Study of thermalization in Yang-Mills theory with use of Husimi function

Hidekazu Tsukiji (YITP)

My talk is based on
PTEP, 083A01 (2015)
(arXiv: 1505.04698)
& arXiv: 1603.04622

Collaborators : Hideaki Iida (Keio U)
Teiji Kunihiro (Kyoto U)
Akira Ohnishi (YITP)
Toru T. Takahashi (Gumma Col.)

Entropy production in pure state

Thermalization $\Longleftrightarrow$ Entropy production

When the system is an isolated system,
von-Neumann entropy dose not change by the quantum time evolution.

We consider coarse-graining for entropy production.

Ex) Partial trace is one way of coarse-graining.

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho$$

→Entanglement entropy $S_A = -\mathrm{Tr}\rho_A \log \rho_A$

Definition in gauge theory [Aoki et. al.(2015); Ghosh et. al.(2015) etc.]

Probe of confinement [I.R.Klebanov et. al.(2008)]

Lattice simulation [Velytsky(2008); Buividovich,Polikarpov(2008); Y.Nakagawa et. al.(2008,2010); E.Ito et. al.(2015). etc.]

Instability, Chaos & Entropy production

Classical system

Instability
&
Chaoticity

Quantum system

Entropy
production

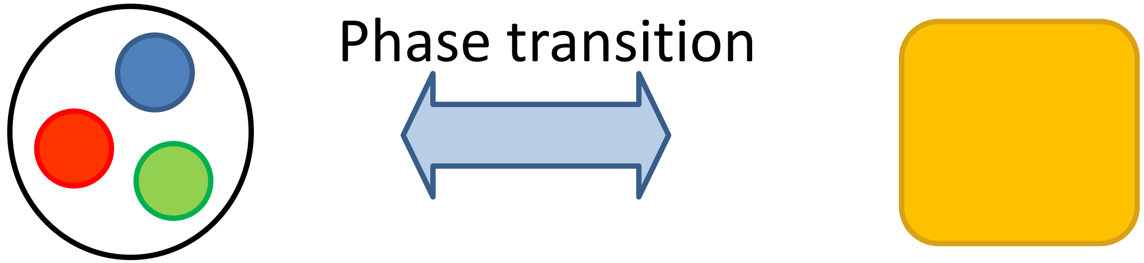
Ex) Entanglement entropy

Semi-classical treatment

What is the relation between them?

In this talk, we consider coarse-graining on “phase space” and define another entropy.

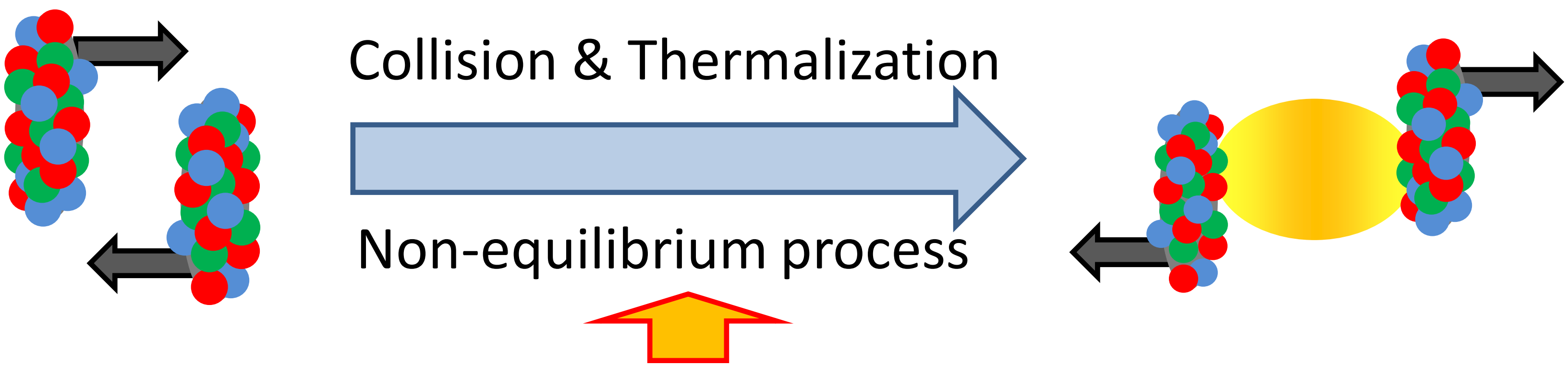
Motivation by phenomenology



Hadron
(Confined)

Quark gluon plasma
(Deconfined)

Relativistic heavy ion collisions



Strong gluon field is dominant.
It is well described by semi-classical Yang-Mills field.

Outline

Motivation

To explore the thermalization in semi-classical systems

How to detect chaoticity

Semi-classical time evolution

Another coarse graining

Entropy

Numerical methods

Entropy production in YM field

Outline

Motivation

To explore the thermalization in semi-classical systems

How to detect chaoticity

Semi-classical time evolution ・・・ *Wigner function*

Another coarse graining ・・・ *Husimi function*

Entropy ・・・ *Husimi-Wehrl entropy*

Numerical methods

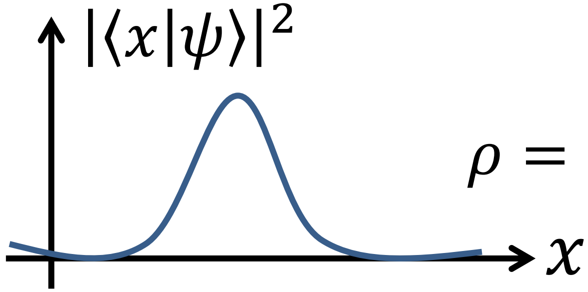
Entropy production in YM field

Wigner function

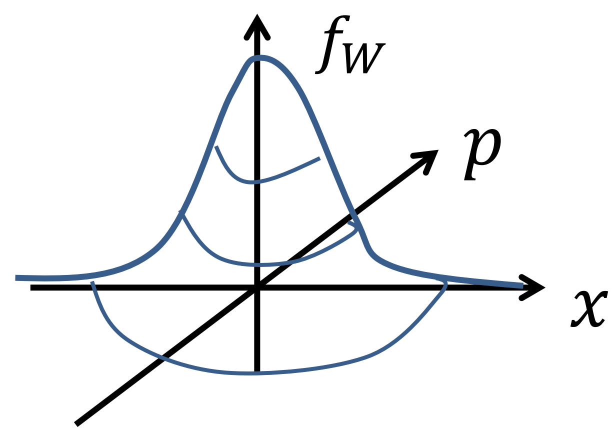
Wigner function [Wigner(1932)]

$$f_W(\vec{p},\vec{q};t) = \int d\vec{\eta}\exp(-i\vec{p}\cdot\vec{\eta}/\hbar)\langle\vec{q}+\vec{\eta}/2|\rho|\vec{q}-\vec{\eta}/2\rangle$$

Wigner function is the density matrix in Wigner representation.



$\rho = |\psi\rangle\langle\psi|$ $\Rightarrow$ f_W



(Quasi-)distribution function

$$\langle\hat{A}\rangle = \int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} f_W(\vec{p},\vec{q};t)A_W(\vec{p},\vec{q};t)$$

Semi-classical time evolution

Time evolution of Wigner function $f_W(\vec{p},\vec{q};t) = \int d\vec{\eta} \exp(-i\vec{p}\cdot\vec{\eta}/\hbar)\langle \vec{q}+\vec{\eta}/2|\rho|\vec{q}-\vec{\eta}/2\rangle$

$$\underbrace{\frac{\partial}{\partial t} f_W = \sum_i^n \frac{\partial V}{\partial q_i}\frac{\partial f_W}{\partial p_i} - \sum_i^n \frac{p_i}{m}\frac{\partial f_W}{\partial q_i}}_{\sim O(\hbar^0)} + O(\hbar^2)$$

In semi-classical approximation,

$$\frac{d}{dt} f_W(\vec{p},\vec{q};t) = 0$$

With classical EOM $\dot{q}_i = \frac{p_i}{m}, \dot{p}_i = -\frac{\partial V}{\partial q_i}$

The semi-classical time evolution of Wigner function reflects the classical dynamics, the chaotic behaviors.

Another coarse-graining & Entropy

Another coarse-graining : Gaussian smearing

Husimi function [Husimi(1940)]

$|\vec{\alpha}\rangle$; coherent state

$\rho = |\phi\rangle\langle\phi|$

$$f_H(\Gamma;t) = \langle\vec{\alpha}|\hat{\rho}|\vec{\alpha}\rangle = |\langle\vec{\alpha}|\phi\rangle|^2 \geq 0$$

$$= \int \frac{d\Gamma'}{(\pi\hbar)^n} \exp(-\frac{1}{\hbar}(\Gamma - \Gamma')^2) f_W(\Gamma';t)$$

Where $\Gamma = (\vec{p}, \vec{q})$ is a point on the "phase space" in Wigner rep..

Gaussian smearing

Wigner function ･･･ not positive definite

Husimi function ･･･ semi-positive definite

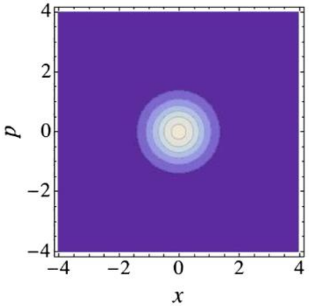
Husimi-Wehrl (HW) entropy [Wehrl(1978)]

$$S_{HW}(t) = -\int \frac{d\Gamma}{(2\pi\hbar)^n} f_H(\Gamma;t) \log f_H(\Gamma;t)$$

Detect chaoticity

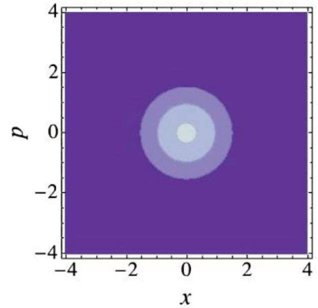
The figures are transferred from [Kunihiro-Muller-Ohnishi-Schafer (2009)].

Wigner function



Gaussian smearing

Husimi function



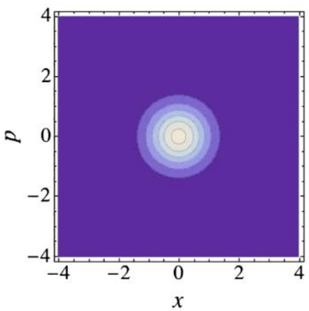
HW entropy

Small

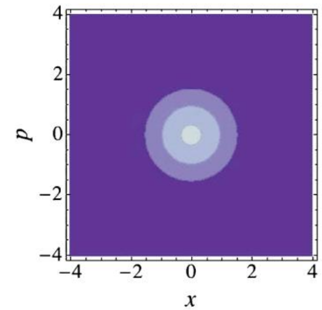
Detect chaoticity

The figures are transferred from [Kunihiro-Muller-Ohnishi-Schafer (2009)].

Wigner function | Husimi function | HW entropy

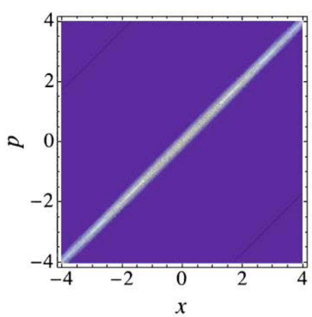


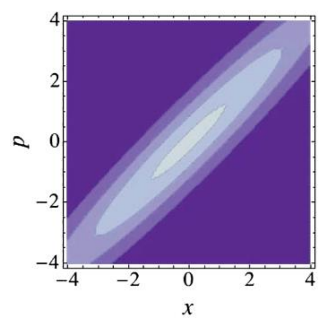
Gaussian smearing



Small

Time evolution
Chaos and instability





Big!

Gauss smearing makes entropy production
when the system has chaoticity and instability.

