Holographic Entanglement Entropy for a General State in $CFT_2$

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Based on [arXiv: 1605.06753 ]
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In this work, we study the entanglement entropy for a **general state** in 2d CFT with holographic dual. In semi-classical limit, the field theory calculation is the same as the Ryu-Takayanagi formula in gravity. From our calculation, it implies that in three dimensional gravity the **Chern-Simons** formalism has more direct relation with conformal field theory.
For a CFT with holographic correspondence, it is suggested that the entanglement entropy can be evaluated by the Ryu-Takayanagi formula \( S_{\text{HEE}}(A) = \frac{\text{Area}(\Sigma_A)}{4G} \) 

The holographic entanglement entropy can be proved by using replica trick in gravity 

A.Lewkowycz, J.Maldacena (2013)
For $AdS_3/CFT_2$ case

- In $AdS_3/CFT_2$, both gravity and field theory will be simplified.
- In semi-classical limit, the pure gravity will dual to a large central charge CFT with sparse light spectrum [Hartman arXiv: 1303.6955]
- In gravity side, the holographic entanglement entropy and Rényi entropy can be studied in multi-interval case or torus case for classical and 1-loop order [Faulkner arXiv: 1303.7221] [T.Barrella, X.Dong, Hartnoll, Martin 1306.4682]
- The result match with field theory calculation
  - [Hartman arXiv: 1303.6955]
In field theory side, by replica trick, the Rényi entropy transform to twist operators’ correlation function

$$S_n = -\frac{1}{n - 1} \log \langle T(z_1) T(z_2) \ldots \rangle$$

and can be decomposed by conformal blocks.

In semi-classical limit, the conformal block can be evaluated by solving a monodromy problem A. B. Zamolodchikov (1988).

When $n \to 1$, the monodromy problem can be solved explicitly and calculate the multi-interval entanglement entropy for vacuum state Hartman 1303.6955.
The monodromy prescription can also be used to study the entanglement entropy in other systems:

- shock wave background
  - P. Caputa, J. Simon, A. Stikonas, T. Takayanagi 1410.2287
  - P. Caputa, J. Simon, A. Stikonas, T. Takayanagi, K. Watanabe 1503.08161

- higher spin system
  - de Boer, A. Castro, E. Hijano, J. I. Jottar, P. Kraus 1412.7520
  - B. Chen, J-q. Wu 1604.03644

- black hole background
  - C. Asplund, A. Bernamonti, F. Galli, T. Hartman 1410.1392

- black hole formation problem

A discussion for entanglement entropy in any general state is absent
We will use the monodromy prescription to study the classical order entanglement entropy for a general state. We only pay attention to single interval case. The state can be described by stress tensor expectation value:

\[ T_0(z) = \frac{\langle \hat{T}(z) \ldots \rangle}{\langle \ldots \rangle} \]

By replica trick, the entanglement entropy transform to two light operators' correlation function \( 1 \ll h_\phi \ll c \):

\[ \langle \phi(z_1)\phi(z_2)\ldots \rangle \]
We assume the vacuum module states dominate in the correlation function

- Null state: \( |\chi\rangle = (L_{-2} + \frac{c}{6} L_{-1}^2) |\hat{\psi}\rangle \quad h_\psi = -\frac{1}{2} - \frac{9}{2c} \)
- Inserting the null state into correlation function

\[
\langle \chi(z_1)\phi(z_1)\phi(z_2)\phi(z_3)\rangle = 0
\]

\[
\psi''(z) + \frac{6}{c} T(z)\psi(z) = 0
\]

\[
T(z) = \frac{\langle \hat{T}(z)\phi(z_1)\phi(z_2)\phi(z_3)\rangle}{\langle \phi(z_1)\phi(z_2)\phi(z_3)\rangle}; \quad \psi(z) = \frac{\langle \hat{\psi}(z)\phi(z_1)\phi(z_2)\phi(z_3)\rangle}{\langle \phi(z_1)\phi(z_2)\phi(z_3)\rangle}
\]

\[
T(z) \sim \frac{h}{(z - z_1)^2} + \frac{\gamma_1}{z - z_1} + \ldots; \quad T(z) \sim \frac{h}{(z - z_2)^2} + \frac{\gamma_2}{z - z_2}
\]

- We need to tune \( \gamma_1 \), \( \gamma_2 \) such that \( \psi(z) \) has trivial monodromy around \( z_1, z_2 \)
- By wald identity, \( \frac{\partial}{\partial z_i} \log \langle \phi(z_1)\phi(z_2)\phi(z_3)\rangle = \gamma_i \), we can solve the correlation function
The second order differential equation can be written in a compact way

\[
\partial v(z) = a(z)v(z)
\]

\[
a(z) = \begin{pmatrix} 0 & -\frac{6}{c} T(z) \\ 1 & 0 \end{pmatrix} \quad v(z) = \begin{pmatrix} -\partial \psi_1(z) & -\partial \psi_2(z) \\ \psi_1(z) & \psi_2(z) \end{pmatrix}
\]

