Linearity of Holographic Entanglement Entropy

1606.xxxxx [AA, X. dong, B. Swingle]

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Where does this come up?

- Thermodynamics of Black Holes [Bekenstein; Hawking..]
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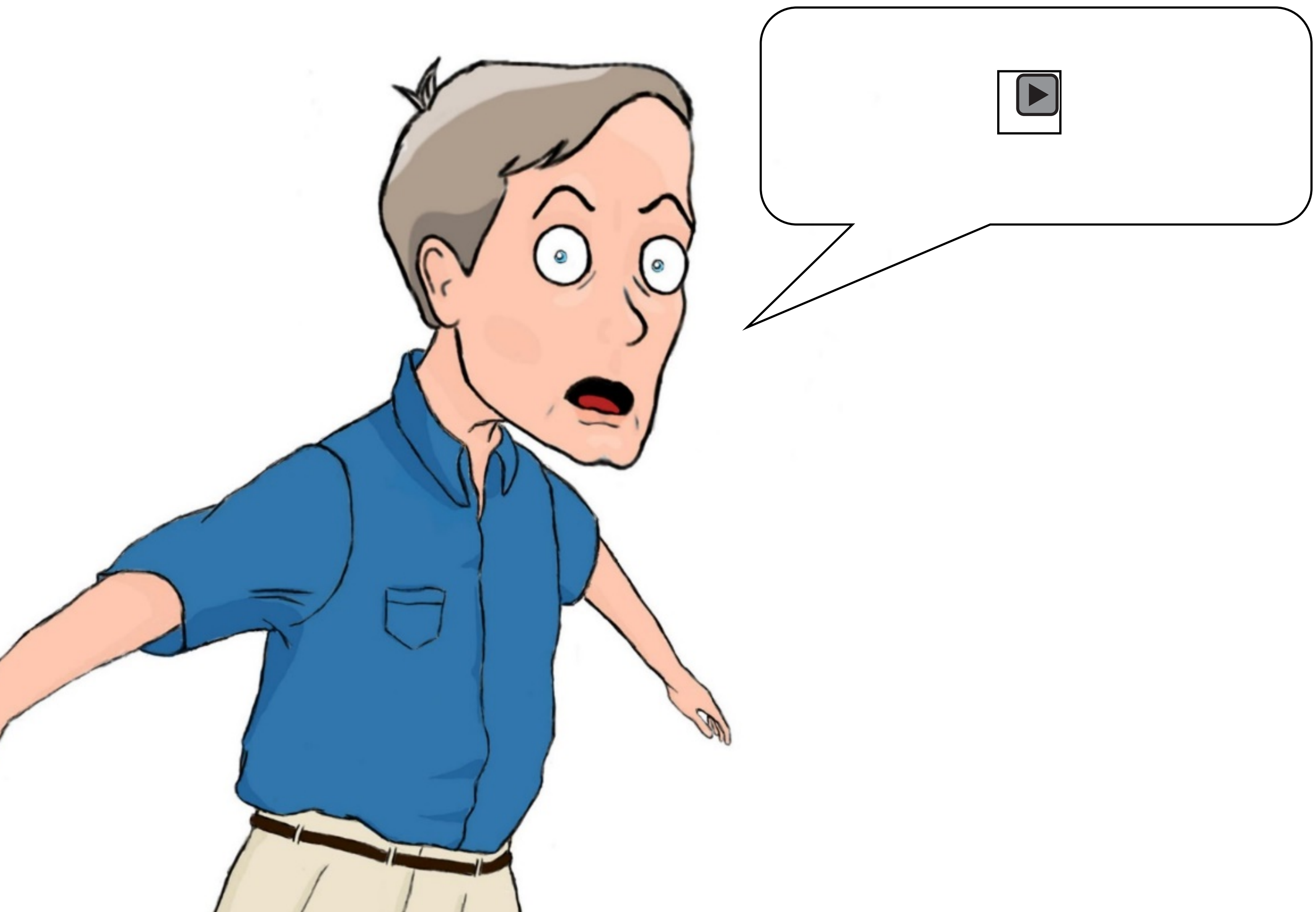
This is NOT fine: This is a microscopic measure of entanglement




Operator equals entropy… You Ryu-Takayanagi people have to fix that!

What do we have to fix?

- Entanglement Entropy (EE):

$$S_R\left(|\psi\rangle\right) = -Tr_R \rho_R \ln \rho_R$$

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- But it CAN be written as the expectation value of a linear operator!
- However, this operator gives the correct result only within this specific state.

Let’s demonstrate this with an example…

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- Since the entropy in ANY product state is zero, it is easy to convince yourself that it must be the <u>zero operator</u>.
- But then, what about entangled states: $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$

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- Does this mean that the area operator is a <u>'nonlinear' and 'state dependent'</u> operator? But what about the intuition from canonical quantum gravity?
- Is this perhaps a new insight into quantum gravity?!!

No…

I will

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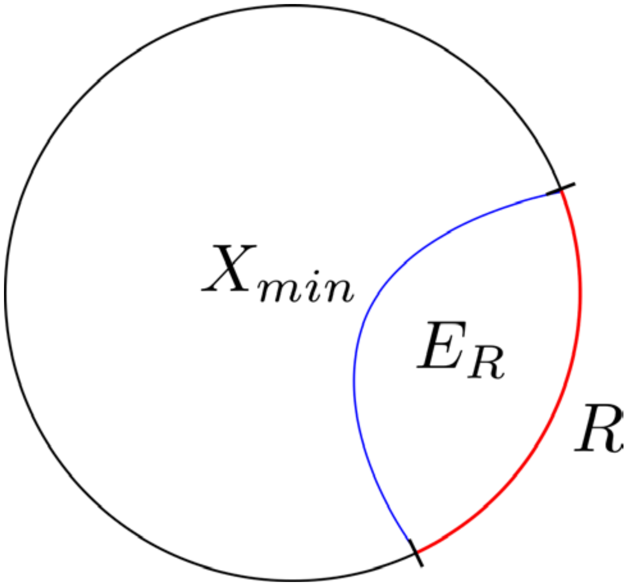
The goal would be to compare the results on both sides of the duality. I will focus on AdS_3/CFT_2

On the Bulk: Motivate how I expect the area operator to behave.