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How to detect chaoticity

Semi-classical time evolution ・・・ Wigner function

Another coarse graining ・・・ Husimi function

Entropy ・・・ Husimi-Wehrl entropy

Numerical methods

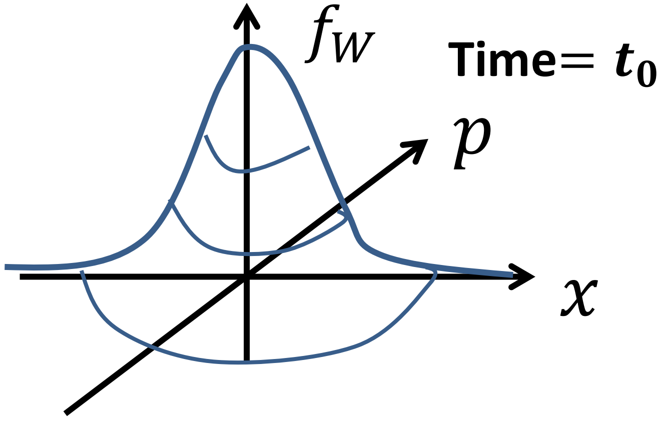
Entropy production in YM field

Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Husimi-Wehrl entropy in term of Wigner function

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2) \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'}, \vec{q'}; t)$$
$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p} + \vec{p'} - \vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q} + \vec{q'} - \vec{q''})^2) f_W(\vec{p''}, \vec{q''}; t)$$



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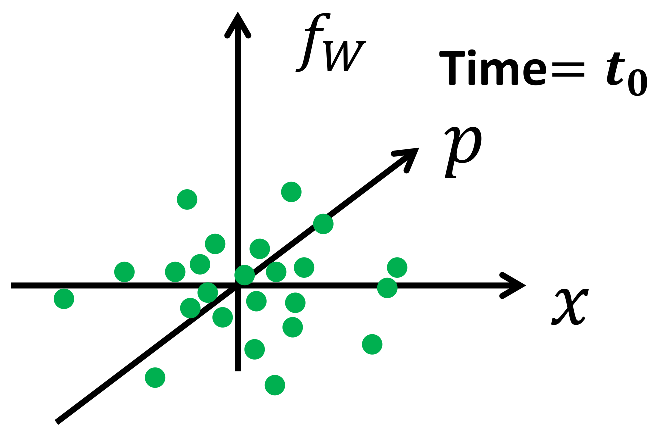
Test particles
[C.Y.Wong (1982)]



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$$
\begin{aligned}
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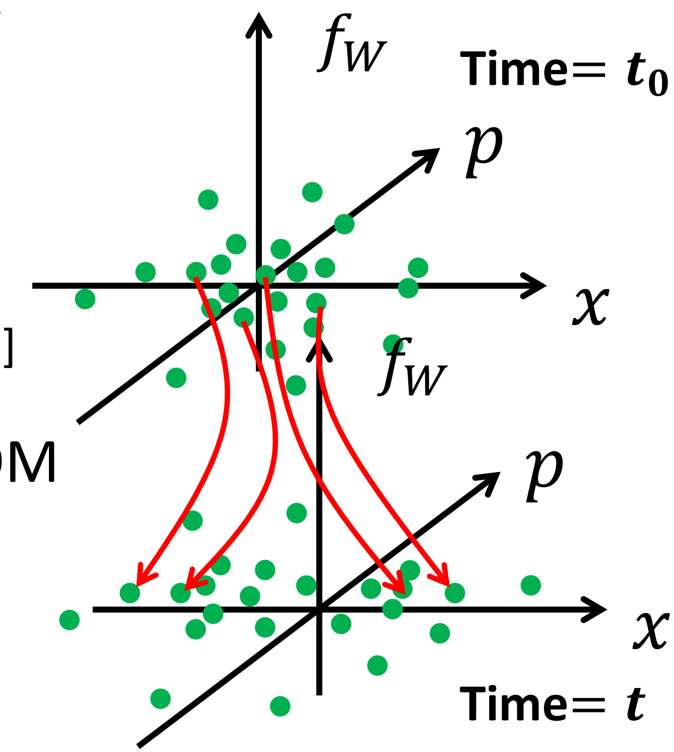
\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp\left(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2\right) f_W(\vec{p''},\vec{q''};t)
\end{aligned}
$$

Test particles

[C.Y.Wong (1982)]

Classical EOM



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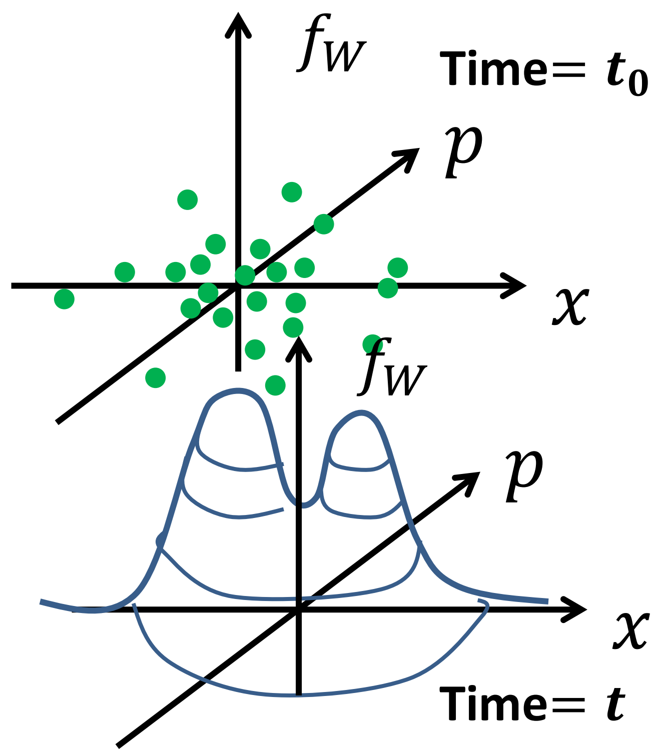

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Two sets of test particles $\{\Gamma'_i\}$ $\{\Gamma''_i\}$

Same set of test particles:

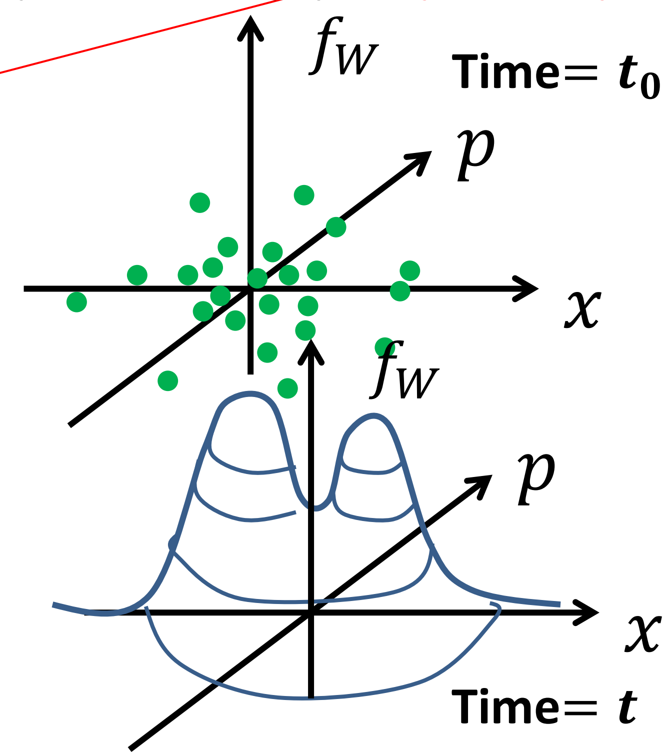
$$\{\Gamma'_i\} = \{\Gamma''_i\}$$

Test particle(TP) method

Different sets of test particles:

$$\{\Gamma'_i\} \neq \{\Gamma''_i\}$$

Parallel test particle(pTP) method



Examples in quantum mechanics

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

$$m = 1, g = 1, \epsilon = 0.1$$

Classical system is chaotic.

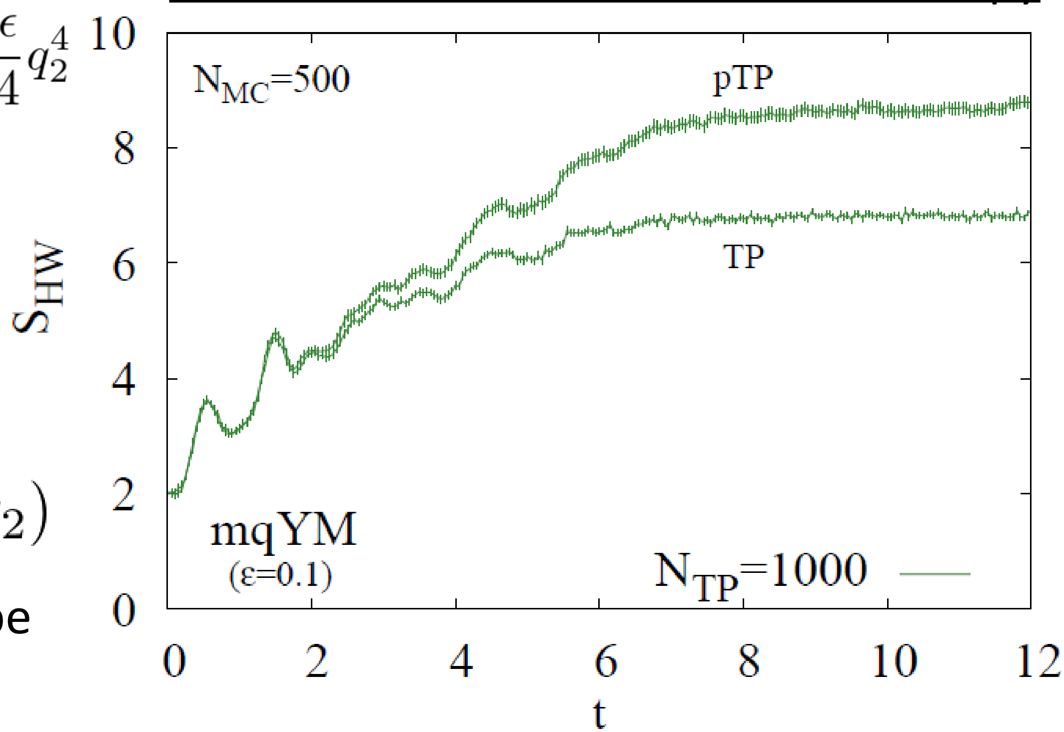
Initial condition: coherent state

$$f_W(\Gamma, t=0) = \exp[-\frac{1}{\hbar}\Gamma^2]$$

$$\Gamma = (p_1 - 10, p_2 - 10, q_1, q_2)$$

Our two numerical methods describe the entropy production.

Time evolution of Husimi-Wehrl entropy



The results in TP and pTP methods approach each other from below and above, respectively.
We can guess the converged value between them.

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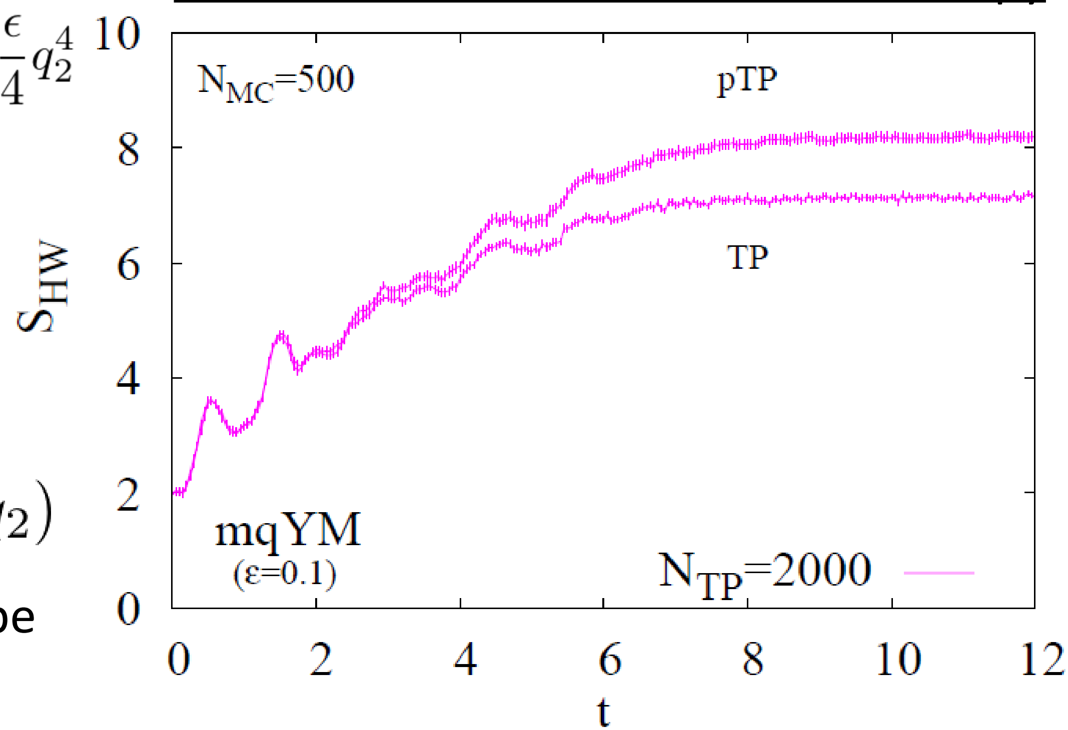

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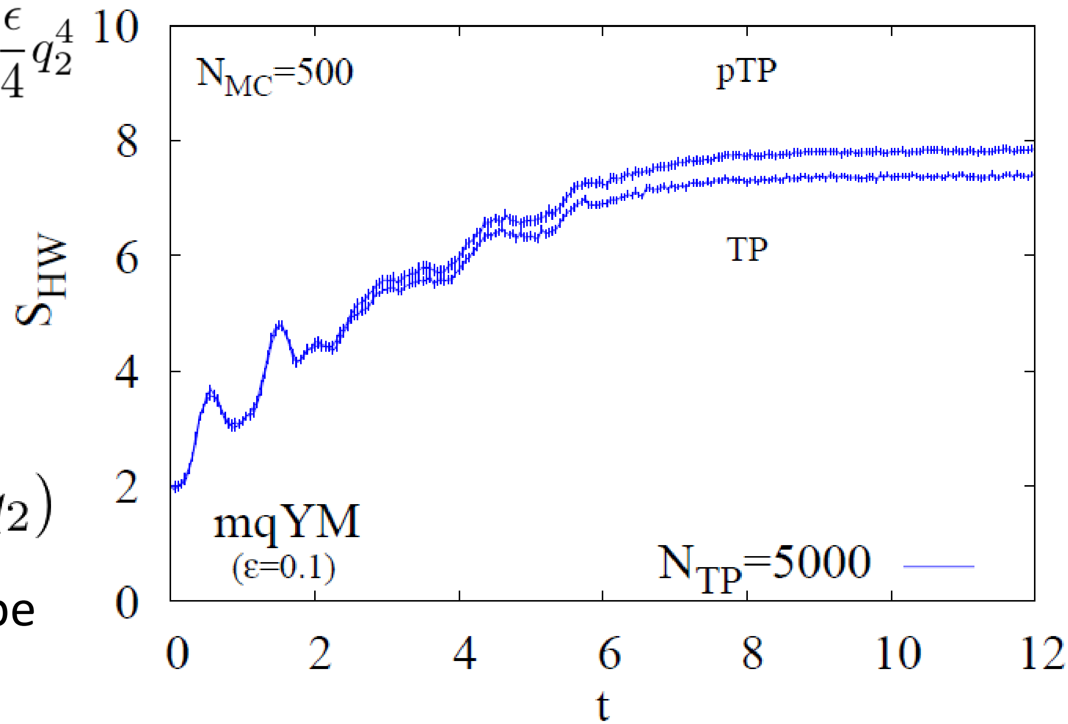

