quantum information metrics in QFT

Cédric Bény

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distinguishability between

$$\rho = \frac{1}{Z}\, e^{-\beta H}$$

$$\rho_\lambda = \frac{1}{Z_\lambda}\, e^{-\beta H + \lambda V}$$

result depends on cutoff

cutoff

vs

resolution

QFT

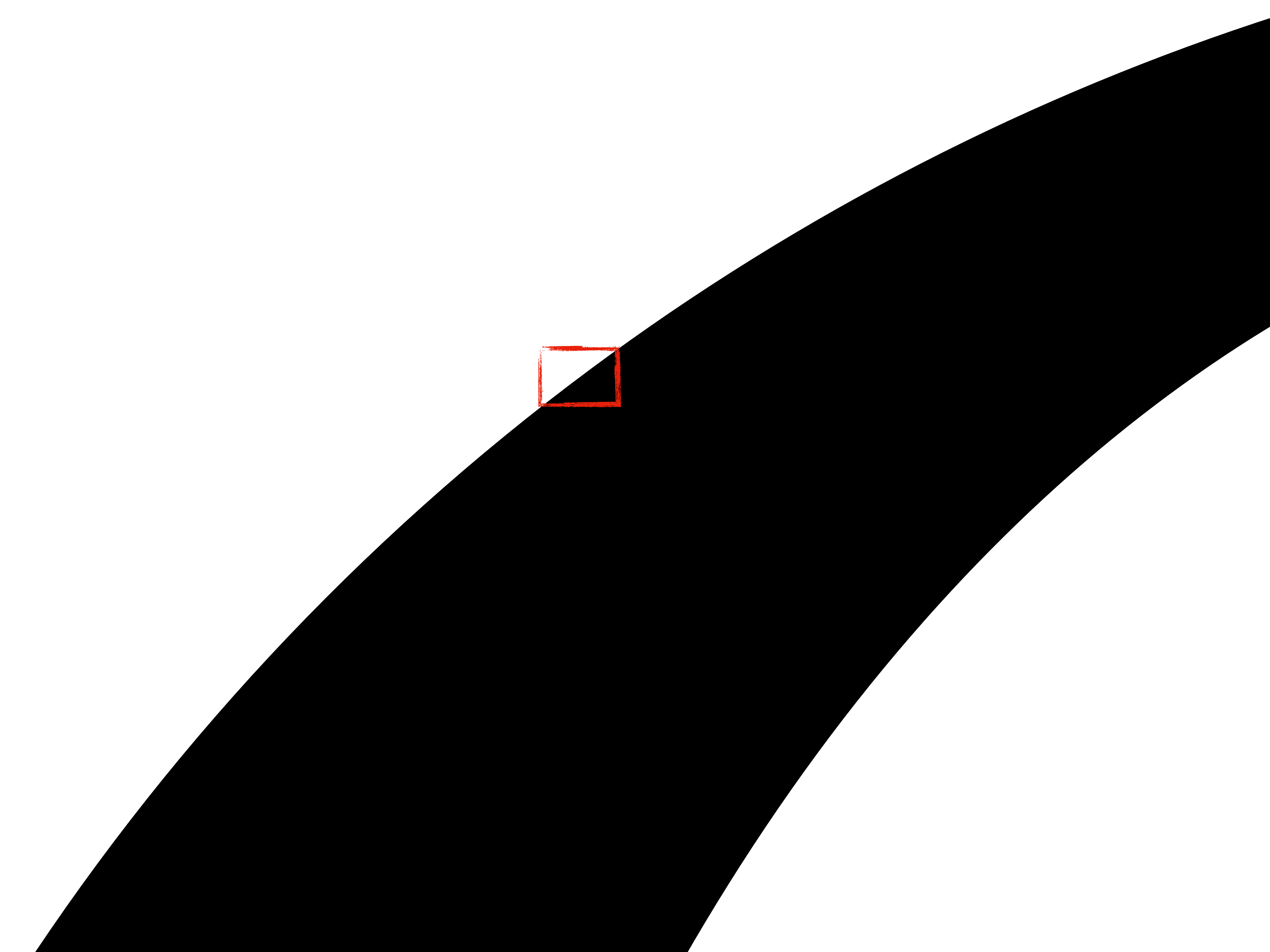
local quantum theory in a continuous space