On the Boundary: Rely heavily on 1+1 holographic CFT techniques to compute the EE using the replica trick.

The RT Proposal

[Ryu, Takayanagi]



- Ryu-Takayanagi: EE of subregion R is given by the area of the minimal area surface anchored and homologous to R

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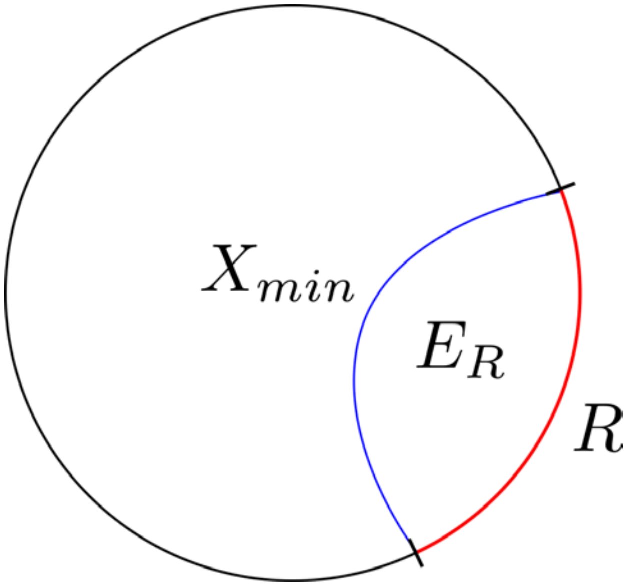

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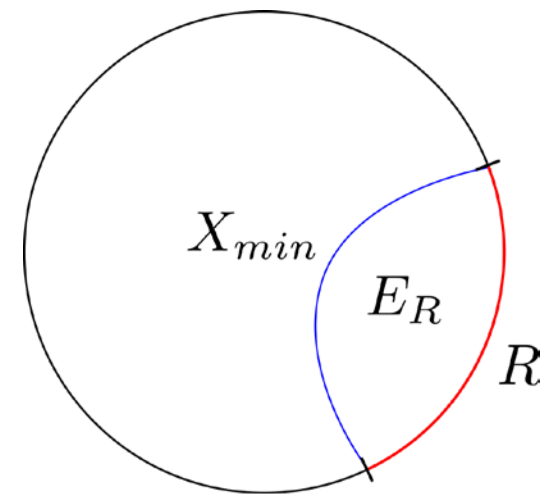
- The question: How do the Entropy and Area behave under superpositions of geometries?

The Area Operator of RT

$$\hat{A}(X_{min})$$

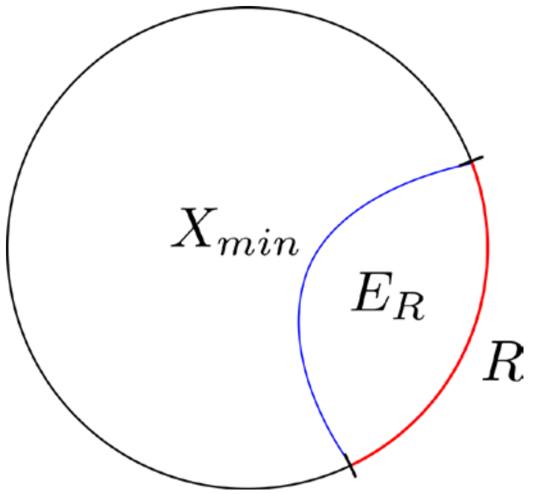
The Area Operator of RT is…



The Area Operator of RT is…

1. Gauge Invariant

- Guaranteed by the minimality condition



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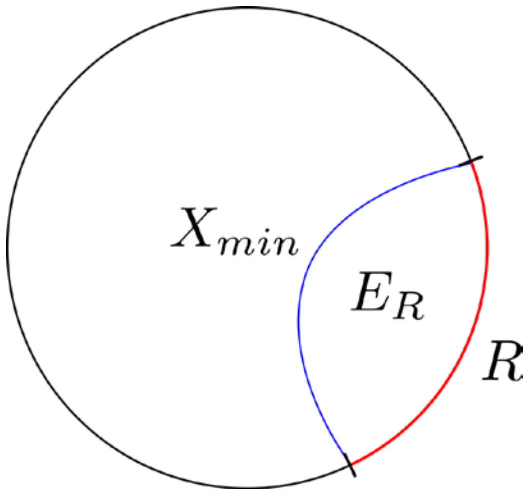

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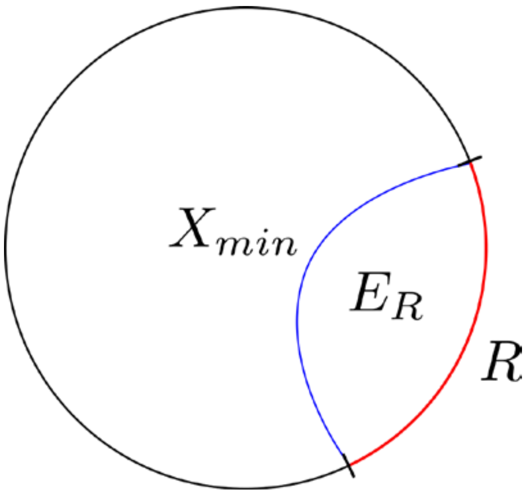

