Fixed Point Matrix Product States and 1+1D Topological Quantum Field Theory

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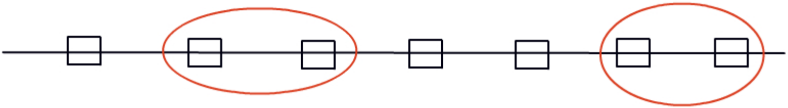
Outline

- Classification of Gapped Phases in 1+1D
- Matrix Product States (MPS), real-space RG, and fixed points
- Lattice TQFT realization of fixed point MPS
- Generalizations: symmetric MPS and fermionic MPS

Gapped Hamiltonians in 1+1D

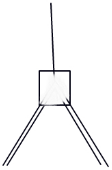
- state space $\mathcal{H} = \bigotimes_s^N \mathcal{A}_{(s)}$, local Hamiltonian $H = \sum_s h_{s,s+1}$

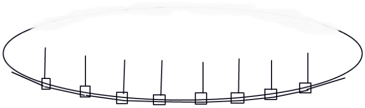


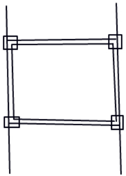
- gapped: energy gap persists in thermodynamic limit
- exhaustive example: local commuting projectors (LCP)

Matrix Product States

- physical space $\mathcal{A} \simeq \mathbb{C}^d$, virtual space $V \simeq \mathbb{C}^D$
- MPS tensor $\mathcal{P} : V \otimes V^* \to \mathcal{A}$, adjoint $T = \mathcal{P}^\dagger$
- defines state $|\mathcal{P}\rangle \in \mathcal{A}^N$ and LCP Hamiltonian $h_{\mathcal{P}} |\mathcal{P}\rangle = 0$





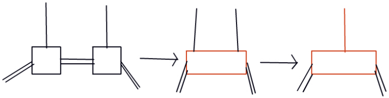


$$\left|\psi_T^X\right\rangle = \sum_{i_1 \cdots i_N = 1}^{d} \mathrm{Tr}[X T(e_{i_1}) \cdots T(e_{i_N})] \, |i_1 \cdots i_N\rangle$$

- MPS efficiently approximate ground states of gapped local Hamiltonians [Hastings 07]

Real-space RG for MPS

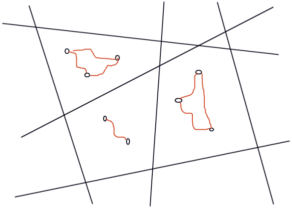
- $\mathcal{P}' := (TT)^\dagger : V \otimes V^* \to \mathcal{A} \otimes \mathcal{A}$
- truncate to dim $\mathrm{im}\mathcal{P} \leq D^2$, RG converges



- Fixed point: $\mathcal{A}$ inherits an algebra structure (of operator products), V is a module over $\mathcal{A}$.
- Canonical form [Perez-Garcia et al 06]: the $T(e_i)$ can be simultaneously block-diagonalized $\longrightarrow$ $\mathcal{A}$ is semisimple
- GSD is # of distinct blocks of T = number of simple components of $\mathcal{A}$ = dimension of $Z(\mathcal{A})$

Gapped Phases in 1+1D

- A Gapped Phase is an equivalence class of gapped systems. Two systems lie in the same phase if they can be smoothly connected without closing the gap:

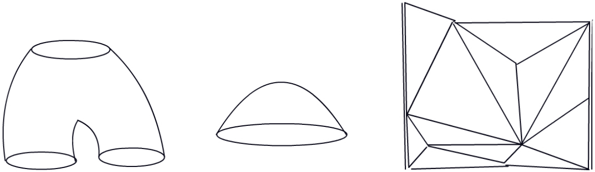


$$(\mathcal{A}_0, H_0) \sim (\mathcal{A}_1, H_1)$$
$$\exists H'_{t\in[0,1]}$$
$$\mathcal{A}' \simeq \mathcal{A}_0 \oplus \mathbb{C}^{n_0} \simeq \mathcal{A}_1 \oplus \mathbb{C}^{n_1}$$
$$H'_{t=i} = H_i \oplus \mathbb{1}_{n_i} \text{ for } i = 0, 1$$

- 1+1D gapped phases are boring (no top. order / LRE)! They are characterized entirely by their GSD [Schuch et al 10]
- However, adding symmetry, we get SPTs. Adding fermions, we get fermionic (invertible) topological order.