QFT

local quantum theory in a continuous space

QF

QF

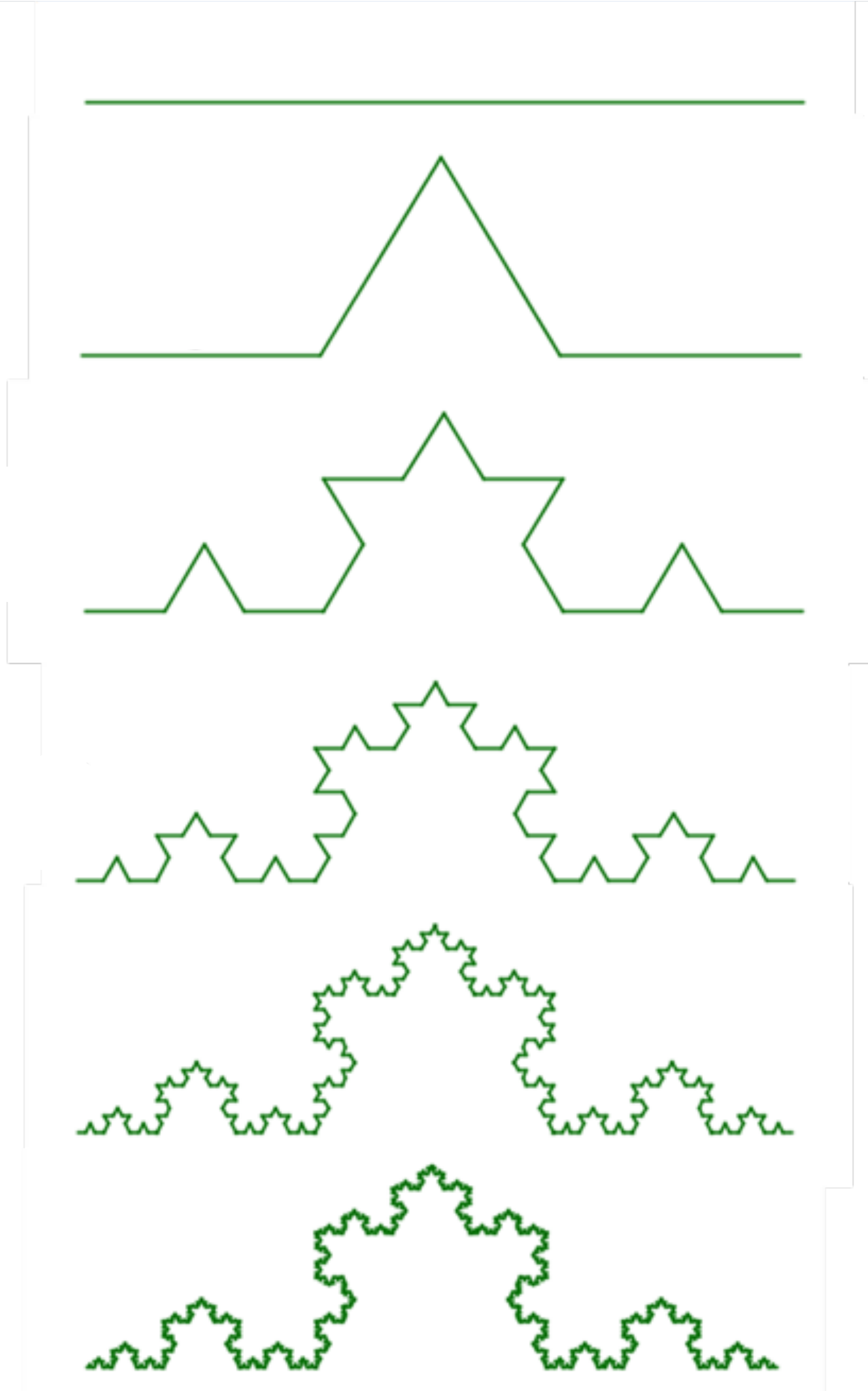


$$\partial_x^2 \phi(x) + m^2 \phi(x) = 0$$

finite resolution

arbitrary

finite resolution



$$\pi = \left\{3, \frac{31}{10}, \frac{314}{100}, \ldots\right\}$$

$$x_1, x_2, \ldots \in \mathbb{Q} \quad \text{is Cauchy if}$$

$$\forall\, \epsilon > 0 \quad \exists\, N \ \text{ such that } \ \forall\, i, j > N, \ \|x_i - x_j\| < \epsilon$$

$\Lambda \mapsto \psi_\Lambda$ is "Cauchy" if

$$\forall\, \sigma, \epsilon \quad \exists\, \Lambda_0 \text{ such that } \forall\, \Lambda, \Lambda' > \Lambda_0,\ d_\sigma(\psi_\Lambda, \psi_{\Lambda'}) < \epsilon$$

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mathematics

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mathematics

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information

$$d_\sigma(\psi_\Lambda, \psi_{\Lambda'})$$

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distinguishability

$$\rho_1 \otimes \cdots \otimes \rho_1 \qquad \text{or} \qquad \rho_2 \otimes \cdots \otimes \rho_2$$

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$$\text{minimize} \quad \begin{cases} p_1 = \mathrm{Tr}(P\,\rho_2^{\otimes N}) \\ p_2 = 1 - \mathrm{Tr}(P\,\rho_1^{\otimes N}) \end{cases} \quad \text{over} \quad P$$

$$\min_{p_2=\epsilon} \; p_1 \quad \simeq \quad e^{-NS(\rho_2 \| \rho_1)}$$

relative entropy
(Kullback–Leibler divergence)

$$\min_{p_2 = \epsilon} \; p_1 \quad \simeq \quad e^{-NS(\rho_2 \| \rho_1)}$$

$$S(\rho_2 \| \rho_1) = \mathrm{Tr}[\rho_2(\ln \rho_2 - \ln \rho_1)]$$

entropy

$$S(\rho) = \ln d - S(\rho \| \tfrac{1}{d} I)$$

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mutual information

$$I(A:B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$$

quantum channels

$$\mathcal{N}(\rho)$$

$$\mathcal{N}(\rho) = \qquad\qquad \rho$$

$$\mathcal{N}(\rho) = \qquad \rho \otimes \rho'$$

$$\mathcal{N}(\rho) = U\,(\,\rho \otimes \rho'\,)\,U^\dagger$$

$$\mathcal{N}(\rho) = \mathrm{Tr}'\, U\,(\,\rho \otimes \rho'\,)\,U^\dagger$$

distinguishability
must be monotonic

$$S(\mathcal{N}(\rho_2)\|\mathcal{N}(\rho_1)) \;\leq\; S(\rho_2\|\rho_1)$$

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$$\mathrm{Tr}(P\mathcal{N}(\rho)) = \mathrm{Tr}(\mathcal{N}^\dagger(P)\,\rho)$$

Information metrics

Riemannian metrics on manifold of states

$$\mathcal{M} = \{\rho : \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

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which are monotonic

$$d(\mathcal{N}(\rho_1), \mathcal{N}(\rho_2)) \leq d(\rho_1, \rho_2)$$

for all channels $\mathcal{N}$

classically: unique

$$\langle X, Y \rangle_\rho = \int \frac{X(x)Y(x)}{\rho(x)}\, dx$$

$\rho + \epsilon X$

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quantumly: classified by Petz

$$\langle X, Y \rangle_\rho = \mathrm{Tr}(X \Omega_\rho^{-1}(Y))$$

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$$\Omega_\rho(Y) \sim \rho Y$$