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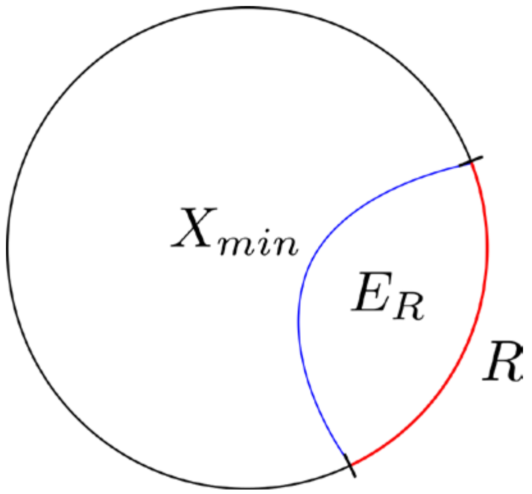
4. Fairly Diagonal in a Semi-Classical Basis
   - Fluctuations in semi-classical states vanish as $G_N \sim \frac{1}{c} \to 0$



$$\hat{A}_{ij} = A_i \delta_{ij} + \frac{1}{c^{\#}} e^{S\left(\frac{E_i+E_j}{2}\right)} R_{ij}$$

The Area Operator of RT
A Prediction

- The takeaway:

  Since the off-diagonal terms are suppressed, the expectation value of the area operator in a superposition will simply be the average of the area in each branch of the wavefunction.

- Extending the RT formula, we make the prediction:

$$\lim_{c \to \infty} \frac{S_R\Big(\sum_i \alpha_i |\psi_i\rangle\Big) - \sum_i |\alpha_i|^2 S_R\Big(|\psi_i\rangle\Big)}{c} = 0$$

EE of a Superposition

EE of a Superposition

- Now let’s discuss what the EE looks like for a superposition of macroscopically distinct semi-classical states. We want to compare this to how the area operator behaves.

- We will consider two cases:
  - Superpositions of TFD’s of different temperature.
  - Superpositions of single sided pure states.

EE of a Superposition
TFDs of Different Temperature

- Let's consider the state $|\Psi\rangle = \sum_{i=1}^{M} \alpha_i |\beta_i\rangle$

  where $|\beta_i\rangle = \frac{1}{\sqrt{Z}} \sum_{E} e^{\frac{-\beta_i E}{2}} |E\rangle_L |E\rangle_R$

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[Lindem, Popescu, Smolin]

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- Each $S_i \sim \mathcal{O}(c)$. The magnitude of the second term $\sim \ln M$
- So long as $M \ll e^c$ we have

$$S_R = \sum_{i=1}^{M} |\alpha_i|^2 S_i + \ldots$$

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Approximations made are not valid when $M \sim e^c$.

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- Averaging breaks down once VI Block dominance fails. This occurs when $M \sim e^c$

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- In the TFD case we found this 'entropy of mixing' contribution. This is a new contribution to RT.
- The entropy of mixing piece is truly a nonlinearity: One can never end up with logarithms of amplitudes when taking expectation values of linear operators.
- RT seems very resilient! I hoped to find something wrong with the Area piece without resorting to e^c states!

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$$S(\mathcal{I}, |\Psi\rangle) = \langle\Psi|\hat{\mathcal{A}}_{\mathcal{I}}|\Psi\rangle$$

  and consider such a state defined on a product Hilbert space of two CFTs

$$|\beta\rangle = \frac{1}{\sqrt{Z}} \sum_E e^{\frac{-\beta E}{2}} |E\rangle_L |E\rangle_R$$

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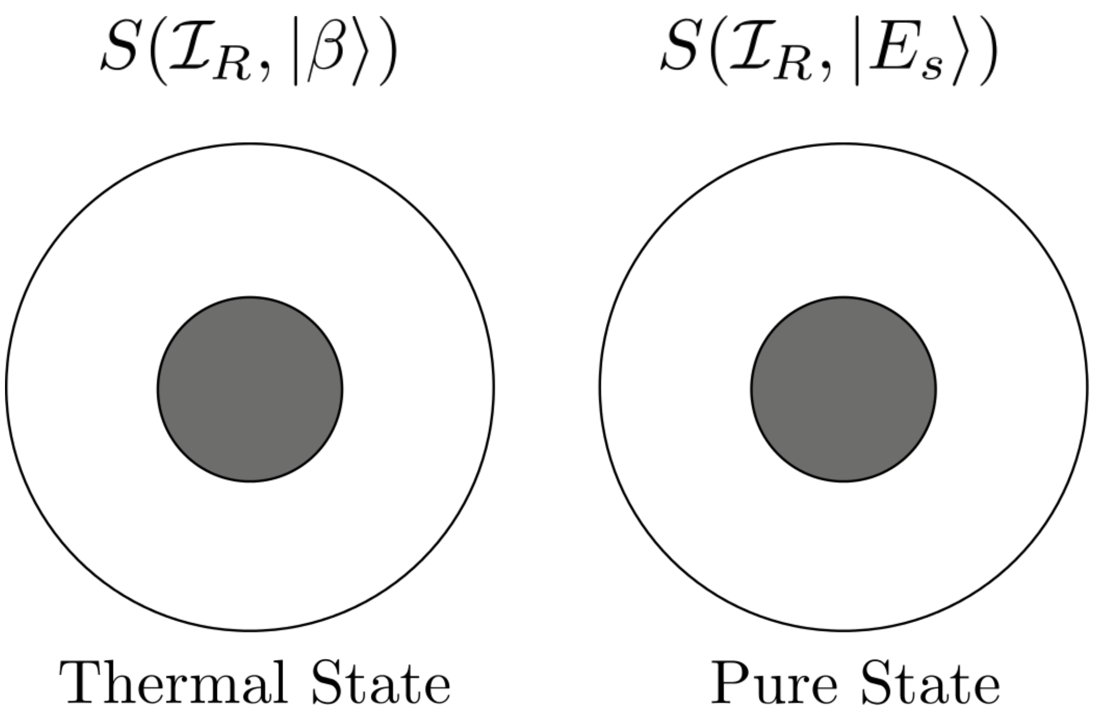