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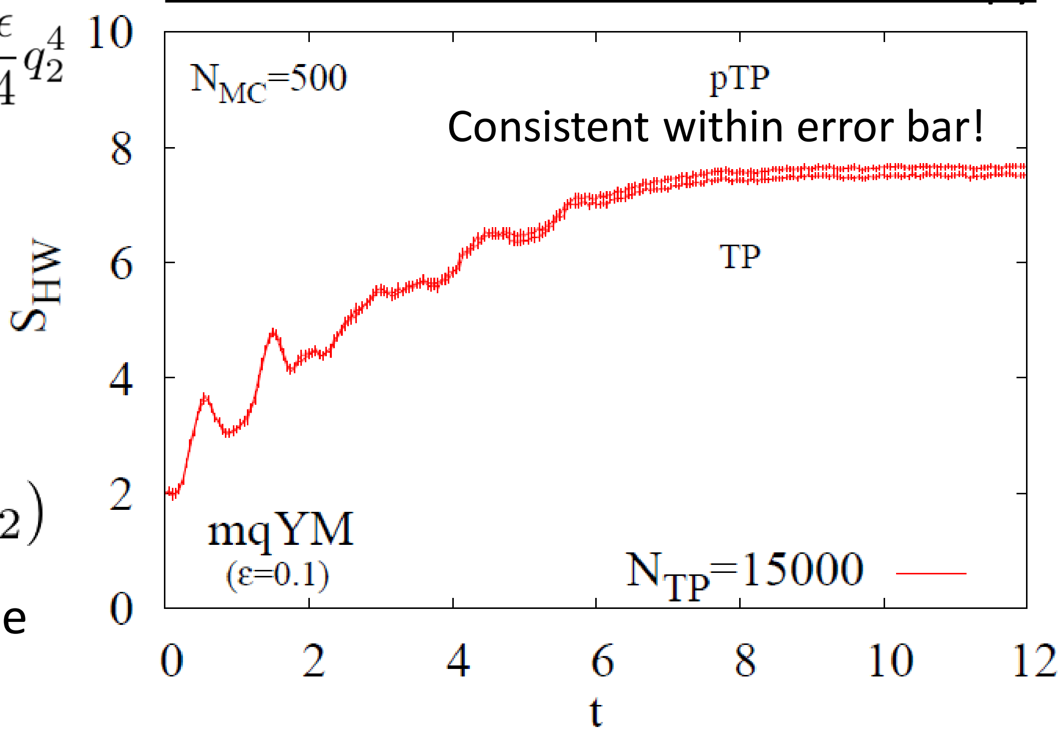

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Entropy production in YM field

Yang-Mills field

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

We will work in temporal gauge $A_0^a = 0$

Then Hamiltonian in a non-compact formalism is given by

$$H = \frac{1}{2}\sum_{x,a,i} E_i^a(x)^2 + \frac{1}{4}\sum_{x,a,i,j} F_{ij}^a(x)^2$$

$$F_{ij}^a = \partial_i A_j^a(x) - \partial_j A_i^a(x) + \sum_{b,c} f^{abc} A_i^b(x) A_j^c(x)$$

Canonical variables are $(A_i^a(x), E_i^a(x))$

Classical EOM

$$\dot{A}_i^a(x) = E_i^a(x)$$

$$\dot{E}_i^a(x) = \sum_j \partial_j F_{ij}^a(x) + \sum_{b,c,j} f^{abc} A_j^b(x) F_{ji}^c(x)$$

For the extension,

$$(q,p) \longrightarrow (A_i^a(x), E_i^a(x))$$

c.f. S. Mrowczynski, B. Muller(1994) (in a scalar field case)

HW entropy in Yang-Mills theory

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

We set inner product of fields
The extension is straightforward.

$$AB = \sum_{i,a} \int d^3x A_i^a(x) B_i^a(x)$$

Husimi functional

$$f_H[A, E; t] = \int \frac{DA'DE'}{(\pi\hbar)^{N_D}} \exp[-\frac{1}{\hbar\Delta}(A - A')^2 - \frac{\Delta}{\hbar}(E - E')^2] f_W[A', E'; t]$$

Husimi-Wehrl entropy

$$S_{HW}(t) = -\int \frac{DADE}{(2\pi\hbar)^{N_D}} f_H[A, E; t] \log f_H[A, E; t]$$

Initial condition of Wigner functional

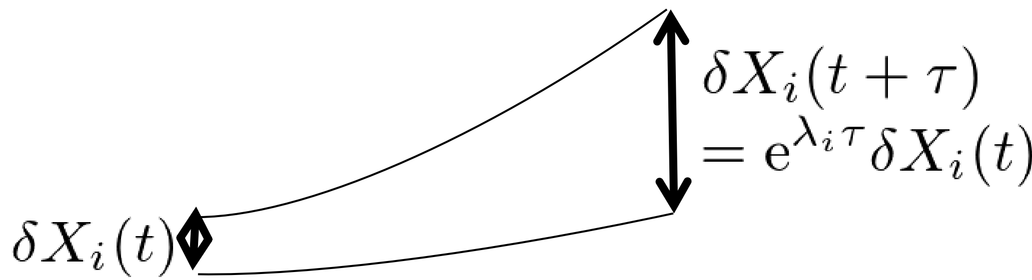
$$f_W[A, E : t = 0] = 2^{N_D} \exp[-\frac{1}{\hbar\omega} A^2 - \frac{\omega}{\hbar} E^2]$$

Chaoticity in classical Yang-Mills field

Chaotic systems have a sensitivity to initial value.

Lyapunov exponents λ_i

$$\lambda_i = \lim_{\tau\to\infty} \log \frac{\delta X(t+\tau)}{\delta X(t)}$$



$\delta\vec{X}$: distance between classical trajectories

It is given by the eigenvalue of a time evolution operator about distance in phase space;

$$U(t, t+\tau) = \mathcal{T}[\exp(\int_t^{t+\tau} \mathcal{H}(t')dt')] \qquad \mathcal{H} \text{ : Hessian}$$

- τ is infinite;

  Lyapunov exponent
- τ is intermediate time scale;

  Intermediate Lyapunov exponent(ILE)
- τ is infinitesimal;

  Local Lyapounov exponent(LLE)

In classical Yang-Mills field,
the LLE and ILE were calculated.

[Kunihiro-Muller-Ohnishi-Schafer-Takahashi-Yamamoto (2010)]
[Iida-Kunihiro-Muller-Ohnishi-Schafer-Takahashi (2013)]

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Another coarse graining ･･･ Husimi function

Entropy ･･･ Husimi-Wehrl entropy

Numerical methods

Test particle & Parallel test particle methods

Entropy production in YM field

YM field is a chaotic system. Is HW entropy is created?

Product ansatz

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

In higher dimension, we need a larger number of test particles.

⇨ Product ansatz to converge numerical results

Husimi function → The product of that of a single DOF

DOF : degrees of freedom

$$f_H(q,p;t) = \prod_i^D h_i(q_i,p_i;t)$$

Husimi-Wehrl entropy → The sum of that of a single DOF

$$S_{HW}^{(PA)} = -\sum_i^D \int \frac{dq_i dp_i}{2\pi\hbar} h(q_i,p_i;t) \log h(q_i,p_i;t)$$

$$\geq S_{HW}$$ (From subadditivity of entropy)

⇨ Upper bound of HW entropy

Check in the case of quantum mechanical system

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

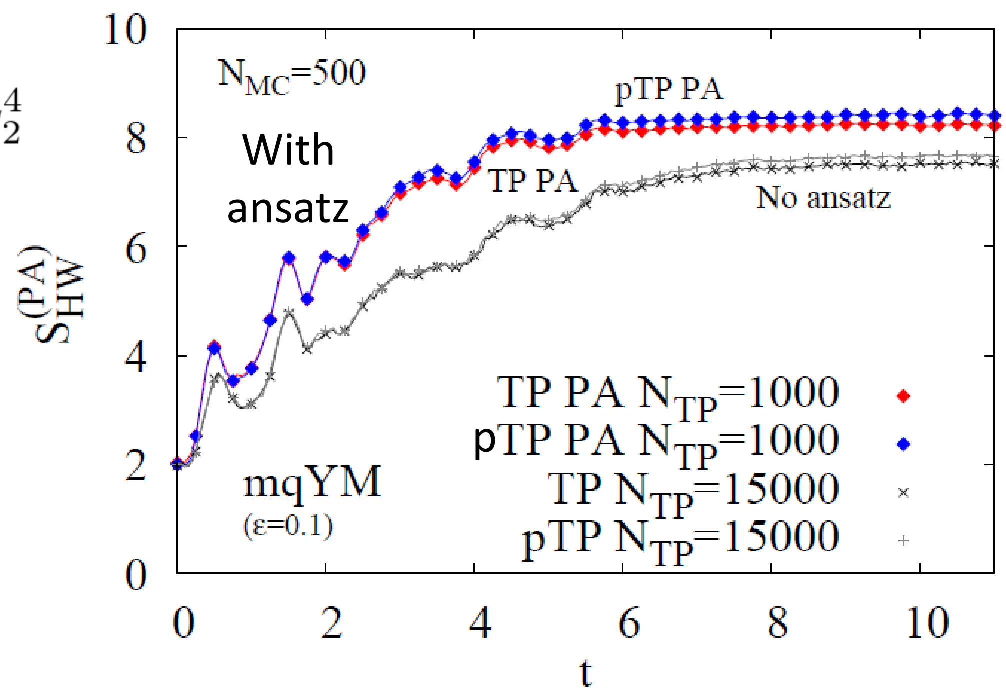
$$m = 1, g = 1, \epsilon = 0.1$$

Initial condition

$$f_W(\Gamma, t=0) = \exp[-\frac{1}{\hbar}\Gamma^2]$$

$$\Gamma = (p_1 - 10, p_2 - 10, q_1, q_2)$$

Time evolution of Husimi-Wehrl entropy



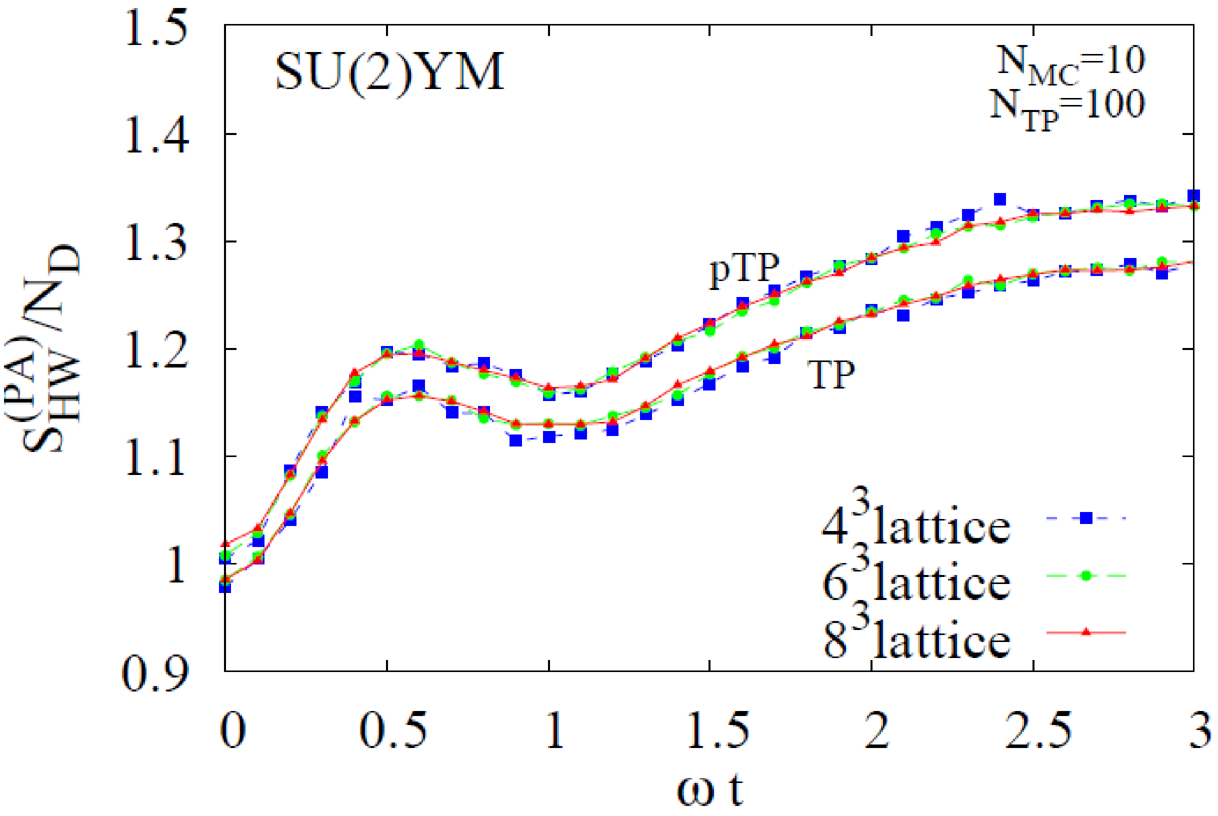
Product ansatz gives the upper bound of entropy and consistent results within 10% error bar.