Topological Quantum Field Theory

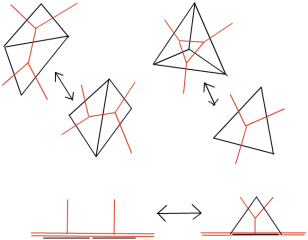
- Lagrangian description of fixed points of gapped systems
- assigns state spaces to circles, linear maps to bordisms
- alg. description: commutative Frobenius algebra [Atiyah 89]



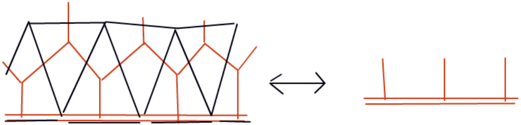
- lattice TQFT: combinatorial, local [Fukuma et al 94]
- label edges by $e_i \in \mathcal{A}$, sum over labelings with weight $\prod_\Delta C_{ijk}$
- symmetric Frobenius + unitarity = semisimple algebra
- lattice TQFT with $\mathcal{A} \longrightarrow$ continuum TQFT with $Z(\mathcal{A})$

Tensor Networks from TQFTs

- LTQFT on triangulation $\longrightarrow$ tensor network on dual complex
- tensors $C^k{}_{ij}$ on interior vertices, $T^{\mu}{}_{\nu i}$ on boundary
- topological invariance $\Rightarrow$ tensor network is a fixed point

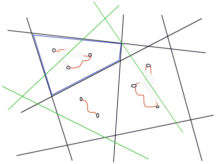


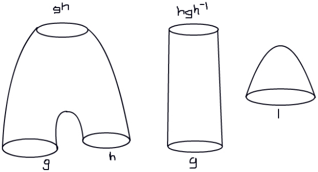
- cylinder with boundary condition T (a module) on one end has MPS $|T\rangle$ on the other



Generalization: Symmetric MPS from Equivariant TQFT

- H_t sym. $\rightarrow$ gapped sym. phases (SPTs, SETs) [Gu, Wen 09]
- response to b.g. gauge field $\rightarrow$ equivariant TQFT





- equivariant state-sum / tensor network: group labels on edges
- equivariant state-sum on cylinder $\rightarrow$ symmetric MPS
- twisted sectors: $\sum \mathrm{tr}\left[\rho(g)_V T^{i_1} \cdots T^{i_N}\right] |i_1 \cdots i_N\rangle$
- charges (broken sym): $\rho_A(g)^N \left|\psi_X^T\right\rangle = \left|\psi^T_{\rho X \rho^{-1}}\right\rangle$
- also works for anti-unitary symmetries such as time-reversal

Generalization: Fermionic MPS from Spin TQFT

- spin TQFT: sensitive to spin structure η and topology
- on the lattice, model η as a certain non-flat 1-chain
- define a state-sum with Koszul signs [Gaiotto, Kapustin 15]
  - $Z(X,\eta) \sim \sum \prod_{\Delta} C_{ijk} g^{lm} \sigma(\beta) s(\beta,\eta)$
  - supercohomology [Gu, Wen 12] + correction due to η
- on cylinder, get fermionic MPS

$$\sum \mathrm{Tr}[T \cdots T \sigma(\beta) s(\beta,\eta)] \, |\cdots\rangle = \mathrm{superTr}[T \cdots T] \, |\cdots\rangle$$

- extension: supergroup symmetries, supersymmetries?

Classification

- standard TQFT with boundaries
  - determined by sym. Frob. algebra $\mathcal{A}$ and its modules
  - can recover $\mathcal{A}$ from its category of modules
  - TQFT $\simeq$ Morita class of s.s. algebras $\simeq$ center
- TQFTs with additional structure: equivariance, spin, etc
  - determined by "algebra objects" in some sym. mon. category
  - Morita classes of algebra objects gives TQFTs
- fixed point MPS (phases) have the same classification
- invertible phases $=$ invertible Morita classes
  - $\mathrm{Rep}(G)$ recovers group cohomology classification of SPTs
  - sRep(G, ϵ) *should* recover spin cobordism, fermionic SPTs
- Bosonization sRep$(G, \epsilon) \to \mathrm{Rep}(G)$ relates the symmetric bosonic and fermionic constructions and classifications