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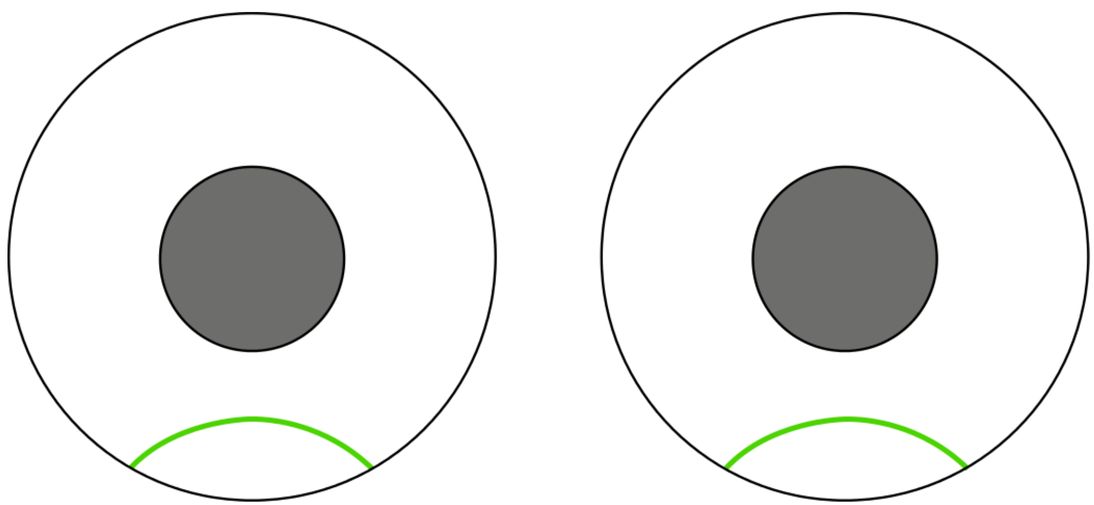

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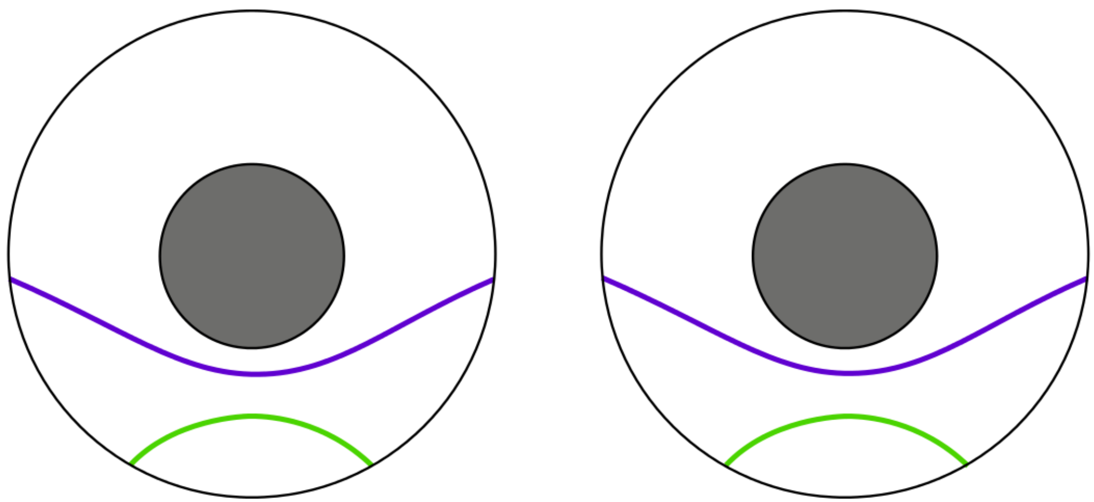

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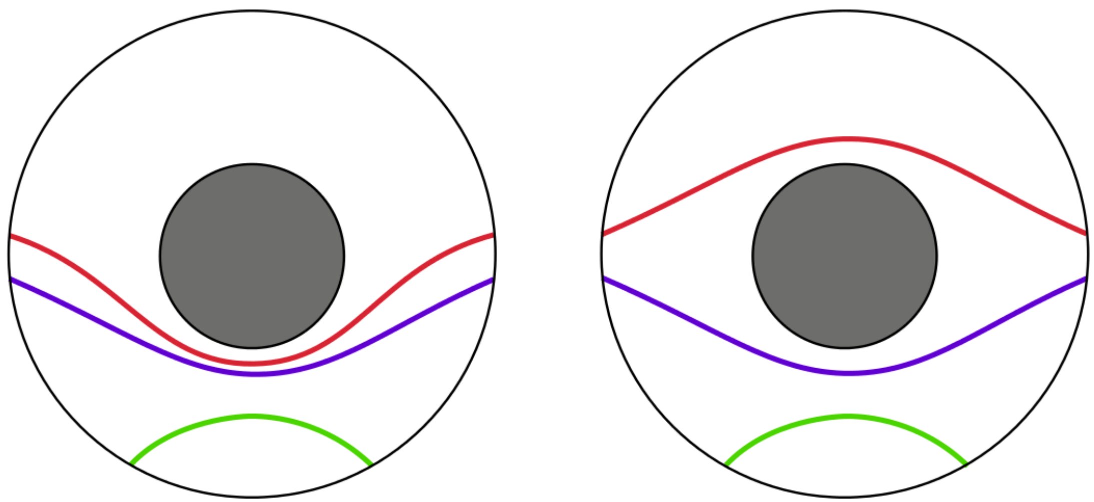