Kubo-Mori metric

$$S(\rho + \epsilon X \| \rho) = \epsilon^2 \mathrm{Tr}(X \Omega_\rho^{-1}(X)) + \mathcal{O}(\epsilon^4)$$

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$$\langle X, X \rangle_\rho = \int_0^1 \mathrm{Tr}(A\rho^s A \rho^{1-s})\, ds = \int_0^1 \langle A(0)\, A(is) \rangle ds$$

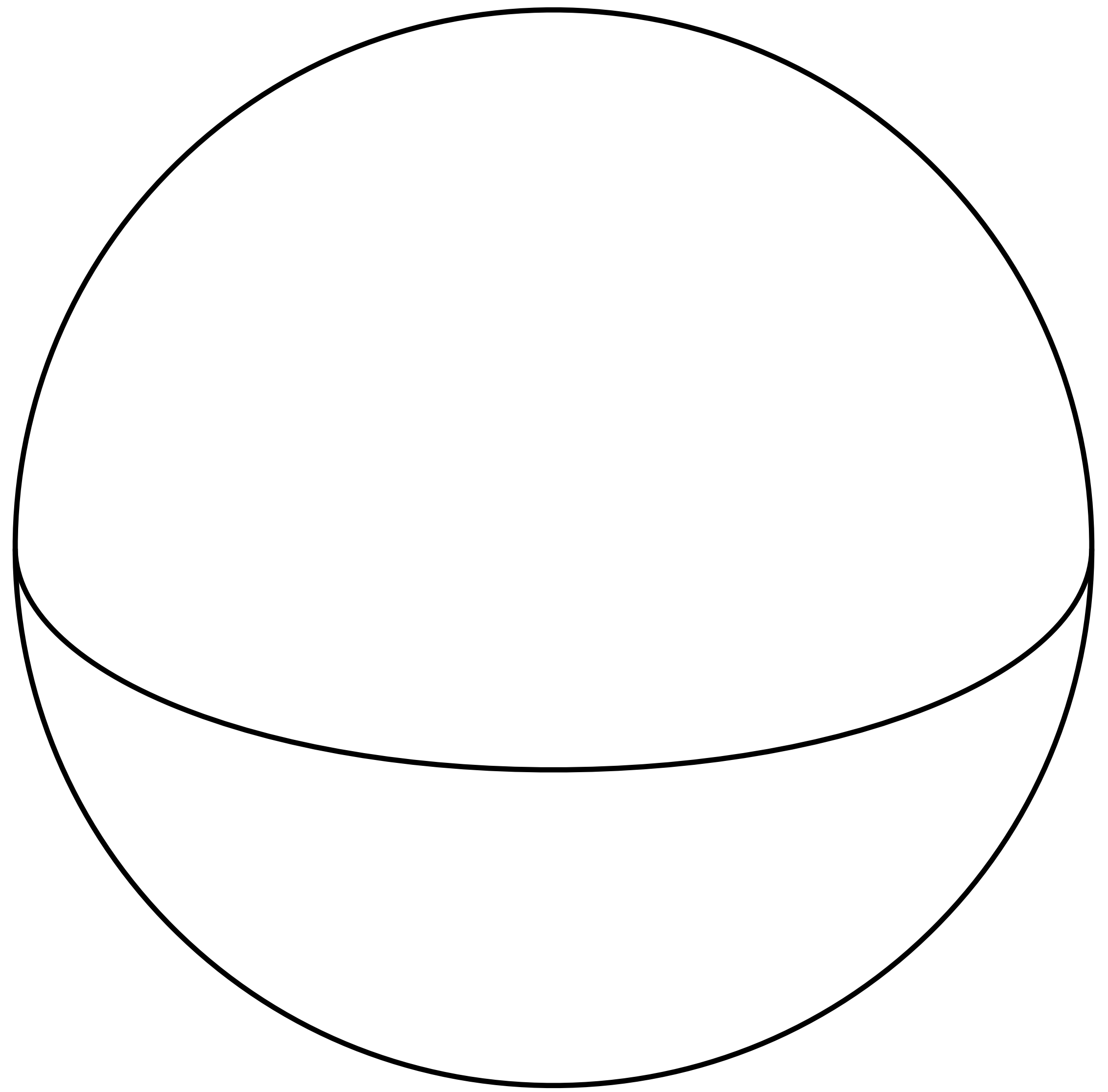
but
this all depends on the cutoff…

coarse-grained distinguishability

$$d_\sigma(\rho_2, \rho_1) = d(\mathcal{N}_\sigma(\rho_2), \mathcal{N}_\sigma(\rho_1))$$

