Holographic Entanglement Entropy for a General State in CFT_2

Jie-qiang Wu (Peking University)

Based on [arXiv: 1605.06753]
with Bin Chen

Quantum Information in String Theory and Many-body Systems, YITP

June 9th, 2016

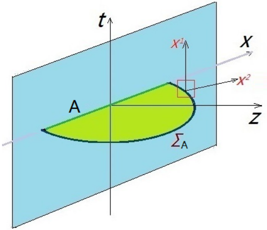
Main Result

- In this work, we study the entanglement entropy for a **general state** in 2d CFT with holographic dual
- In semi-classical limit, the field theory calculation is the same as the Ryu-Takayanagi formula in gravity
- From our calculation, it implies that in three dimensional gravity the **Chern-Simons** formalism has more direct relation with conformal field theory

Holographic entanglement entropy

- For a CFT with holographic correspondence, it is suggested that the entanglement entropy can be evaluated by Ryu-Takayanagi formula S.Ryu T.Takayanagi (2006)

$$S_{HEE}(A) = \frac{\text{Area}(\Sigma_A)}{4G}$$



- The holographic entanglement entropy can be proved by using replica trick in gravity
  A.Lewkowycz, J.Maldacena (2013)

For AdS_3/CFT_2 case

- In AdS_3/CFT_2, both gravity and field theory will be simplified
- In semi-classical limit, the pure gravity will dual to a large central charge CFT with sparse light spectrum Hartman arXiv: 1303.6955
- In gravity side, the holographic entanglement entropy and Rényi entropy can be studied in multi-interval case or torus case for classical and 1-loop order Faulkner arXiv: 1303.7221
  T.Barrella, X.Dong, Hartnoll, Martin 1306.4682
- The result match with field theory calculation
  Hartman arXiv: 1303.6955
  B.Chen J-j.Zhang 1309.5354 B.Chen J-q.Wu 1405.6254

Entanglement entropy in CFT

- In field theory side, by replica trick, the Rényi entropy transform to twist operators' correlation function

$$S_n = -\frac{1}{n-1} \log \langle \mathcal{T}(z_1) \mathcal{T}(z_2) \dots \rangle$$

  and can be decomposed by conformal blocks
- In semi-classical limit, the conformal block can be evaluated by solving a monodromy problem A. B. Zamolodchikov (1988)
- When $n \to 1$, the monodromy problem can be solved explicitly and calculate the multi-interval entanglement entropy for vacuum state Hartman 1303.6955

- The monodromy prescription can also be used to study the entanglement entropy in other systems:
  shock wave background
  P.Caputa, J.Simon, A.Stikonas, T.Takayanagi 1410.2287
  P.Caputa, J.Simon, A.Stikonas, T.Takayanagi, K.Watanabe 1503.08161
  higher spin system
  de Boer, A.Castro, E.Hijano, J.I.Jottar, P.Kraus 1412.7520
  B.Chen, J-q. Wu 1604.03644
  black hole background
  C.Asplund, A.Bernamonti, F.Galli, T.Hartman 1410.1392
  black hole formation problem
  T.Anous, T.Hartman, A.Rovai, J.Sonner 1603.04856
  ...
- A discussion for entanglement entropy in any general state is absent

Monodromy prescription

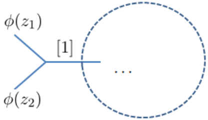
- We will use the monodromy prescription to study the classical order entanglement entropy for a general state
- We only pay attention to single interval case
- The state can be described by stress tensor expectation value

$$T_0(z) = \frac{\langle \hat{T}(z)...\rangle}{\langle ...\rangle}$$

- By replica trick, the entanglement entropy transform to two light operators' correlation function $1 \ll h_\phi \ll c$

$$\langle \phi(z_1)\phi(z_2)...\rangle$$

- We assume the vacuum module states dominate in the correlation function



  - Null state: $|\chi\rangle = (L_{-2} + \frac{c}{6}L_{-1}^2)\,|\hat{\psi}\rangle \quad h_\psi = -\frac{1}{2} - \frac{9}{2c}$
  - Inserting the null state into correlation function

$$\langle \chi(z)\phi(z_1)\phi(z_2)...\rangle = 0$$

$$\psi''(z) + \frac{6}{c}T(z)\psi(z) = 0$$

$$T(z) = \frac{\langle \hat{T}(z)\phi(z_1)\phi(z_2)...\rangle}{\langle \phi(z_1)\phi(z_2)...\rangle};\ \psi(z) = \frac{\langle \hat{\psi}(z)\phi(z_1)\phi(z_2)...\rangle}{\langle \phi(z_1)\phi(z_2)...\rangle}$$

$$T(z) \sim \frac{h}{(z-z_1)^2} + \frac{\gamma_1}{z-z_1} + ...; \quad T(z) \sim \frac{h}{(z-z_2)^2} + \frac{\gamma_2}{z-z_2}$$

  - We need to tune γ_1 γ_2 such that $\psi(z)$ has trivial monodromy around z_1 z_2
  - By wald identity, $\frac{\partial}{\partial z_i}\log\langle\phi(z_1)\phi(z_2)...\rangle = \gamma_i$, we can solve the correlation function