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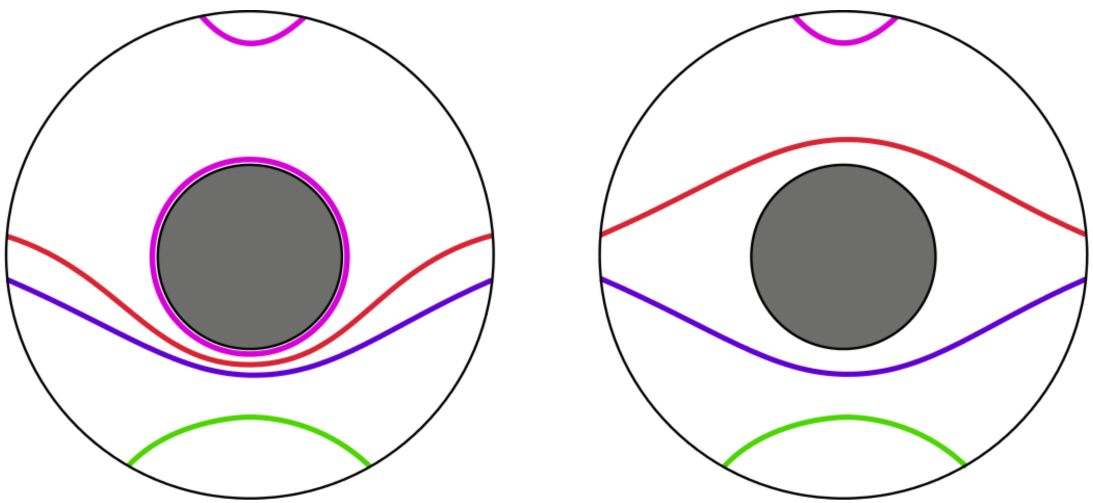

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HOMOLOGY

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$$S(|\psi\rangle) = \langle\psi|\hat{A}|\psi\rangle \ \& \ \text{Homology}$$

- This makes sense: the homology constraint is sensitive to whether the two CFTs are connected via a wormhole. This notion we know is not a linear observable.

Lessons..

- That EE in holographic theories behaves like a linear operator within subspaces of dimension $\ll e^c$.
- Maybe this is a general lesson that ‘State-dependent’ quantities can behave like linear operators within certain subspaces. Is there a Complexity Operator?
- The Homology prescription is not linear in the CFT.
- Some open questions:
  - Single Sided Entropy of Mixing?
  - FLM corrections?
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Homology

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- To study this further, consider the following approximate form of the TFD:

$$|\widetilde{TFD}\rangle = \frac{1}{Z(\beta)} \sum_{i=1}^{e^{2\pi\sqrt{cE_s/3}}} e^{\frac{-\beta E_s}{2}} |E_i\rangle_L |E_i\rangle_R$$

$$= e^{-\frac{\pi^2 c}{3\beta}} \sum_{i=1}^{e^{2\pi^2 c/3\beta}} \mathcal{O}_i^L \otimes \mathcal{O}_i^R |0\rangle_L |0\rangle_R$$

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Actually, let's restrict the number of terms, $M \leq e^{2\pi^2 c/3\beta}$

$$|\mathrm{Mixed}\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} \mathcal{O}_i^L \otimes \mathcal{O}_i^R |0\rangle_L |0\rangle_R$$

and compute the EE of an interval on the right CFT.

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- The entropy of an interval

$$S_{\text{Mixed}} = \text{Min}\left(\frac{c}{3}\ln\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{l\pi}{\beta}\right)\right]\ ;\ \ln M + \frac{c}{3}\ln\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{(2\pi - l)\pi}{\beta}\right)\right]\ \right)$$

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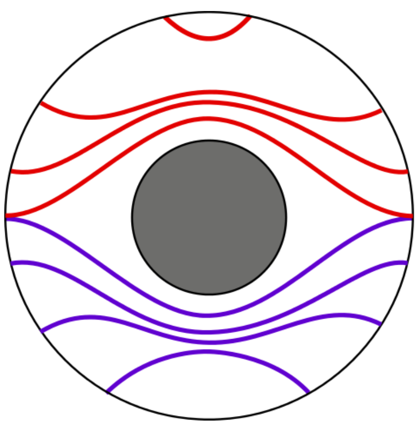

$\ln M \ll \mathcal{O}(c)$

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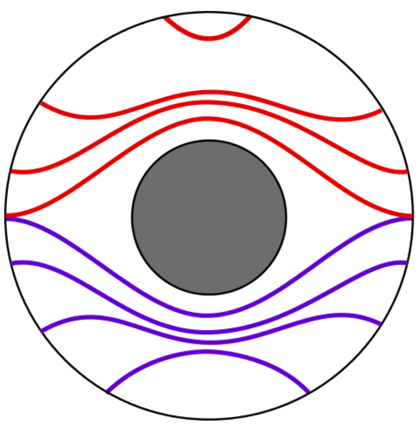

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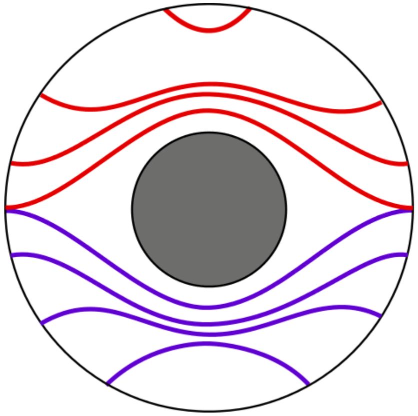

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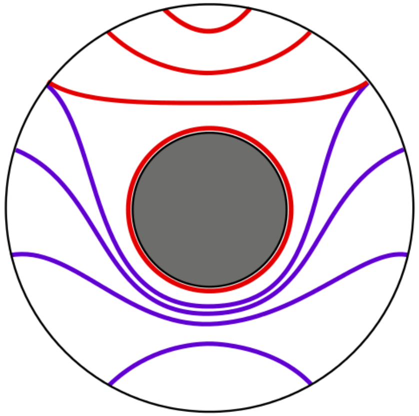

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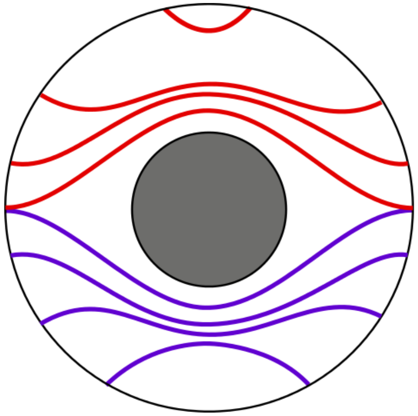