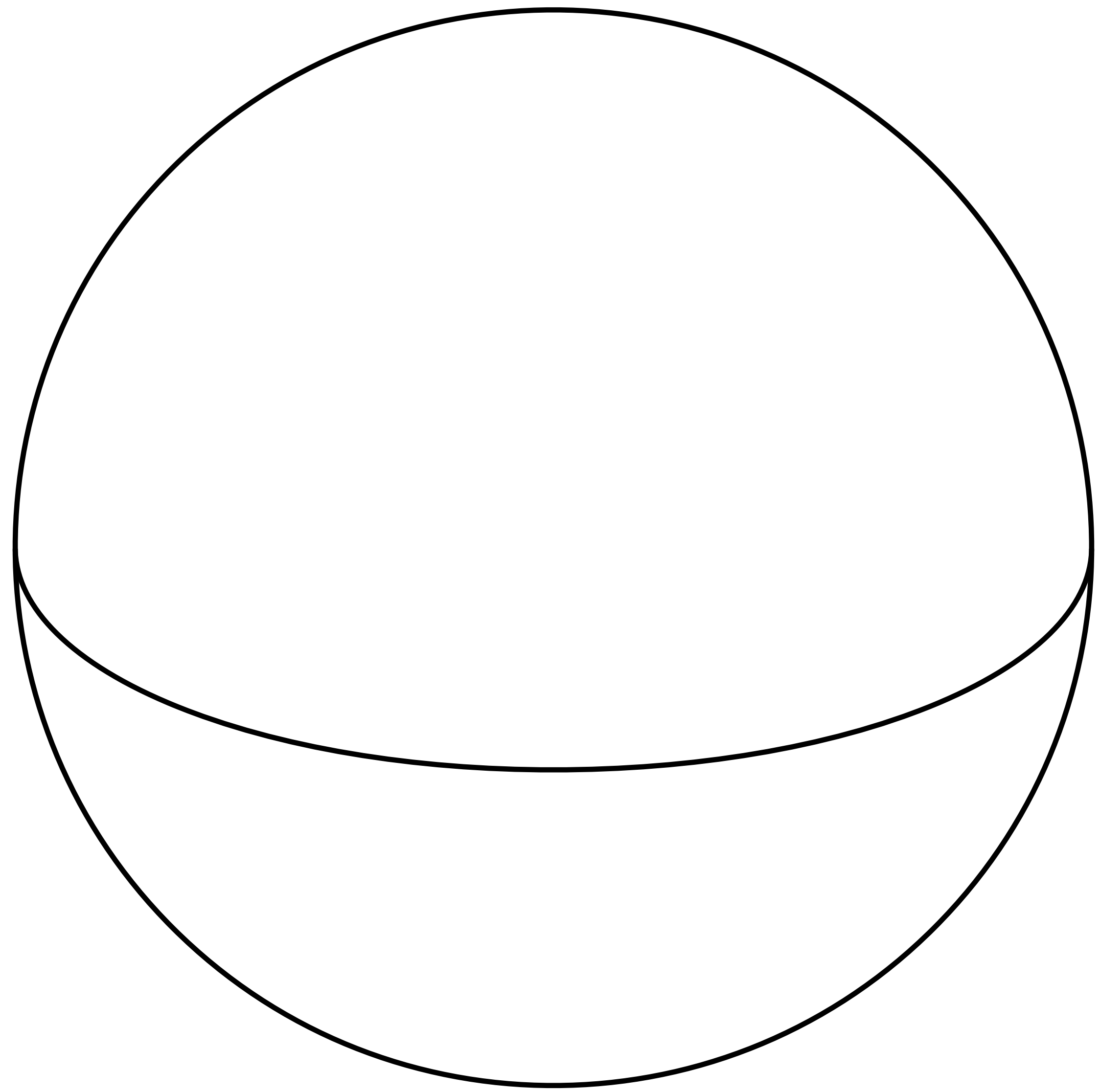
$$d_\sigma(\rho_2, \rho_1) = d(\mathcal{N}_\sigma(\rho_2), \mathcal{N}_\sigma(\rho_1))$$

$$\langle\!\langle X, Y \rangle\!\rangle_\rho^\sigma = \langle \mathcal{N}_\sigma(X), \mathcal{N}_\sigma(Y) \rangle_{\mathcal{N}_\sigma(\rho)}$$



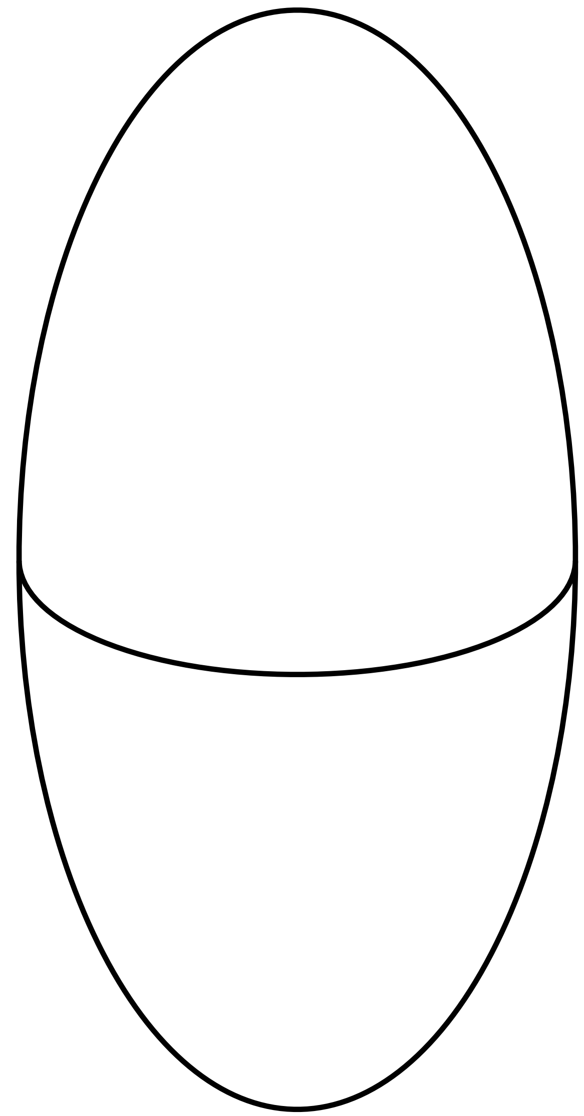
$\langle X, Y \rangle_\rho$

$\sigma = 0\,\mu m$



$\langle\!\langle X, Y \rangle\!\rangle_\rho^\sigma$

$$\sigma = 10\,\mu m$$



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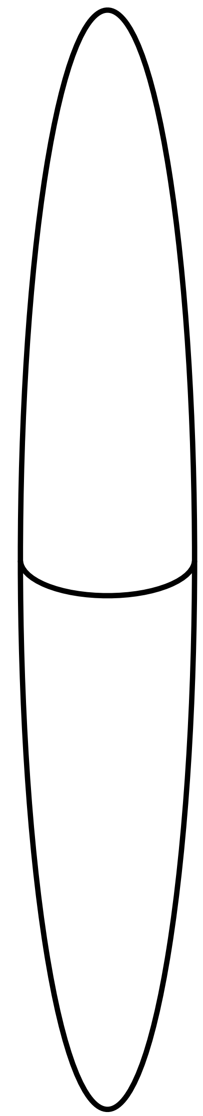