The convergence with the number of the test particles is better.

Entropy production in SU(2) Yang-Mills field

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Time evolution of Husimi-Wehrl entropy per one DOF



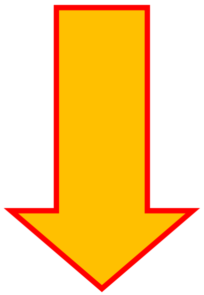
HW entropy is created in Yang-Mills theory.

Entropy production in SU(2) Yang-Mills field

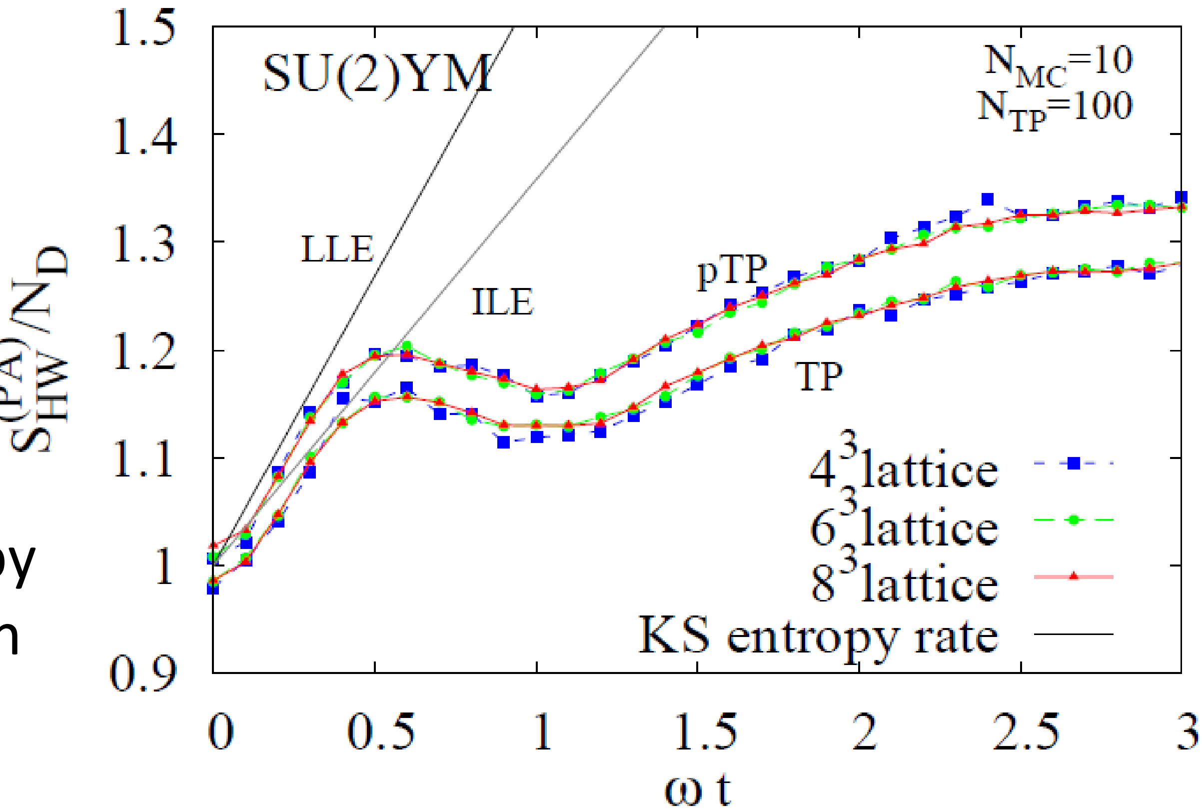
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Time evolution of Husimi-Wehrl entropy per one DOF

Chaotic dynamics



HW entropy production



Growth rate of HW entropy at initial time =
Sum of positive Lyapunov exponents given by [Kunihiro, et. al.(2010)].

Outline

Motivation

To explore the thermalization in semi-classical systems

How to detect chaoticity

Semi-classical time evolution ・・・ Wigner function

Another coarse graining ・・・ Husimi function

Entropy ・・・ Husimi-Wehrl entropy

Numerical methods

Test particle & Parallel test particle methods

Entropy production in YM field

HW entropy is produced in YM field theory.

Entropy growth rate = sum of positive Lyapunov exponents

Summary

- We have calculated entropy in semi-classical Yang-Mills theory for the first time.
- We have proposed to use a coarse graining function, Husimi function and calculated Husimi-Wehrl (HW) entropy.
- The production rate of the HW entropy agrees with the sum of positive Lyapunov exponents.
- This result suggests that thermal entropy has been created in Yang-Mills theory.

Future work

- Calculation with a more realistic setup for relativistic heavy ion collisions.
- Application to other models such as matrix models.

  Cf.) [Asano-Kawai-Yoshida(2015)]
- I'd like to understand the relation to entanglement entropy.

Back up

Examples in quantum mechanics

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

$$m = 1, g = 1, \epsilon = 0.1$$

Classical system is chaotic.

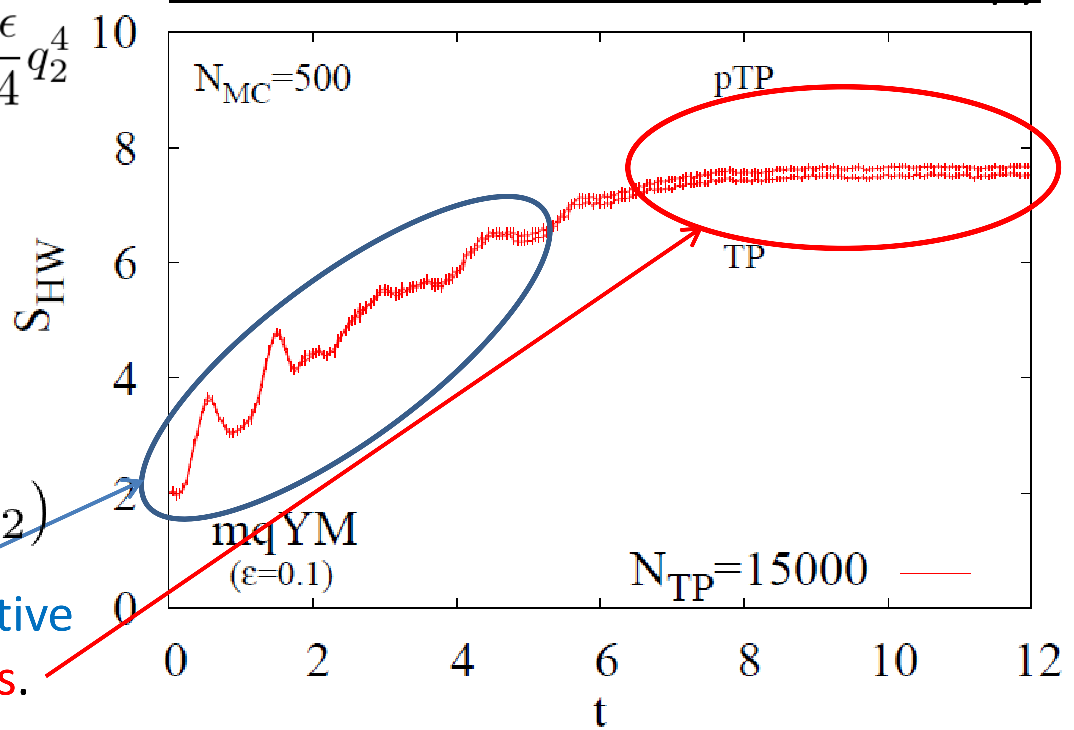
Initial condition: coherent state

$$f_W(\Gamma, t=0) = \exp[-\frac{1}{\hbar}\Gamma^2]$$

$$\Gamma = (p_1 - 10, p_2 - 10, q_1, q_2)$$

Husimi function spreads with collective motion in early time and saturates.

Time evolution of Husimi-Wehrl entropy



The results in TP and pTP methods approach each other from below and above, respectively.
We can guess the converged value between them.

Another coarse-graining

We are interested in
instabilities and chaotic behavior of classical systems.

Explore the quantum thermalization process
in semi-classical approximation

Another coarse-graining in phase space

Thermalization in Yang-Mills(YM) field

• Motivation	Heavy ion collisions
• Thermalization scenario	Chaos & instability
• Quantum effect	Semi-classical time evolution
• Entropy	Husimi-Wehrl entropy
• Numerical methods	Test particle & Parallel test particle methods
• Is entropy produced in YM?	Yes!

Outline

Motivation

To explore the thermalization process in heavy ion collisions

Strategy

Entropy production from chaoticity ･･･ Lyapunov exponent

Semi-classical formalism ･･･ Wigner function

Coarse graining ･･･ Husimi function

Numerical methods

Test particle & Parallel test particle methods, Product ansatz

Conclusion

HW entropy is produced in YM field theory. The growth rate agrees with a sum of positive Lyapunov exponents.

Entropy production in pure state

We consider coarse-graining for entropy production.

Ex) Partial trace is one way of coarse-graining.

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho$$

→Entanglement entropy $S_A = -\mathrm{Tr}\rho_A \log \rho_A$

It is considered as a probe of confinement [I.R.Klebanov et. al.(2008)]
and calculated in lattice simulation. Velytsky(2008); Buividovich,Polikarpov(2008); Y.Nakagawa et. al.(2008,2010); E.Ito et. al.(2015). etc.

In this talk,
we propose another way of coarse-graining in the phase space to evaluate instabilities and chaotic behavior of a system and calculate entropy to explore the thermalization process by using a semi-classical formalism.

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A probe of confinement I.R.Klebanov et. al.(2008)

Lattice simulation Velytsky(2008); Buividovich,Polikarpov(2008); Y.Nakagawa et. al.(2008,2010); E.Ito et. al.(2015). etc.

In this talk,

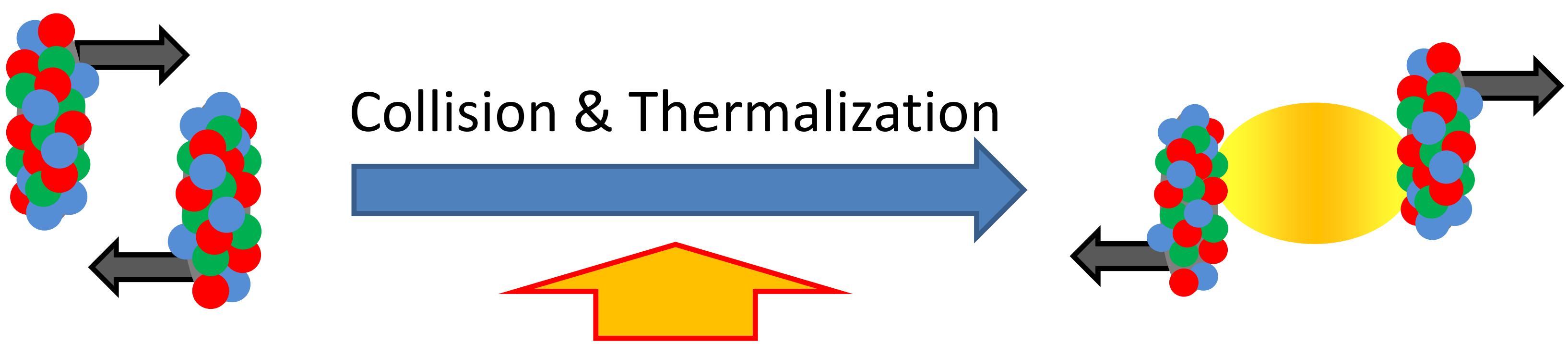
we propose another way of coarse-graining in the phase space to evaluate instabilities and chaotic behavior of a system and calculate entropy to explore the thermalization process by using a semi-classical formalism.

Phenomenological motivation

We explore the thermalization process by calculating entropy.
But the system is an isolated system.
Von-Neumann entropy dose not change by the quantum time evolution.

ma

<u>Relativistic heavy ion collisions</u>



This process is a non-equilibrium process in QCD.
In the process, gluon is dominant
and the semi-classical Yang-Mills field is exact.