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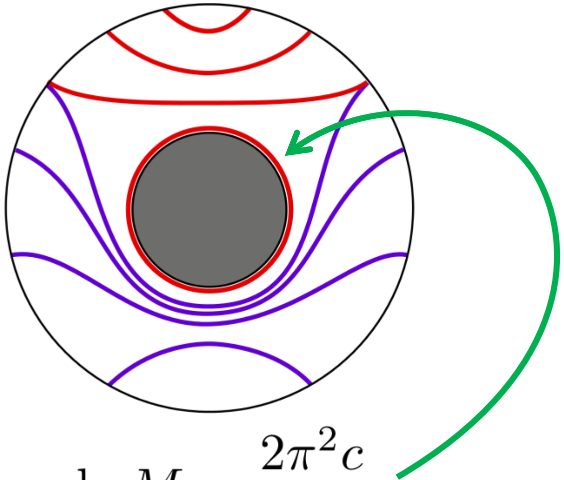

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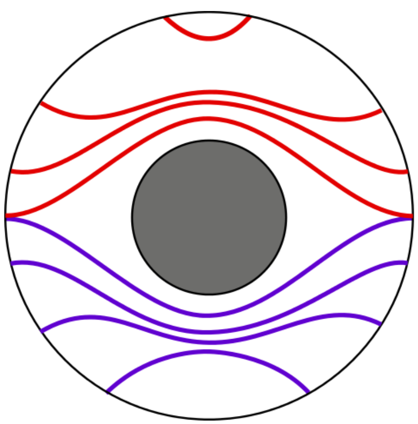

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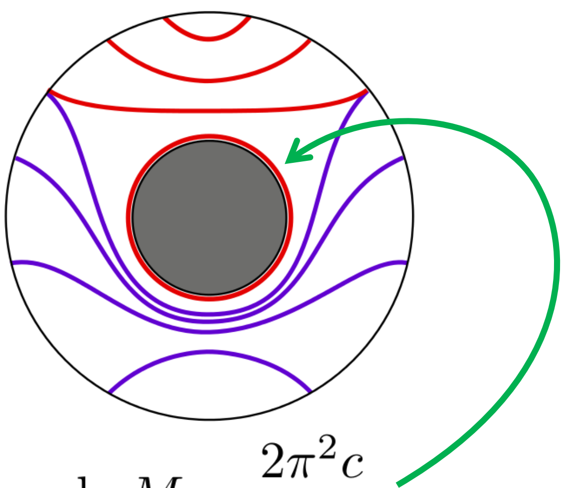

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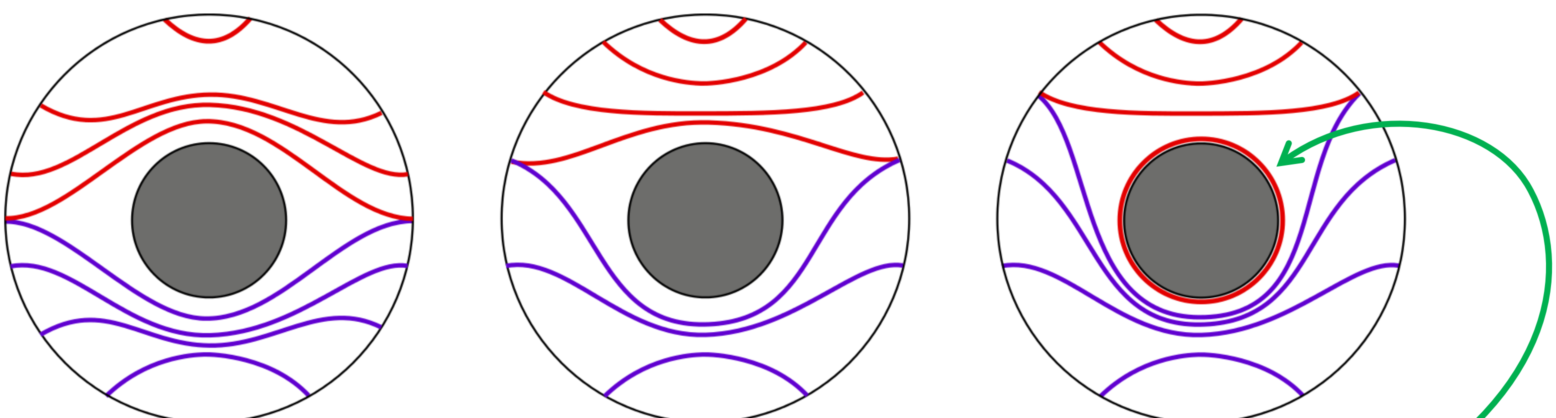

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$\ln M = \dfrac{2\pi^2 c}{3\beta'}$

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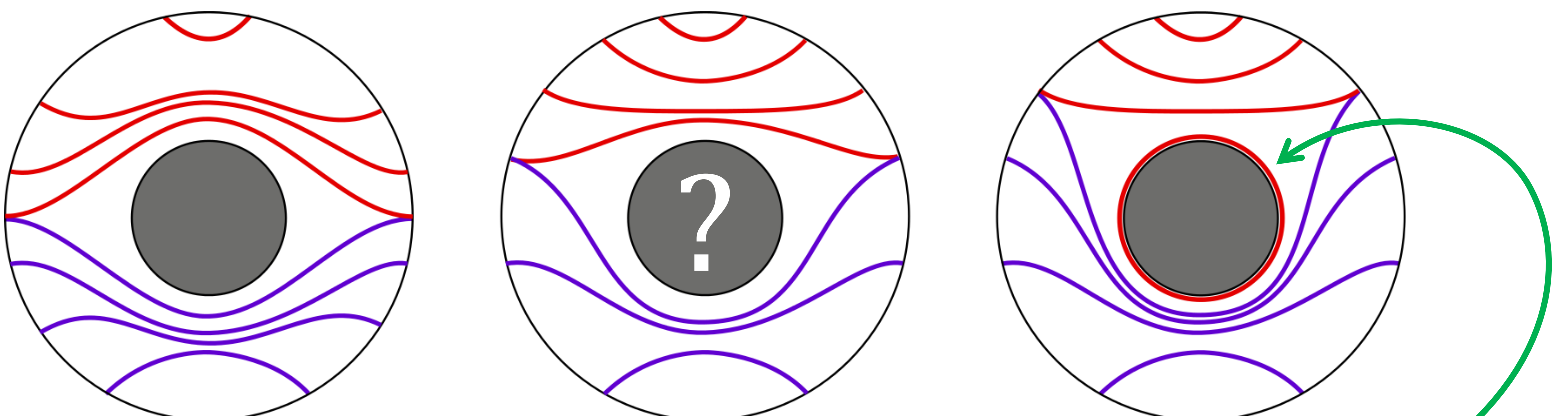