$\langle\!\langle X, Y \rangle\!\rangle_\rho^\sigma$

$$\sigma = 1000\,\mu m$$

$$\langle\!\langle X, Y \rangle\!\rangle_{\rho}^{\sigma}$$

$$\rho_t = \frac{1}{Z_t} e^{-\beta(H+tV)}$$

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$$D(V) := \langle \partial_t \mathcal{N}_\sigma(\rho_t), \partial_t \mathcal{N}_\sigma(\rho_t) \rangle_{\mathcal{N}_\sigma(\rho_t)} \big|_{t=0}$$

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$$d(V) = \lim_{\mathrm{vol}\to\infty} \tfrac{1}{\mathrm{vol}} D(V)$$

example

$f \in$ phase space

$$\mathrm{Tr}\big(\rho\, e^{i\phi(f)}\big) = e^{-\frac{1}{2}(f, Af)}$$

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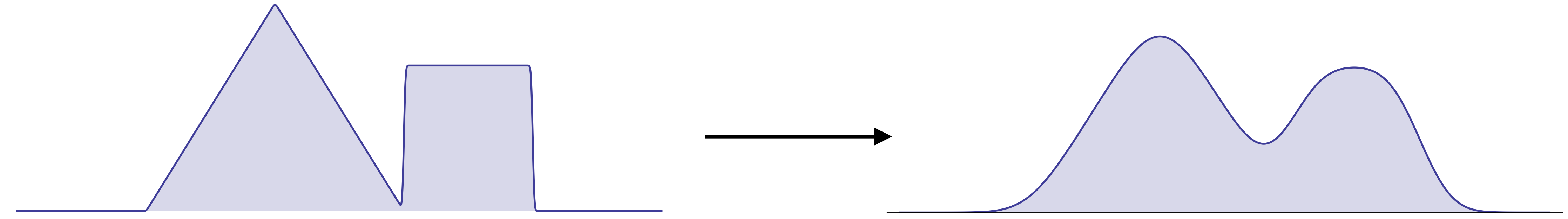
$$A \longrightarrow X^\dagger A X + Y$$