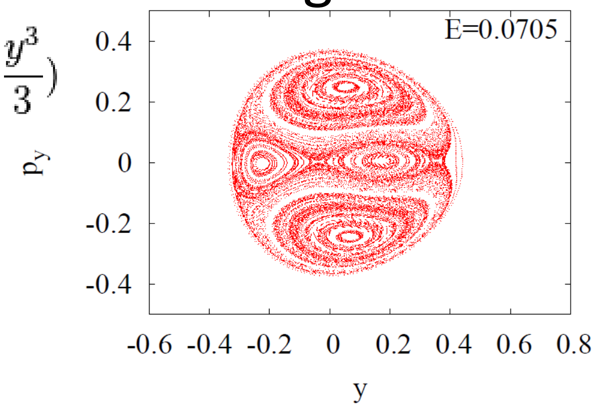
Chaoticity

Hénon-Heiles System

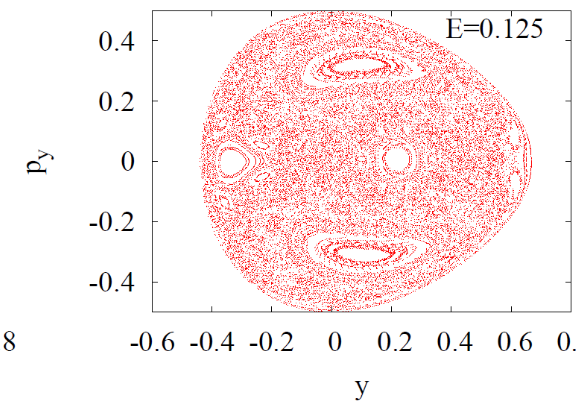
$$H=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}(x^2+y^2)+\lambda(x^2y-\frac{y^3}{3})$$

Henon-Heiles system shows chaotic behavior when the energy is high enough.

Regular



Chaos

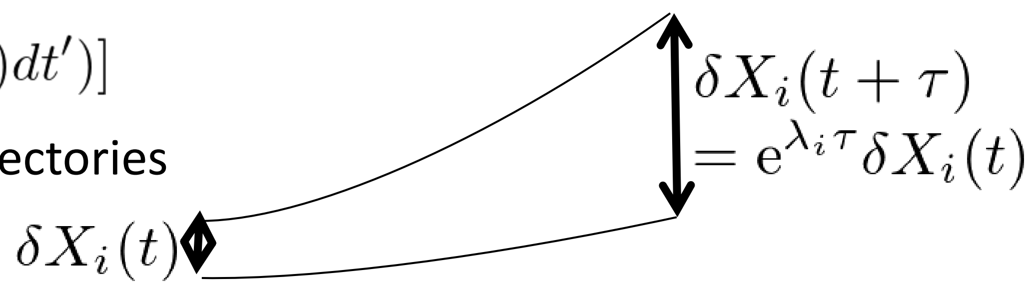


Chaotic systems have a sensitivity to initial value. This property is characterized by **Lyapunov exponents** λ_i, which is given from eigenvalue of a time evolution operator about distance $\delta\vec{X}$ in phase space;

$$U(t,t+\tau)=\mathcal{T}[\exp(\int_t^{t+\tau}\mathcal{H}(t')dt')]$$

$\delta\vec{X}$: distance between classical trajectories

$\mathcal{H}$: Hessian



The sum of positive Lyapunov exponents is positive in classical YM field.

[T.Kunihiro, B.Muller, A.Ohnishi, A.Schafer, T.T.Takahashi, A.Yamamoto, PRD **82**, 114015(2010)]
[H.Iida, T.Kunihiro, B.Muller, A.Ohnishi, A.Schafer, T.T.Takahashi, PRD **88**, 094006(2013)]

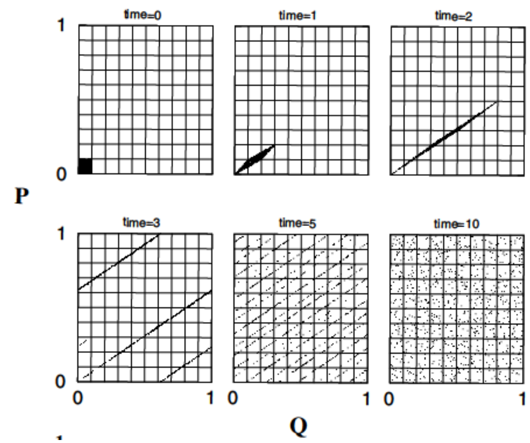
Thermalization scenario based on chaos

V. Latora and M.Baranger (1999);
M. Baranger, V. Latora and A.Rapisarda (2002)

Generalized cat map(chaotic system)

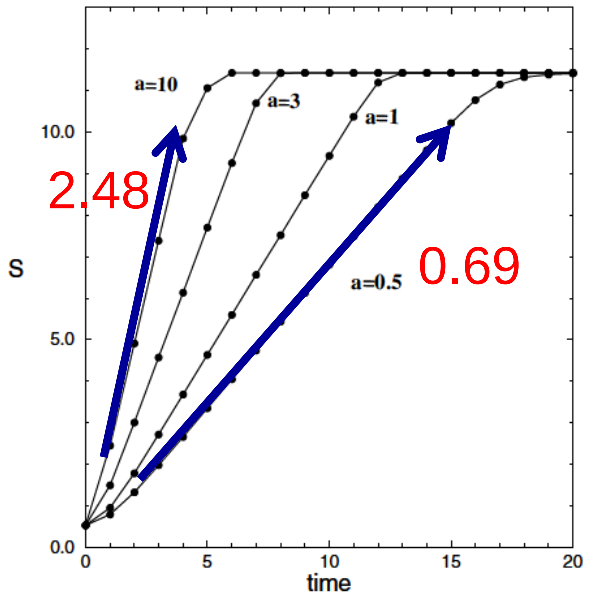
$$P = p + aq \quad (\text{mod } 1),$$

$$Q = p + (1+a)q \quad (\text{mod } 1)$$



Lyapunov exponent

$$\lambda = \log \frac{1}{2}\left(2 + a + \sqrt{a^2 + 4a}\right)$$



Corse-grained Boltzmann Gibbs entropy

$$S(t) = -\sum_{i:\text{cell}} p_i(t) \log p_i(t)$$

$p_i(t)$:probability that the state of the system falls inside cell c_i of phase space at time t

The entropy production rate is consistent with Lyapunov exponent.

$$\boldsymbol{\lambda = 2.48, 1.57, 0.96, 0.69}$$

Husimi function

The figures are transferred from T.Kunihiro, B.Muller, A.Ohnishi, A.Schafer(2009).

<u>Husimi function</u> [Husimi(1940)]

$|\vec{\alpha}\rangle$; coherent state

$$f_H(\Gamma;t) = \langle\vec{\alpha}|\hat{\rho}|\vec{\alpha}\rangle = |\langle\vec{\alpha}|\phi\rangle|^2 \geq 0$$

$$\rho = |\phi\rangle\langle\phi|$$

$$= \int \frac{d\Gamma'}{(\pi\hbar)^n} \exp(-\frac{1}{\hbar}(\Gamma - \Gamma')^2) f_W(\Gamma';t)$$

Where $\Gamma = (\vec{p}, \vec{q})$ is a point on the “phase space” in Wigner rep..

- Husimi function is semi-positive definite.

When Wigner function is lengthened by chaotic behaviors or instabilities, Husimi function spreads in “phase space”.

Husimi-Wehrl (HW) entropy

The figures are transferred from T.Kunihiro, B.Muller, A.Ohnishi, A.Schafer(2009).

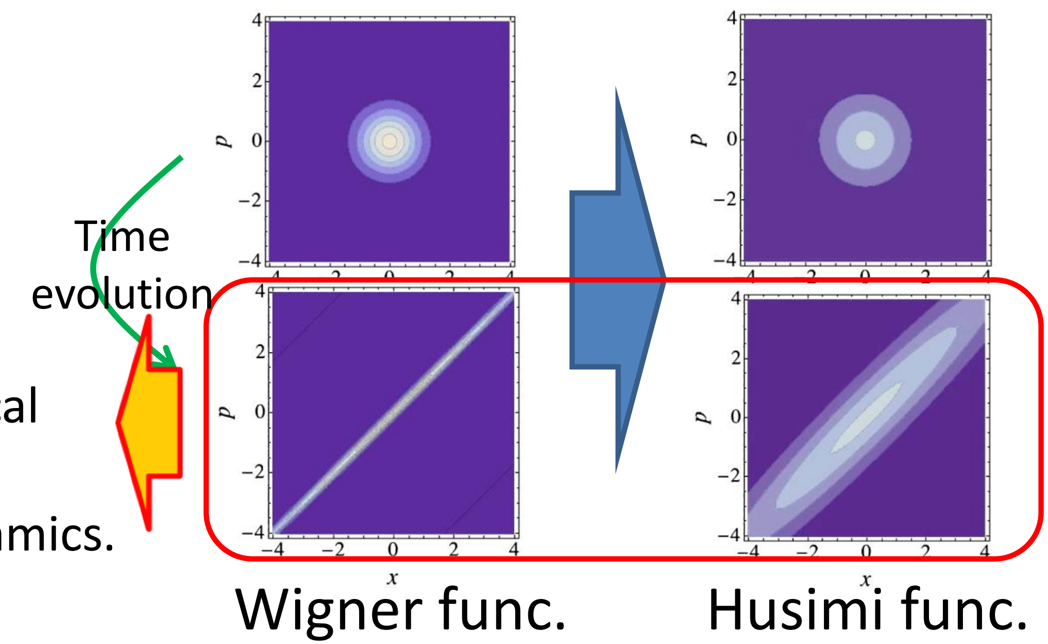
We can define entropy in terms of Husuimi function.

Husimi-Wehrl entropy [Wehrl(1978)]

$$S_{HW}(t) = -\int \frac{d\Gamma}{(2\pi\hbar)^n} f_H(\Gamma;t) \log f_H(\Gamma;t)$$

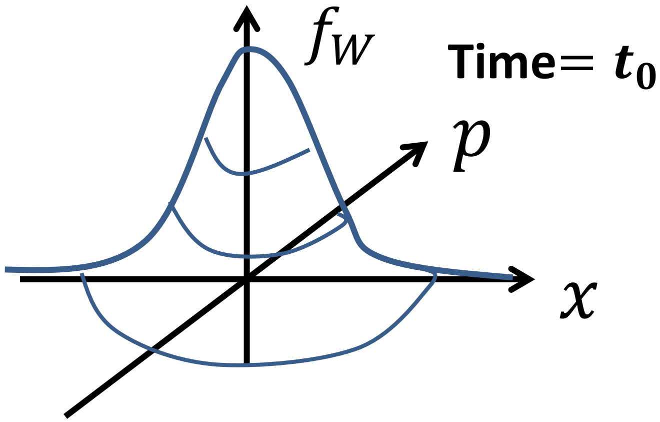
- Husimi function is semi-positive definite.
- Gauss smearing makes entropy production.

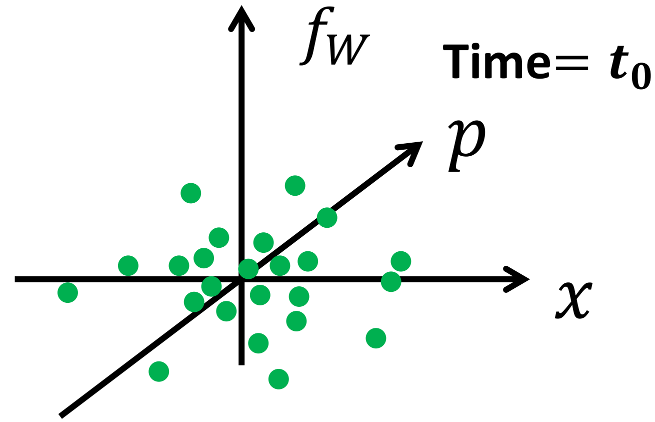
HW entropy is created when classical systems have chaos or instability.
The entropy evaluates chaotic dynamics.



Test particle method

Initial configuration





Numerical methods

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Husimi-Wehrl entropy in term of Wigner function

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2) \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'},\vec{q'};t)$$
$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2) f_W(\vec{p''},\vec{q''};t)$$

We would like to calculate these integrations numerically.