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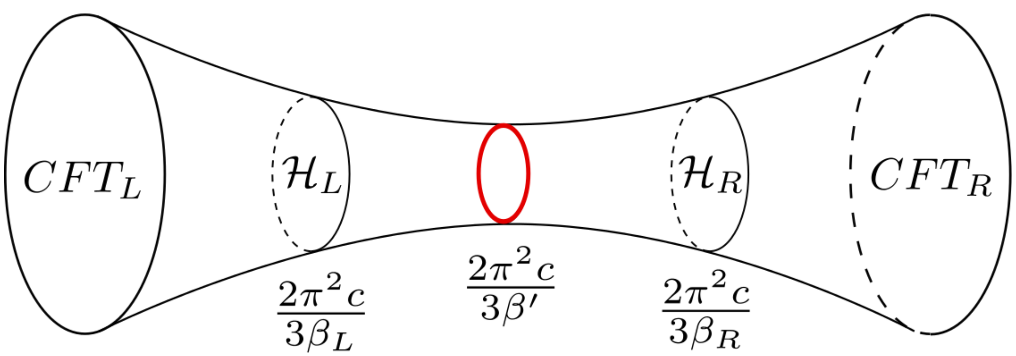
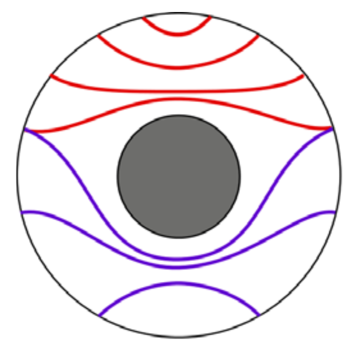
Homology

- Think Shenker-Standford wormholes!
- The RT surface is hidden in the causal shadow.





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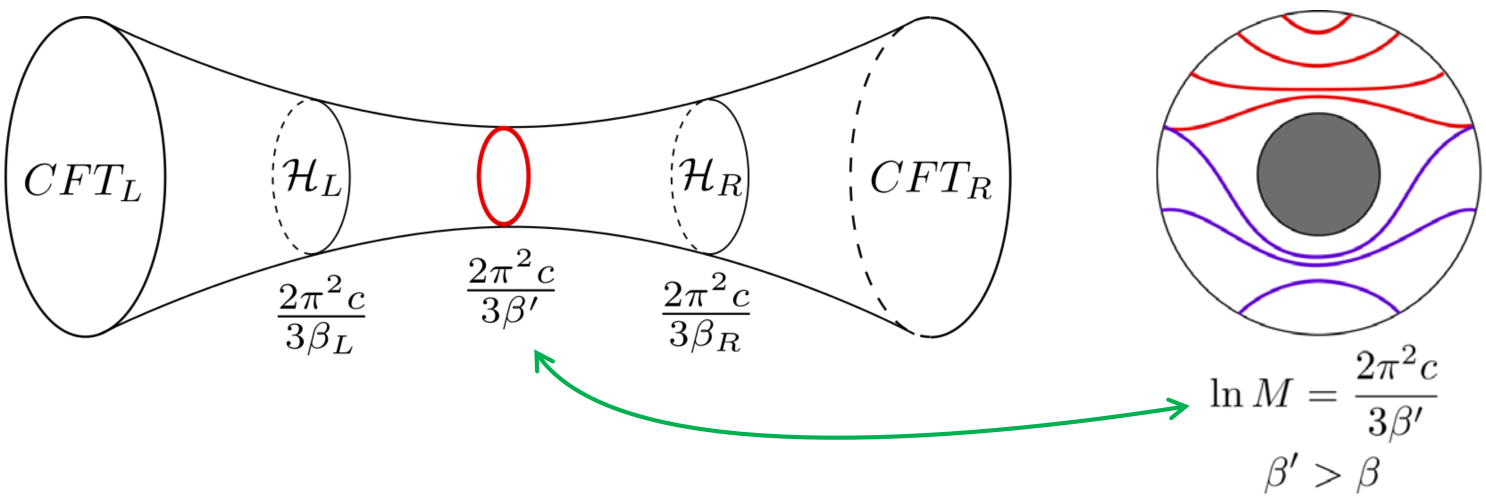