$$(Xf)(x) \propto \int e^{-\frac{(x-y)^2}{2\sigma^2}} f(y)\, dy$$



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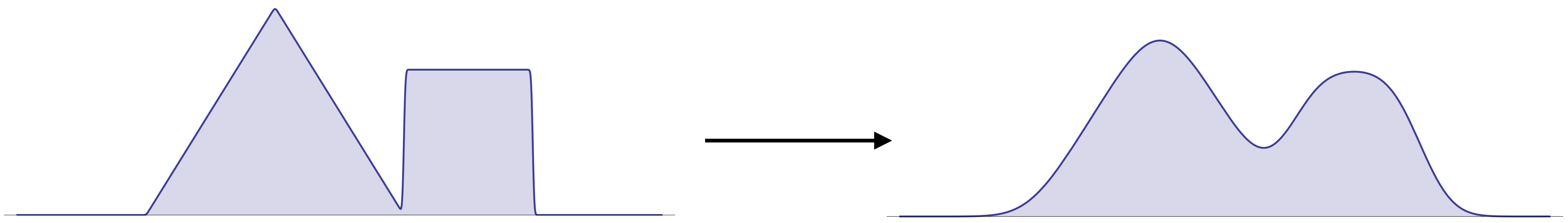
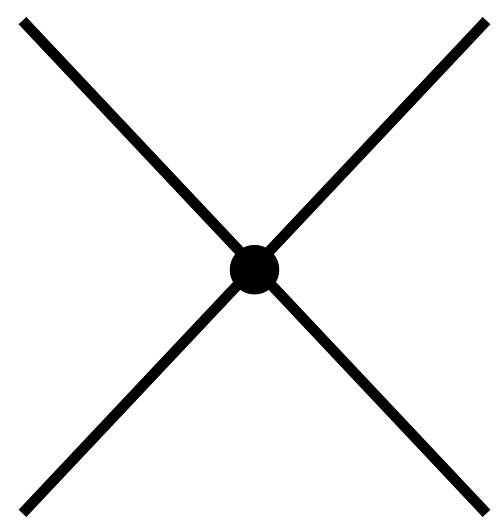
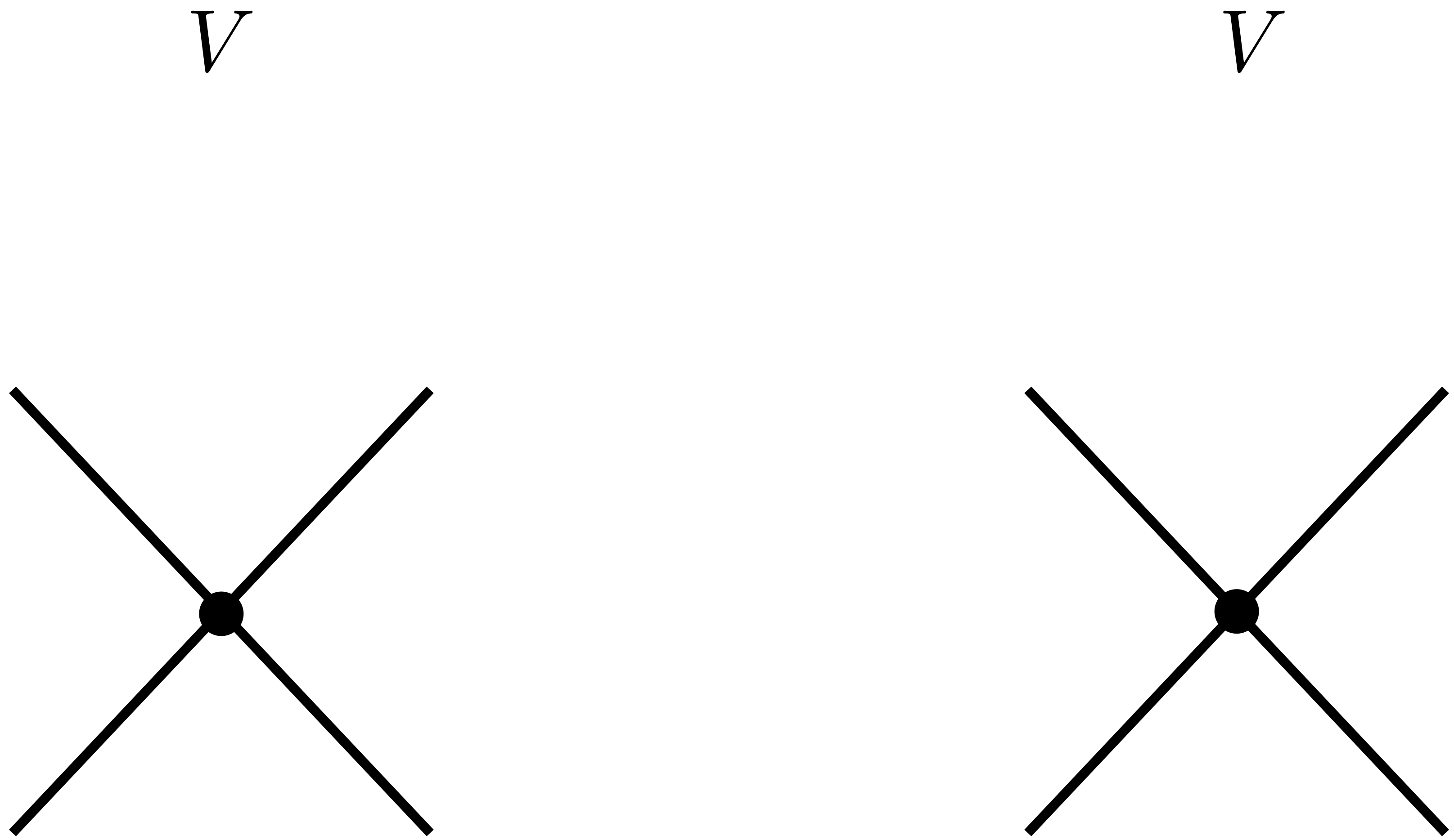
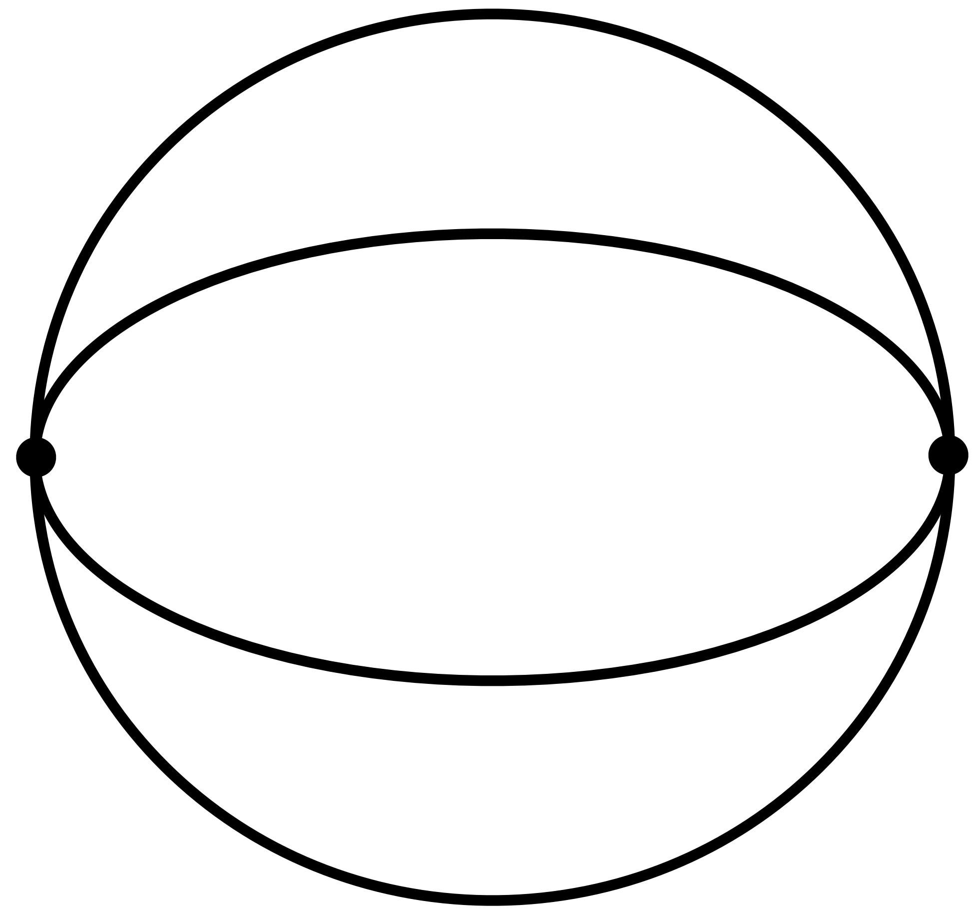