Sets of test particles $\{\Gamma'_i\}$ $\{\Gamma''_i\}$

Using same set of test particles:

$$\{\Gamma'_i\} = \{\Gamma''_i\}$$

Test particle(TP) method

Using two sets of test particles:

$$\{\Gamma'_i\} \neq \{\Gamma''_i\}$$

Parallel test particle(pTP) method

Examples in quantum mechanics

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, PTEP, 083A01 (2015).
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

$$m = 1, g = 1, \epsilon = 0.1$$

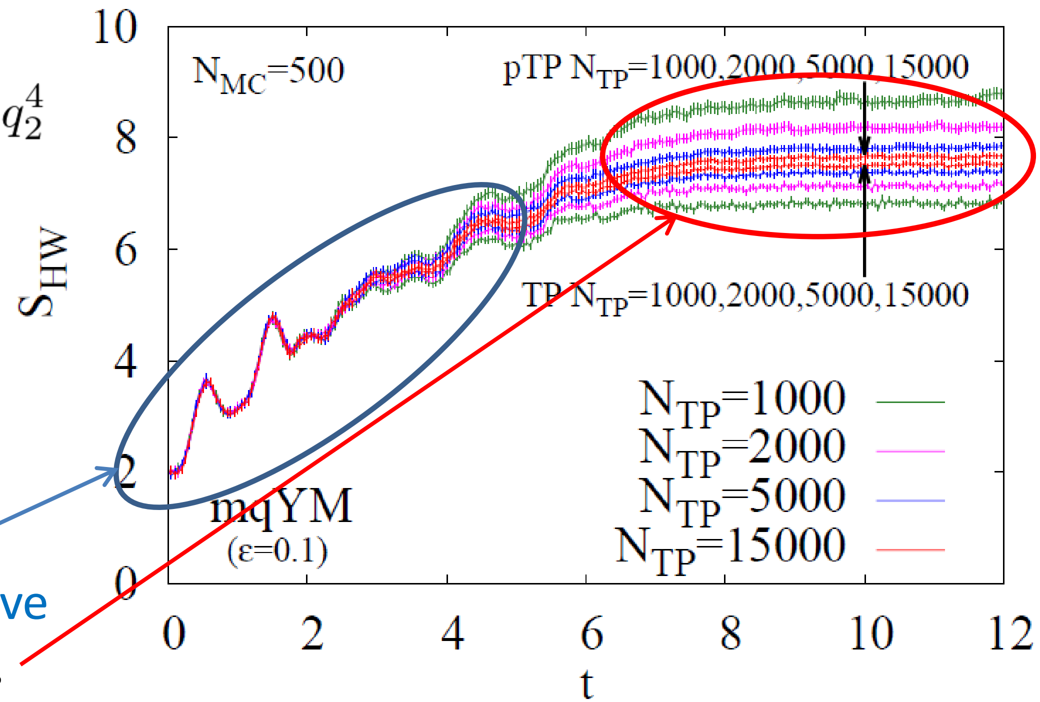
Initial condition: coherent state

$$f_W(\Gamma, t=0) = \exp[-\frac{1}{\hbar}\Gamma^2]$$

$$\Gamma = (p_1, p_2, q_1, q_2)$$

Husimi function spreads with collective motion in early time and saturates.

Time evolution of Husimi-Wehrl entropy



The results in TP and pTP methods approach each other from below and above, respectively. We can guess the converged value between them.

Entropy production in SU(2) Yang-Mills field

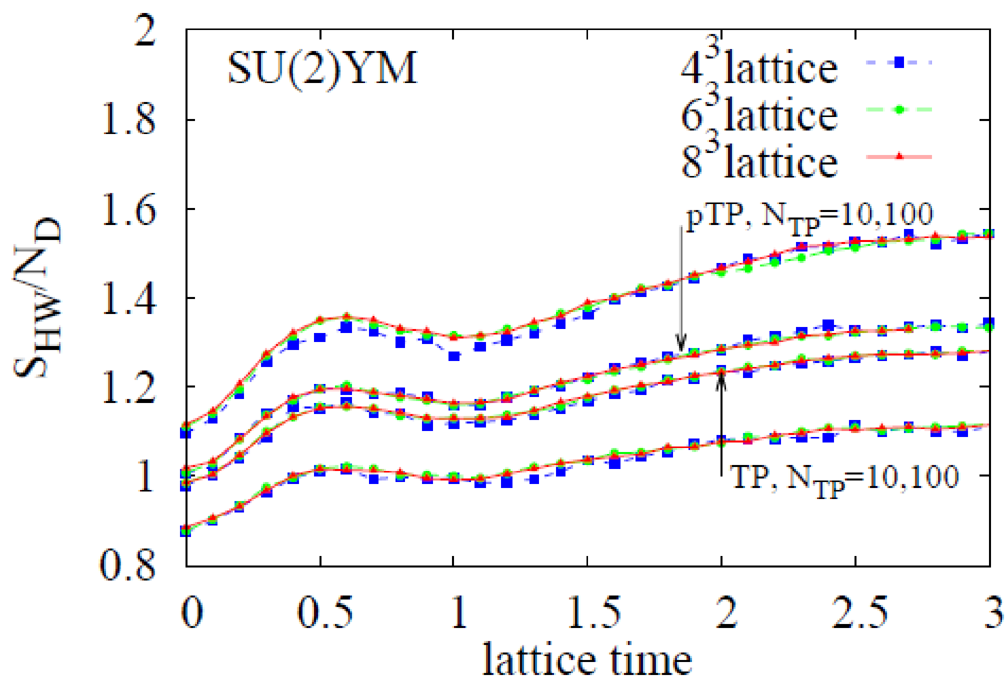
H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

Husimi-Wehrl entropy is produced in YM field!

The results in TP and pTP approach each other from below and above.

The time evolution of the entropy on each lattice size agrees with each other.

Time evolution of Husimi-Wehrl entropy per one degrees of freedom



We see that the HW entropy is created in Yang-Mills theory though in the product ansatz.

Entropy production in SU(2) Yang-Mills field

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi, arXiv: 1603.04622.

The growth rate is consistent with the sum of positive Lyapunov exponents in [Kunihiro, et. al.(2010)]

$$U(t,t+\tau)=\mathcal{T}[\exp(\int_t^{t+\tau}\mathcal{H}(t')dt')]$$

Lyapunov exponents are given by the eigenvalue of the time evolution operator.

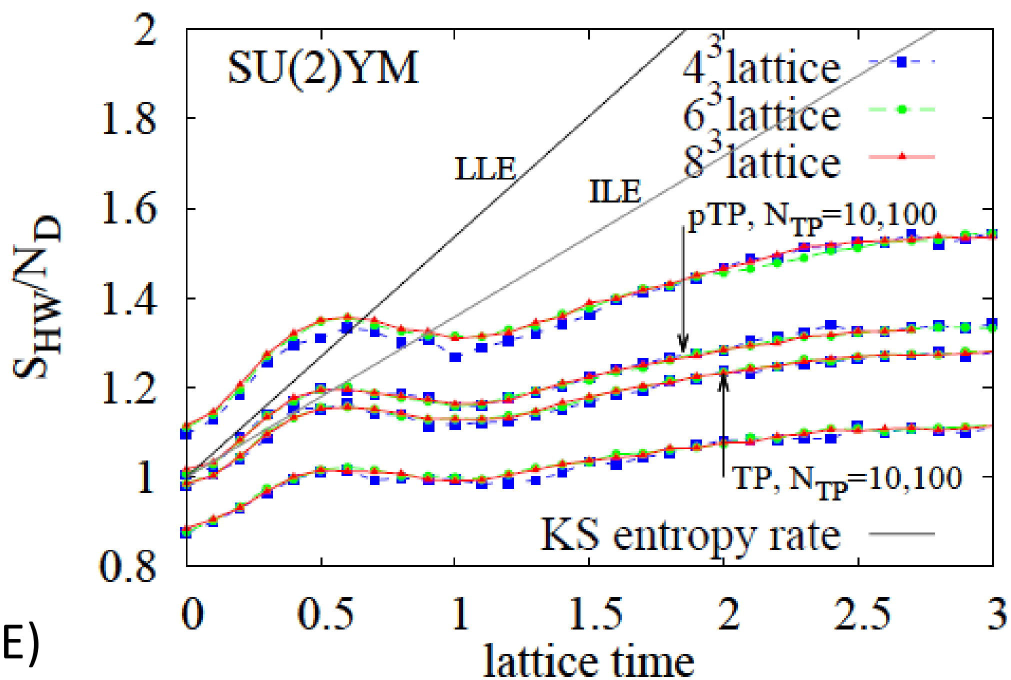
When τ is infinitesimal;

local Lyapounov exponent(LLE)

When τ is intermediate time scale;

intermediate Lyapounov exponent(ILE)

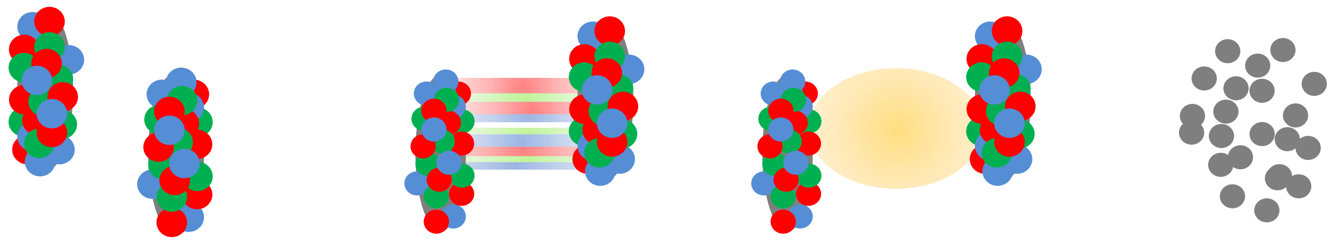
Time evolution of Husimi-Wehrl entropy per one degrees of freedom



The production of Husimi-Wehrl entropy is caused by the chaotic behavior of Yang-Mills field.

Motivation

Relativistic heavy ion collisions



Time evolution

Color Glass Condensate(CGC)

Classical approx. is valid.
[McLerran, Venugopalan(1993)]

Glasma

Classical approx.
[Lappi, McLerran(2006)]

Quark Gluon Plasma (QGP)

Hadron

Ideal liquid

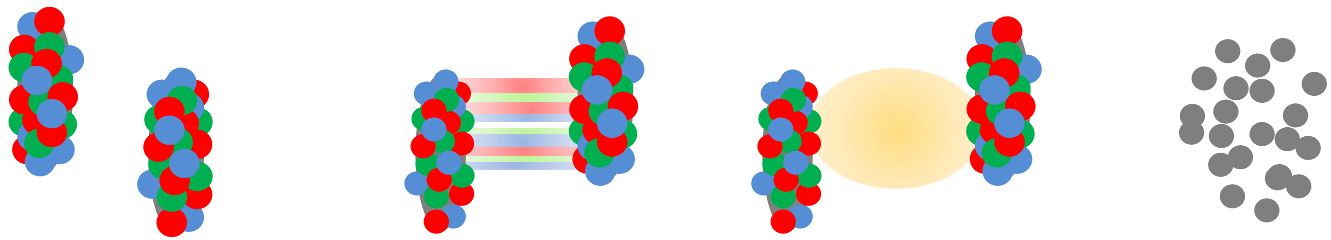
τ_0= 0.6 - 1.0 fm/c

Perturbative QCD cannot explain this time scale. [Baier, Mueller, Schiff and Son(2001,2002)]

Early thermalization [Heinz(2002)]

Motivation

Relativistic heavy ion collisions



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Glasma

Classical approx.
[Lappi, McLerran(2006)]

Classical Yang-Mills field
[Romatschke and Venugopalan(2006)]

Classical YM field
+
Quantum fluctuation
⇒ Instability

Quark Gluon Plasma (QGP)

Ideal liquid

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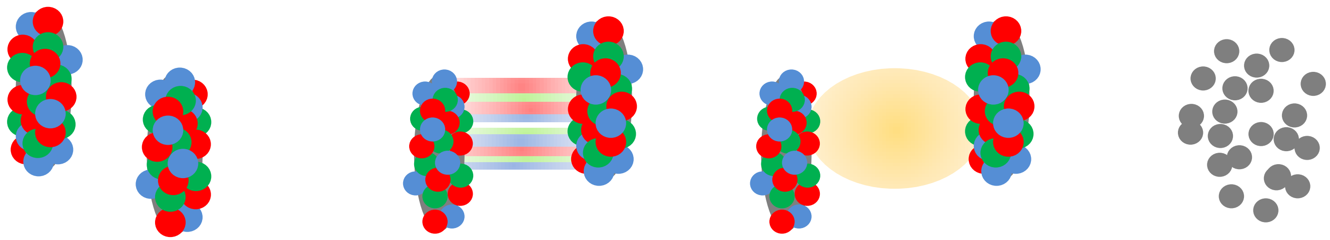

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Classical approx.
[Lappi, McLerran(2006)]

Quark Gluon Plasma (QGP)

Hadron

Classical Yang-Mills field
[Romatschke and Venugopalan(2006)]

Classical Statistical approximation

Early isotropization
[Epelbaum and Gelis(2013)]

Ideal liquid

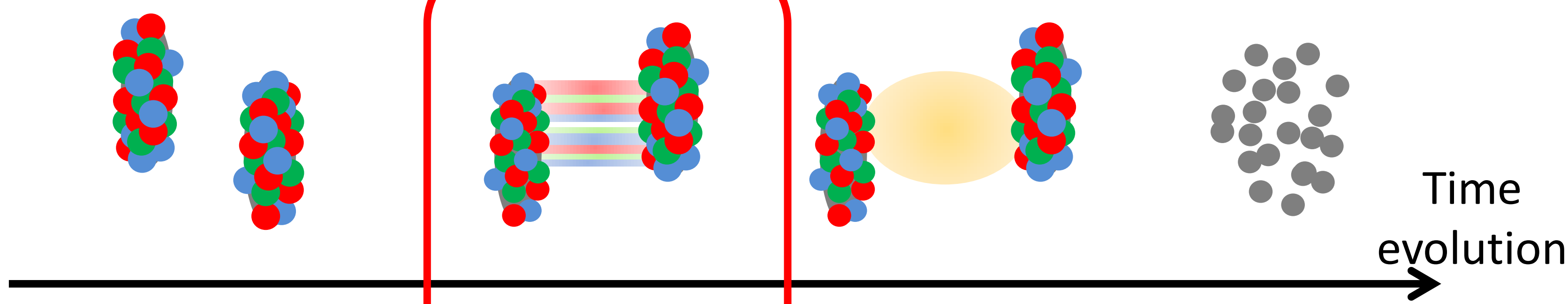
τ_0= 0.6 - 1.0 fm/c

Perturbative QCD cannot explain this time scale.
[Baier, Mueller, Schiff and Son(2001,2002)]

Early thermalization [Heinz(2002)]

Motivation

Relativistic heavy ion collisions



Color Glass Condensate(CGC)

Classical approx. is valid.
[McLerran, Venugopalan(1993)]

Glasma

Classical approx.
[Lappi, McLerran(2006)]

Quark Gluon Plasma (QGP)

Hadron

Ideal liquid

Large amount of entropy is produced in early stage of the collision.