We can take a perturbation beyond the background system

\[
\partial v_0(z) = a_0(z)v_0(z)
\]

\[
a_0(z) = \begin{pmatrix} 0 & -\frac{6}{c} T_0(z) \\ 1 & 0 \end{pmatrix}
\]

Solve the monodromy problem

\[
\log \langle \phi(z_1)\phi(z_2)\ldots \rangle = -2h \log \text{tr} v_0(z_1)v_0^{-1}(z_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

where \( v_0(z_1)v_0(z_2)^{-1} = \mathcal{P} \exp \int_{z_1}^{z_2} a_0(z) \)
Holographic description

- The state in CFT dual to a metric in gravity

\[ ds^2 = d\rho^2 + (e^{2\rho} + e^{-2\rho}) \frac{36}{c^2} T_0(z) \bar{T}_0(\bar{z}) \] \[ dz d\bar{z} - \frac{6}{c} T_0(z) dz^2 - \frac{6}{c} \bar{T}_0(\bar{z}) d\bar{z}^2 \]

which can also be written by Chern-Simons form

\[ g_{\mu\nu} = \frac{1}{2} Tr(A - \bar{A})_\mu (A - \bar{A})_\nu \]

where

\[ A = L_0 d\rho + (e^\rho L_1 + \frac{6}{c} T_0(z) e^{-\rho} L_{-1}) dz \]
\[ = e^{-\rho L_0} \left( a_0(z) + d \right) e^{\rho L_0} \]

\[ \bar{A} = -L_0 d\rho + (e^\rho L_{-1} + \frac{6}{c} \bar{T}_0(\bar{z}) e^{-\rho} L_1) d\bar{z} \]
\[ = e^{\rho L_0} \left( \bar{a}_0(\bar{z}) + d \right) e^{-\rho L_0} \]
The holographic entanglement entropy can be described by Wilson line in Chern-Simons form

M. Ammon, A. Castro and N. Iqbal, 1306.4338
J. de Boer and J.I. Jottar, 1306.4347

\[ \langle \phi(z_1)\phi(z_2)\ldots \rangle = \langle i \mid W_{\mathcal{R}}(C) \mid j \rangle \]

\( W_{\mathcal{R}}(C) \) is in an infinite representation in \( SL(2, R) \times SL(2, R) \) group

The representation can be described by the Hilbert space of one dimensional field theory on the Wilson line

\[ S = \int_{C} ds Tr(PU^{-1} \frac{dU}{ds}) + \lambda(s)(Tr(P^2) - c_2) \]

Couple the 1d field theory with the gauge potential

\[ S = \int_{C} ds Tr(PU^{-1} D_s U) + \lambda(s)(Tr(P^2) - c_2) \]

\( \int [dU][dP] e^{-S} = \langle i \mid W_{\mathcal{R}}(C) \mid j \rangle \)
In semi-classical limit, the path integral can be evaluated by saddle point approximation $Z \approx e^{-S_{\text{on-shell}}}$.

Choosing the curve as geodesic, the on-shell action equals to the geodesic length.

The action can be calculated by Chern-Simons field as

$$S = \text{Tr} \left( \log \text{diag} \left( \mathcal{P} \exp \left( - \int_{f}^{i} A \right) \mathcal{P} \exp \left( \int_{i}^{f} \bar{A} \right) \right) \sqrt{2c_{2}L_{0}} \right)$$

When the end point goes to boundary, the holomorphic and anti-holomorphic part decompose

$$S = -2h(2\rho + \log \text{tr} v_{0}(z_{f})v_{0}(z_{i})^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$+ \log \text{tr} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \bar{v}_{0}(\bar{z}_{i})\bar{v}_{0}^{-1}(\bar{z}_{f}) \right) + \mathcal{O}(e^{-2\rho})$$
In the work, we use monodromy prescription to study the entanglement entropy for a **general state** in 2d large c CFT. The result directly match with holographic calculation by **Wilson line** prescription so also match with Ryu-Takayanagi formula.

In our field theory calculation, the matrix $a_0(z)$ and $\mathcal{P} \exp(\oint a_0(z))$ are related with gauge potential and Wilson line in Chern-Simons form.
Discussion

- Higher spin: the derivation can be extended to higher spin CFT directly

- Multi-interval: the extension to multi-interval case is straightforward

- Global effect: our derivation cannot deal with the global effect
Open question

- Two point function for descendant operators for example $e^{W_{-2y} \phi(z)}$ in higher spin CFT
- Local bulk operator and the relation with HKLL realization
- Back reaction of the Wilson line and relation with shock wave effect
- Couple with matter field
- Higher dimensional case
Thanks for your attention!