$$Y = \begin{pmatrix} y_\phi^2 I & 0 \\ 0 & y_\pi^2 I \end{pmatrix}$$

$$V = \int \phi(x)^4 dx$$




V
V

$d(V) =$ 

classical, $\mathcal{N} = \mathrm{id}$

 $= \quad A$

classical

$$= \quad AX(X^\dagger AX + Y)^{-1}X^\dagger A$$

quantum, $y_\phi y_\pi \gg \hbar$

quantum, $\quad y_\phi y_\pi \gg \hbar$

$$\rule{4cm}{1pt} \quad = \quad (A + \frac{i}{2}\Delta) X B_\beta^{-1} X^\dagger (A + \frac{i}{2}\Delta)$$

quantum, $y_\phi y_\pi \gg \hbar$

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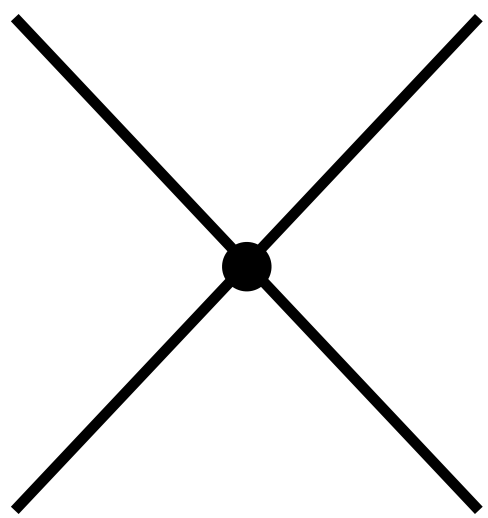
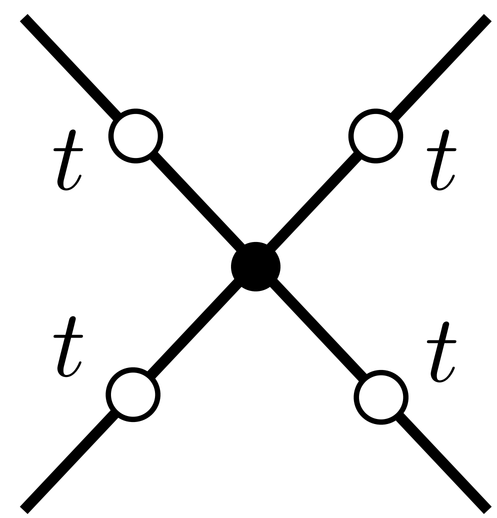


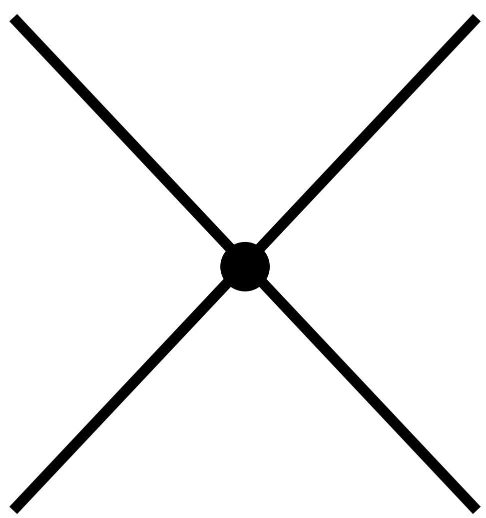
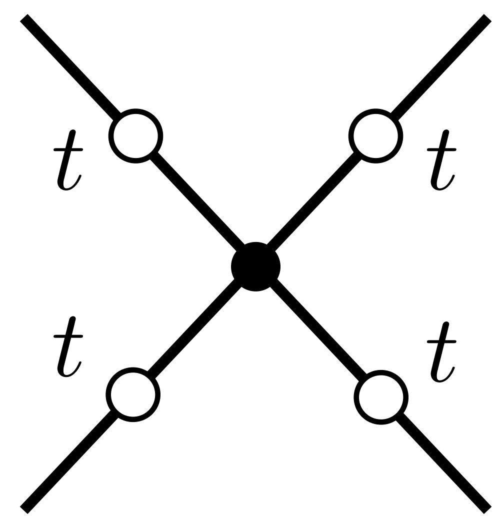
$$= \quad \int_0^\beta$$



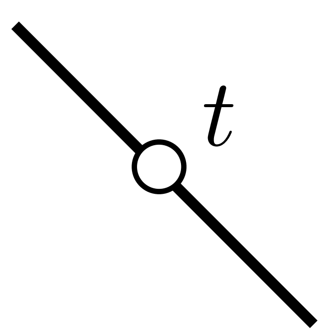
$$dt$$

quantum, $y_\phi y_\pi \gg \hbar$

 $= \quad (A + \frac{i}{2}\Delta) X B_\beta^{-1} X^\dagger (A + \frac{i}{2}\Delta)$