τ_0= 0.6 - 1.0 fm/c

Perturbative QCD cannot explain this time scale.
[Baier, Mueller, Schiff and Son(2001,2002)]

Early thermalization [Heinz(2002)]

We calculate entropy directly by using a semi-classical formalism with initial fluctuations.

Γ

The definition of Husimi function

$$
\begin{aligned}
f_H(\Gamma;t) &= \langle\vec{\alpha}|\hat{\rho}|\vec{\alpha}\rangle = |\langle\vec{\alpha}|\phi\rangle|^2 \geq 0 \\
&= \int \frac{d\Gamma'}{(\pi\hbar)^n} \exp(-\frac{1}{\hbar}(\Gamma-\Gamma')^2) f_W(\Gamma';t) \\
&= \int \frac{dp'dq'}{(\pi\hbar)^n} \exp(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(q-q')^2) f_W(p',q';t)
\end{aligned}
$$

Initial condition of Wigner function

$$
\begin{aligned}
f_W(\Gamma, t=0) &= \exp[-\frac{1}{\hbar}\Gamma^2] \\
&= \exp[-\frac{1}{\hbar\omega}p_1^2 - \frac{1}{\hbar\omega}p_2^2 - \frac{\omega}{\hbar}q_1^2 - \frac{\omega}{\hbar}q_2^2]
\end{aligned}
$$

Why two methods show different behavior?

$$S_{HW}(t) = -\int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n}\exp(-\frac{1}{\Delta\hbar}\vec{p}^2 - \frac{\Delta}{\hbar}\vec{q}^2)\int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} f_W(\vec{p'},\vec{q'};t)$$
$$\times \log \int \frac{d\vec{p''}d\vec{q''}}{(\pi\hbar)^n}\exp(-\frac{1}{\Delta\hbar}(\vec{p}+\vec{p'}-\vec{p''})^2 - \frac{\Delta}{\hbar}(\vec{q}+\vec{q'}-\vec{q''})^2) f_W(\vec{p''},\vec{q''};t)$$
$$= -\frac{1}{N_{MC}}\sum_k^{N_{MC}}\frac{1}{N}\sum_i^N \log\frac{2^n}{N}\sum_j^N \exp[-\frac{1}{\Delta\hbar}(\vec{p}_k+\vec{\bar{p}}^i(t)-\vec{\bar{p}}^j(t))^2 - \frac{\Delta}{\hbar}(\vec{q}_k+\vec{\bar{q}}^i(t)-\vec{\bar{q}}^j(t))^2]$$

In TP method, $\overline{p'}$ ($\overline{q'}$) and $\overline{p''}$ ($\overline{q''}$) cancel at the same number $i = j$.

Coherent state in YM

In quantum mechanics,

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad\qquad (\alpha = \frac{\omega x + ip}{\sqrt{2\hbar\omega}})$$

In Yang-Mills field theory,

$$|\{\alpha\}\rangle = \prod_i^{N_D} |\alpha_i\rangle \qquad |\alpha_i\rangle = \sum_{n_i=0}^{\infty} \frac{\alpha_i{}^n}{\sqrt{n_i!}} |n_i\rangle \qquad (\alpha_i = \frac{\omega A_i + iE_i}{\sqrt{2\hbar\omega}})$$

Physical time scale

H.T., H.Iida, T.Kunihiro, A.Ohnishi, and T.T.Takahashi , in preparation.

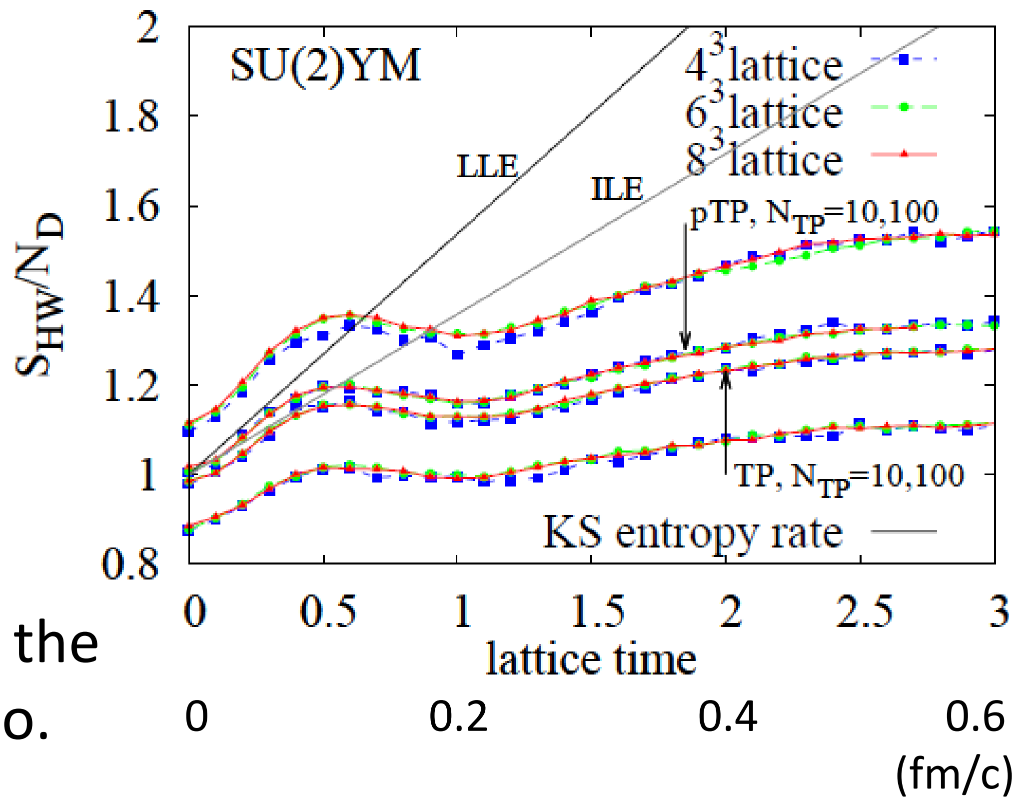
When we assume the range of Wigner function (lattice spacing a) is equal to the flux tube at early stage of the heavy ion collisions;

$$a = Q_S = 0.2\text{[fm/c]}$$

(@ RHIC),

the result suggests the early entropy production.

Time evolution of Husimi-Wehrl entropy per one degrees of freedom

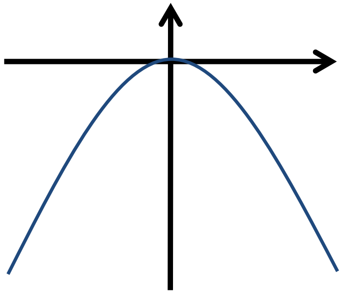


This is encouraging results for the early thermalization scenario.

We should calculate in the more realistic initial condition.

Inverted harmonic oscillator

T. Kunihiro, B. Muller, A. Ohnishi, A. Shafer(2009) (Analytic solution)



Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{1}{2}\lambda^2 q^2$$

Initial condition of Wigner function

$$f_W(p, q; t = 0) = 2\exp(-\frac{1}{\hbar\omega}p^2 - \frac{\omega}{\hbar}q^2)$$

Analytic solution of H-W entropy is given by

$$\boxed{S(t) = \frac{1}{2}\log\frac{A(t)}{4} + 1}$$

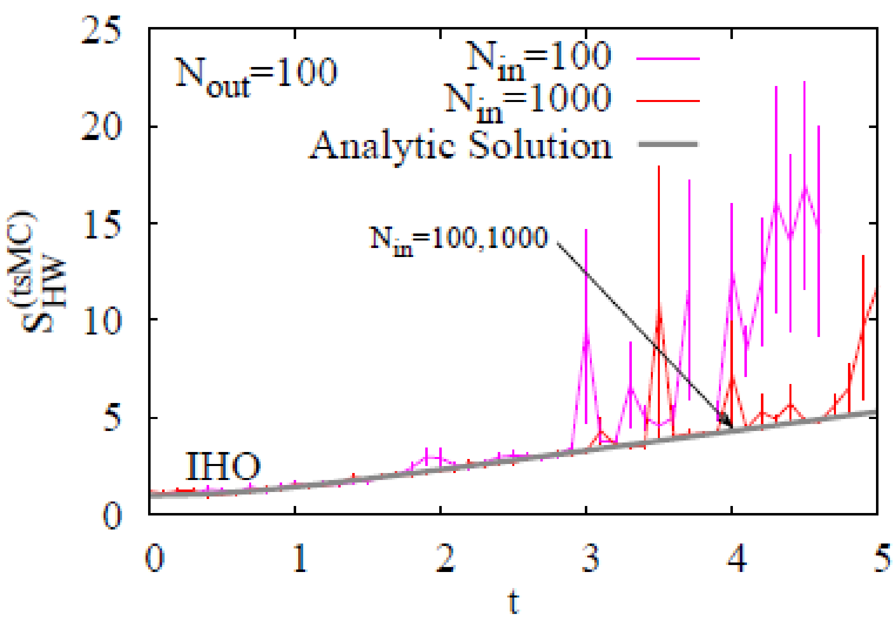
with $\quad A(t) = 2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')$

$$\sigma = \frac{\lambda^2 + \omega^2}{2\lambda\omega}, \delta = \frac{\lambda^2 - \omega^2}{2\lambda\omega}, \rho = \frac{\Delta^2 + \lambda^2}{2\Delta\lambda}, \delta' = \frac{\Delta^2 - \lambda^2}{2\Delta\lambda}$$

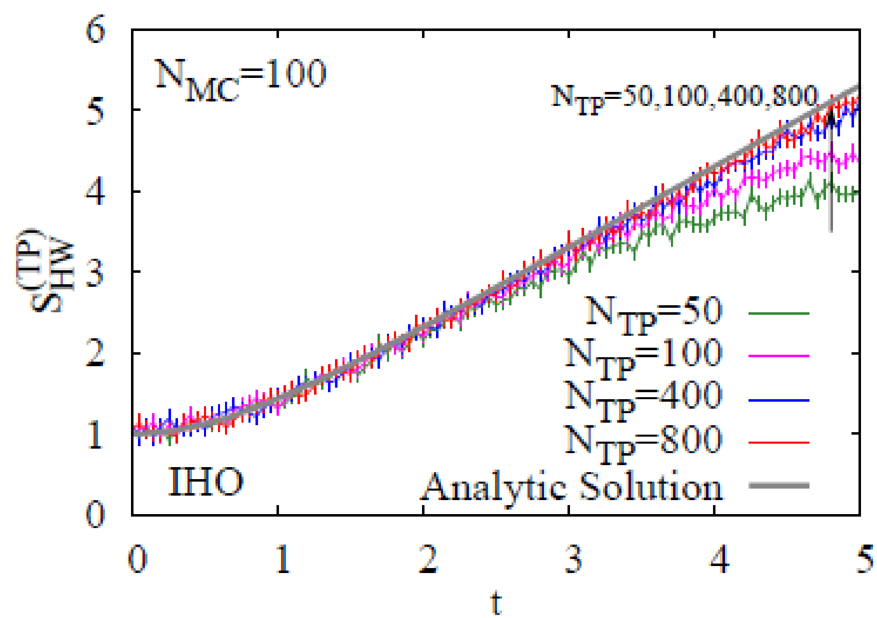
Inverted harmonic oscillator

T. Kunihiro, B. Muller, A. Ohnishi, A. Shafer(2009) (Analytic solution)

Two step Monte-Carlo method



Test particle method



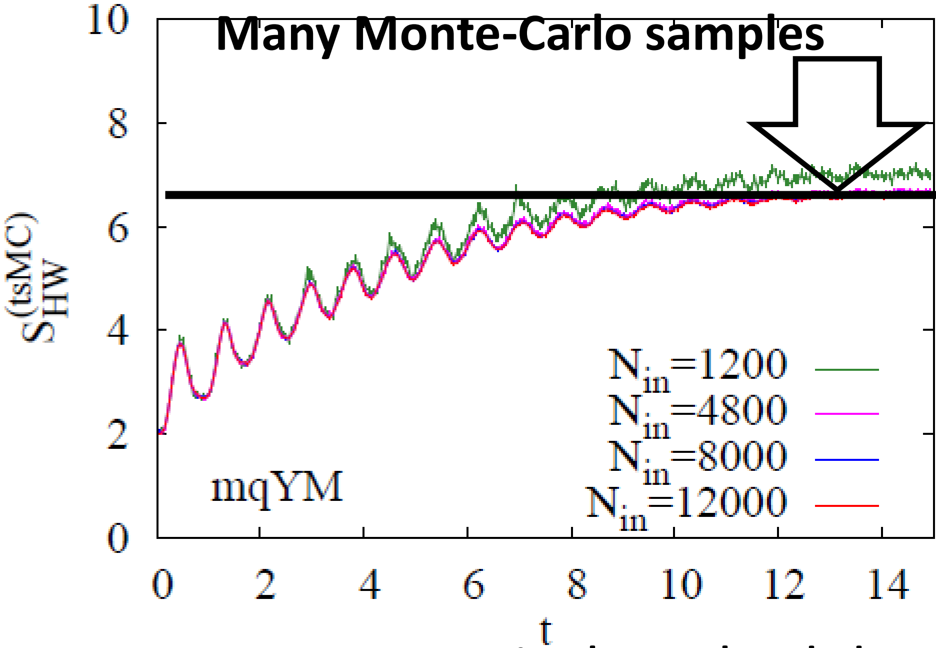
Numerical results are consistent with analytic solutions in large number of samples or test particles.