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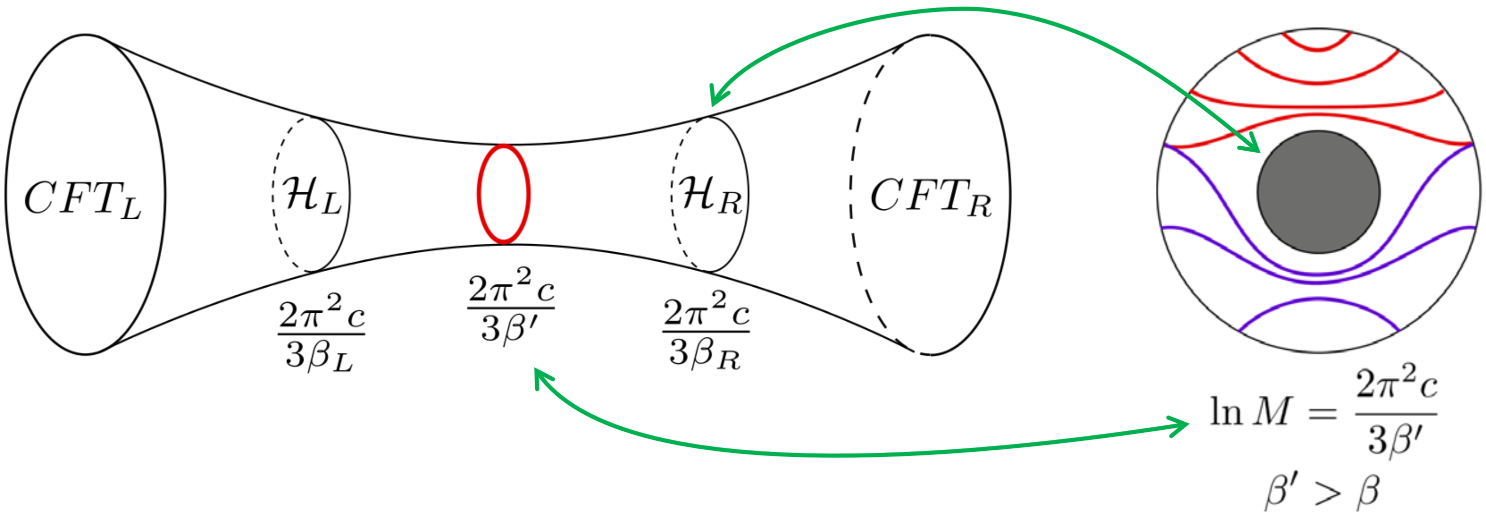

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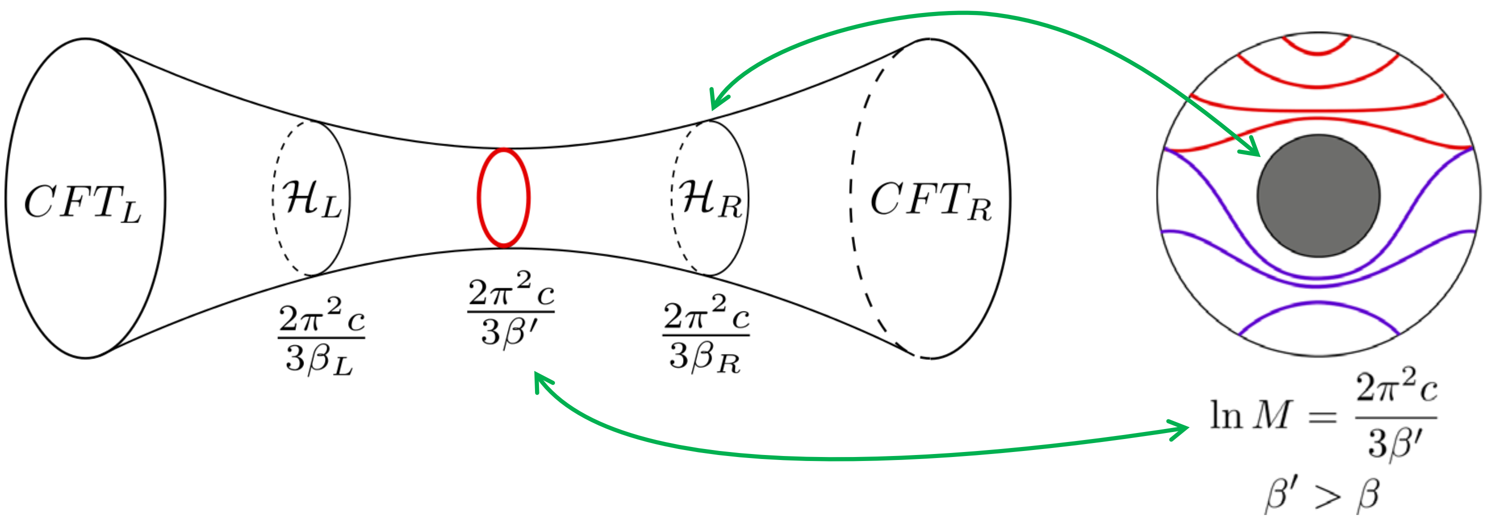
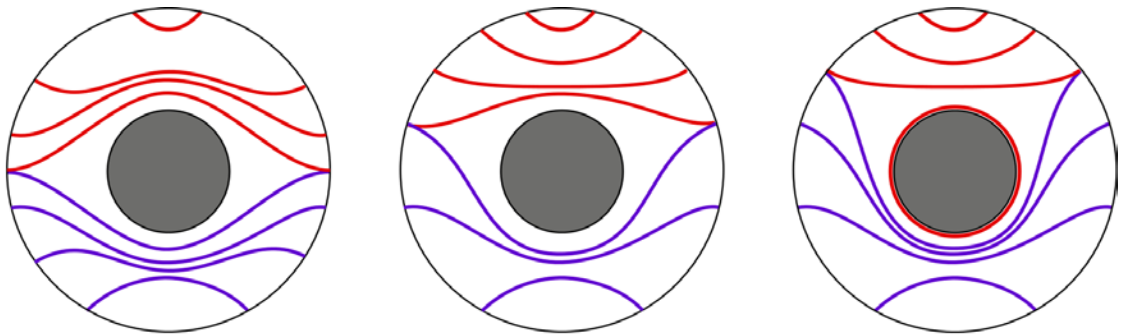


- This is just an analogy. We're not sure exactly what the dual looks like.

Homology



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- <u>Upshot</u>: A different Area operator needs to be used for different $\ln M$. Each operator is linear, but the prescription of choosing the 'right' operator is not.

Lessons..

- That EE in holographic theories behaves like a linear operator within subspaces of dimension $\ll e^{c}$.
- Maybe this is a general lesson that ‘State-dependent’ quantities can behave like linear operators within certain subspaces. Is there a Complexity Operator?
- The Homology prescription is not linear in the CFT.
- Some open questions:
  - Single Sided Entropy of Mixing?
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