- The second order differential equation can be written in a compact way

$$\partial v(z) = a(z)v(z)$$

$$a(z) = \begin{pmatrix} 0 & -\frac{6}{c}T(z) \\ 1 & 0 \end{pmatrix} \quad v(z) = \begin{pmatrix} -\partial\psi_1(z) & -\partial\psi_2(z) \\ \psi_1(z) & \psi_2(z) \end{pmatrix}$$

- We can take a perturbation beyond the background system

$$\partial v_0(z) = a_0(z)v_0(z)$$

$$a_0(z) = \begin{pmatrix} 0 & -\frac{6}{c}T_0(z) \\ 1 & 0 \end{pmatrix}$$

- Solve the monodromy problem

$$\log\langle\phi(z_1)\phi(z_2)...\rangle = -2h\log \mathrm{tr} v_0(z_1)v_0^{-1}(z_2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $v_0(z_1)v_0(z_2)^{-1} = \mathcal{P}exp\int_{z_1}^{z_2} a_0(z)$

Holographic description

- The state in CFT dual to a metric in gravity

$$ds^2 = d\rho^2 + (e^{2\rho} + e^{-2\rho}\frac{36}{c^2}T_0(z)\bar{T}_0(\bar{z}))dzd\bar{z} - \frac{6}{c}T_0(z)dz^2 - \frac{6}{c}\bar{T}_0(\bar{z})d\bar{z}^2$$

which can also be written by Chern-Simons form

$$g_{\mu\nu} = \frac{1}{2}Tr(A-\bar{A})_\mu(A-\bar{A})_\nu$$

where

$$\begin{aligned} A &= L_0 d\rho + (e^\rho L_1 + \frac{6}{c}T_0(z)e^{-\rho}L_{-1})dz \\ &= e^{-\rho L_0}(a_0(z)+d)e^{\rho L_0} \end{aligned}$$

$$\begin{aligned} \bar{A} &= -L_0 d\rho + (e^\rho L_{-1} + \frac{6}{c}\bar{T}_0(\bar{z})e^{-\rho}L_1)d\bar{z} \\ &= e^{\rho L_0}(\bar{a}_0(\bar{z})+d)e^{-\rho L_0} \end{aligned}$$

- The holographic entanglement entropy can be described by Wilson line in Chern-Simons form

  M. Ammon, A. Castro and N. Iqbal, 1306.4338

  J. de Boer and J.I. Jottar, 1306.4347

$$\langle\phi(z_1)\phi(z_2)...\rangle = \langle i \mid W_{\mathcal{R}}(\mathcal{C}) \mid j\rangle$$

- $W_{\mathcal{R}}(\mathcal{C})$ is in an infinite representation in $SL(2,R)\times SL(2,R)$ group
- The representation can be described by the Hilbert space of one dimensional field theory on the Wilson line

$$S = \int_{\mathcal{C}} ds Tr(PU^{-1}\frac{dU}{ds}) + \lambda(s)(Tr(P^2) - c_2)$$

- Couple the 1d field theory with the gauge potential

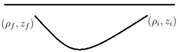
$$S = \int_{\mathcal{C}} ds Tr(PU^{-1}D_s U) + \lambda(s)(Tr(P^2) - c_2)$$

$$\int [dU][dP]e^{-S} = \langle i \mid W_{\mathcal{R}}(\mathcal{C}) \mid j\rangle$$

- In semi-classical limit, the path integral can be evaluated by saddle point approximation $Z \approx e^{-S_{\text{on-shell}}}$
- Choosing the curve as geodesic, the on-shell action equals to the geodesic length
- The action can be calculated by Chern-Simons field as

$$S = Tr(\log diag(\mathcal{P}exp(-\int_f^i A)\mathcal{P}exp(\int_i^f \bar{A}))\sqrt{2c_2}L_0)$$

- When the end point goes to boundary, the holomorphic and anti-holomorphic part decompose



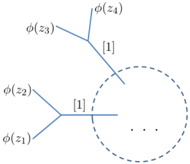
$$\begin{aligned} S &= -2h(2\rho + \log \mathrm{tr} v_0(z_f) v_0(z_i)^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &+ \log \mathrm{tr} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \bar{v}_0(\bar{z}_i) \bar{v}_0^{-1}(\bar{z}_f)) + \mathcal{O}(e^{-2\rho}) \end{aligned}$$

Conclusion

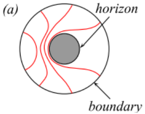
- In the work, we use monodromy prescription to study the entanglement entropy for a **general state** in 2d large c CFT
- The result directly match with holographic calculation by **Wilson line** prescription so also match with Ryu-Takayanagi formula
- In our field theory calculation, the matrix $a_0(z)$ and $\mathcal{P}\exp(\int a_0(z)))$ are related with gauge potential and Wilson line in Chern-Simons form.

Discussion

- Higher spin: the derivation can be extended to higher spin CFT directly
- Multi-interval: the extension to multi-interval case is straightforward



- Global effect: our derivation cannot deal with the global effect



Open question

- Two point function for descendant operators for example $e^{W_{-2}y}\phi(z)$ in higher spin CFT
- Local bulk operator and the relation with HKLL realization
- Back reaction of the Wilson line and relation with shock wave effect
- Couple with matter field
- Higher dimensional case

Thanks for your attention!