 $= \quad \int_0^\beta$ 

dt



= imaginary time evolution in phase space

$$d(\int \phi^4)$$

$$\sigma^D d(\int \phi^4)$$

$$\alpha = \frac{\partial \log(\sigma^D d(\int \phi^4))}{\partial \log \sigma}$$

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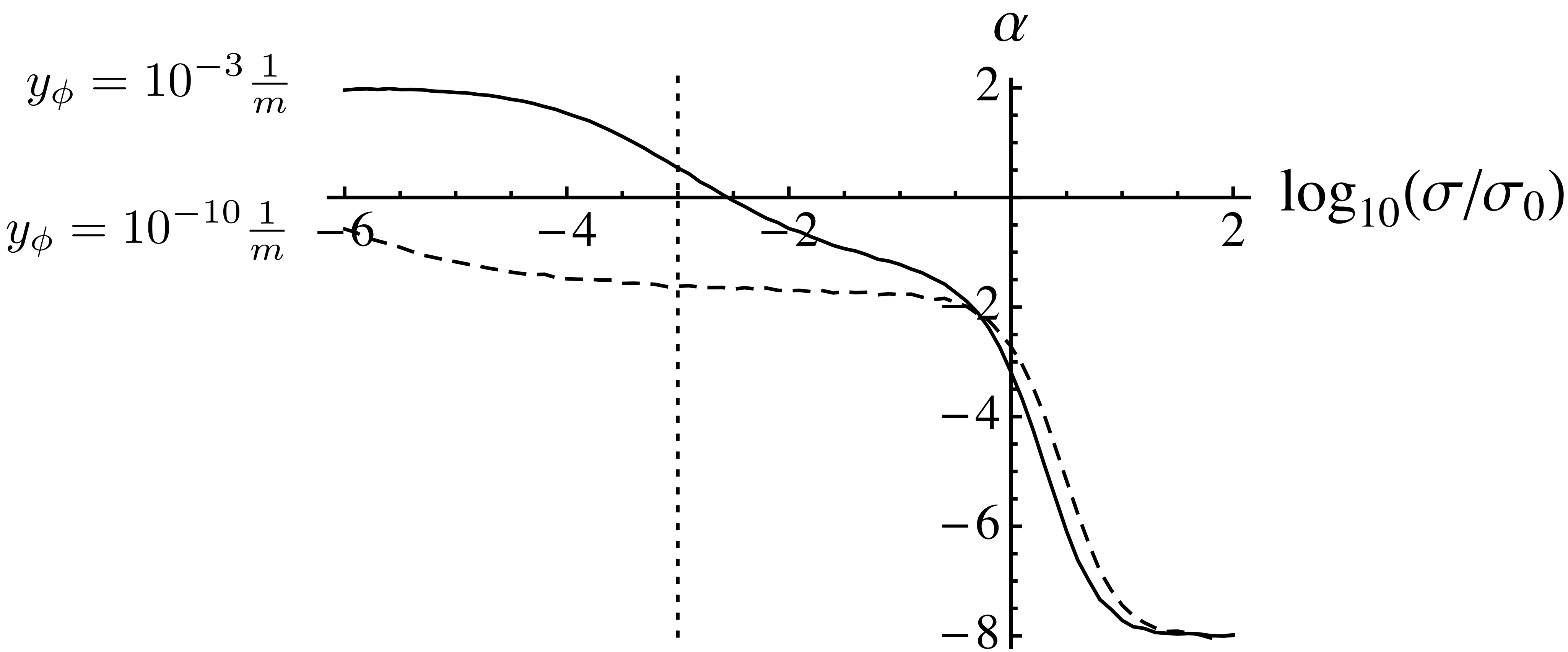
4+1 dimensions



$$\alpha = \frac{\partial \log(\sigma^D d(\int \phi^4))}{\partial \log \sigma}$$

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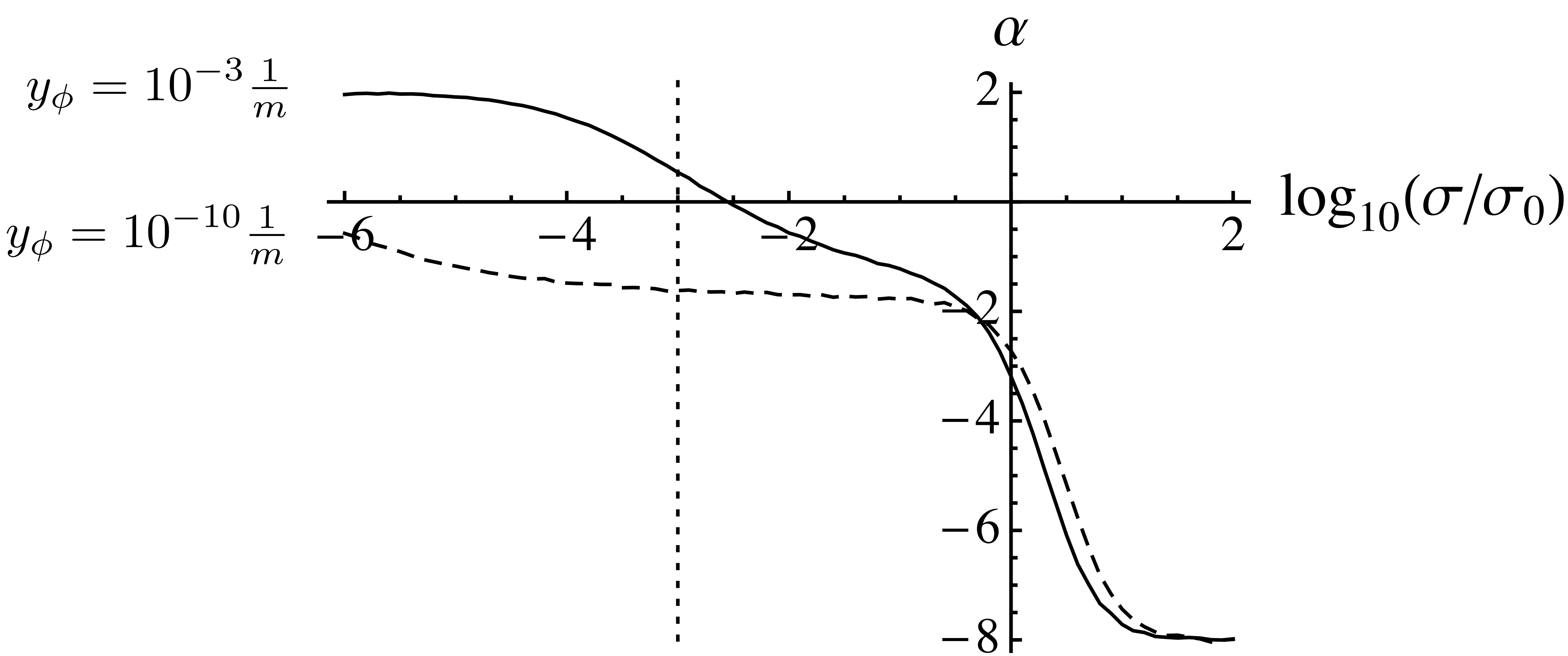

Thanks!