Simulations in quantum mechanics

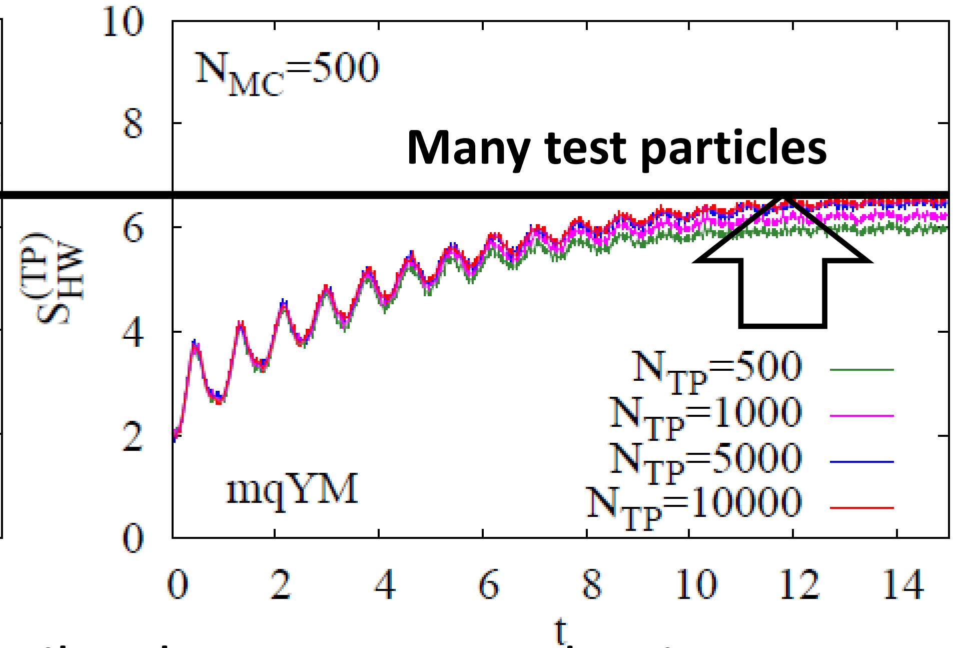
$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

We set $m = 1, g = 1, \epsilon = 1$

Two step Monte-Carlo method



Test particle method

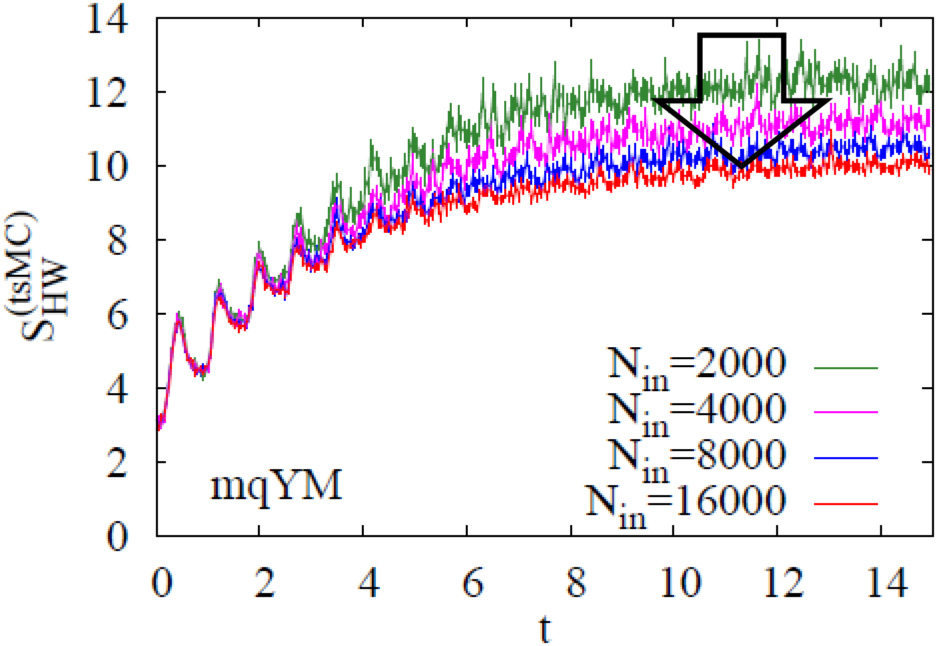


Two numerical method describe the entropy production.
Both results are consistent within error bars.

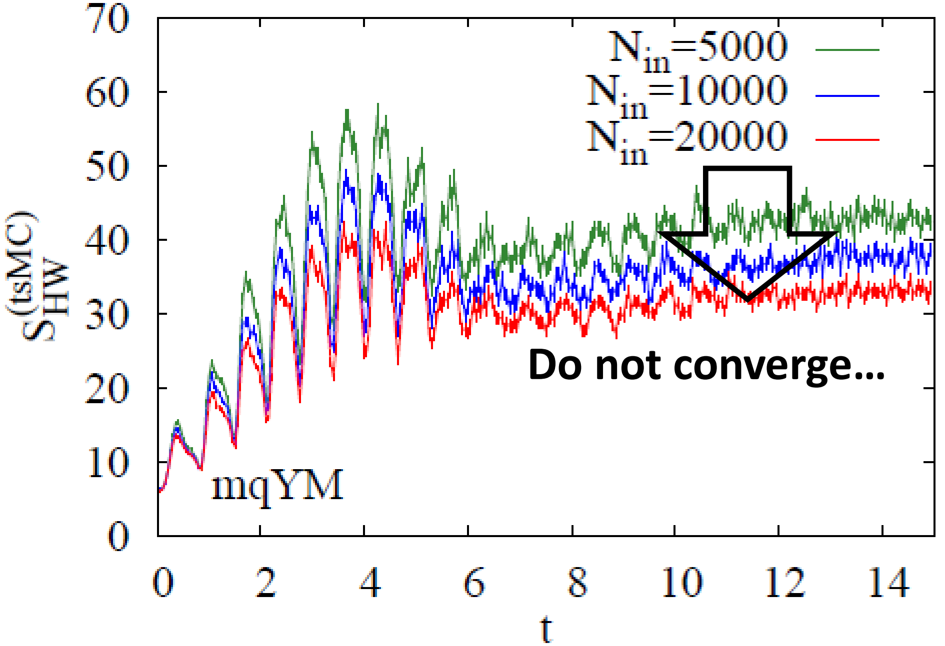
Quantum mechanics in higher dim.

$$H = \sum_i^D \frac{p_i^2}{2m} + \frac{g^2}{2}\sum_{i\neq j}^D q_i^2 q_j^2 + \frac{\epsilon}{4}\sum_i^D q_i^4$$

3-dim case



6-dim case



We need more samples and test-particles in higher dimension.
In a large dimensional case like quantum field theory, we apply some approximations.

Product ansatz

We assume that Husimi function is decomposed into the production of that of 1-dim degree of freedom.

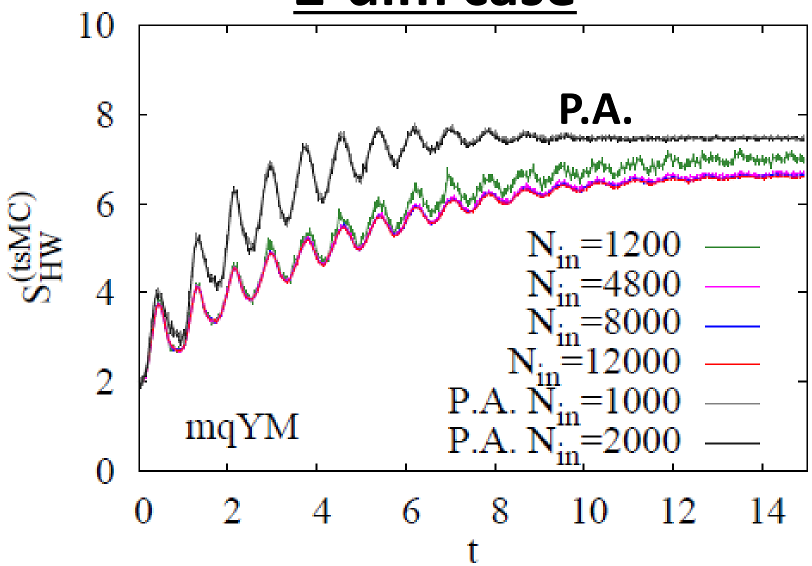
$$f_H(q,p;t) = \prod_i^D h_i(q_i,p_i;t)$$

Then Husimi-Wehrl entropy is written by

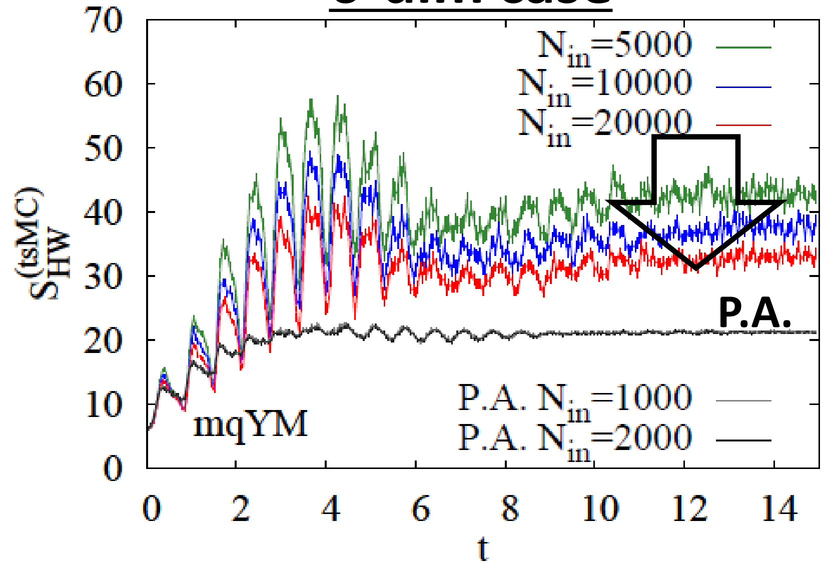
$$S_{HW} \simeq -\sum_i^D \int \frac{dq_i dp_i}{2\pi\hbar} h(q_i,p_i;t) \log h(q_i,p_i;t)$$

Check in the case of quantum mechanical systems.

2-dim case

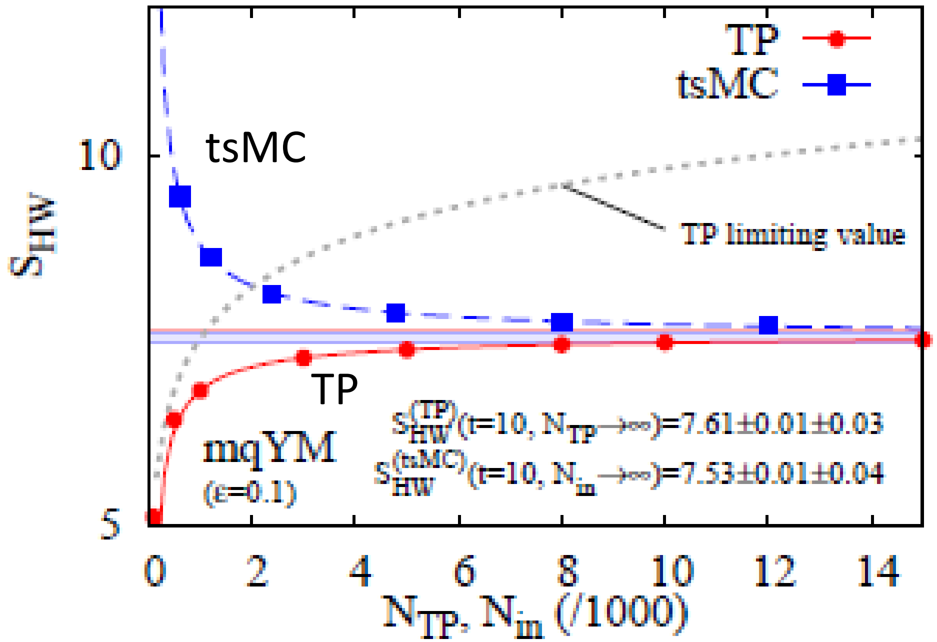


6-dim case



Product ansatz gives consistent results within 10% error bar.
The results of product ansatz converge even in higher dimension.

Large N value at time t=10



Both results are consistent within error bars.