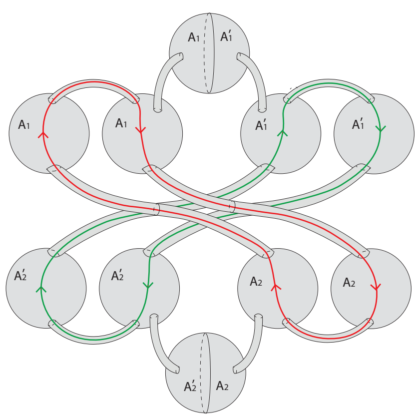
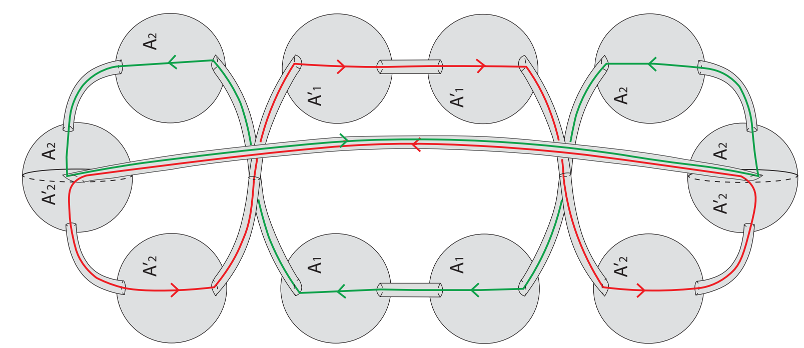
Topological Entanglement Negativity in Chern-Simons theories





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05/31/2016, YITP

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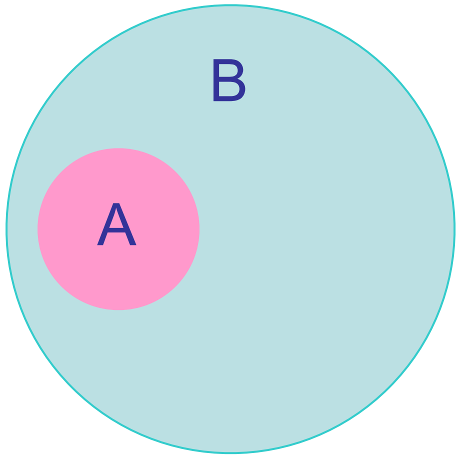
In collaboration with Xueda Wen and Shinsei Ryu

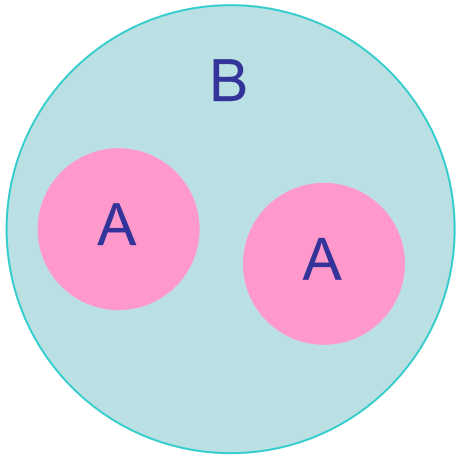
Outline

- Introduction: What is the entanglement negativity?
- Methods: How to compute the entanglement negativity in Chern-Simons theories? (Surgery method)
- Results: Understanding 3-manifolds!

Entanglement negativity

Entanglement (Renyi) entropy: bipartite systems

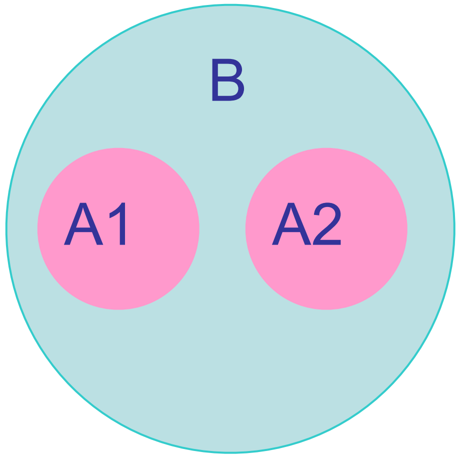


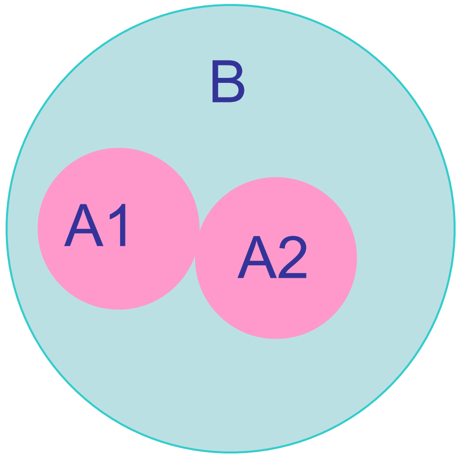


A and B are always adjacent to each other.

Beyond bipartite systems: entanglement between A1 and A2

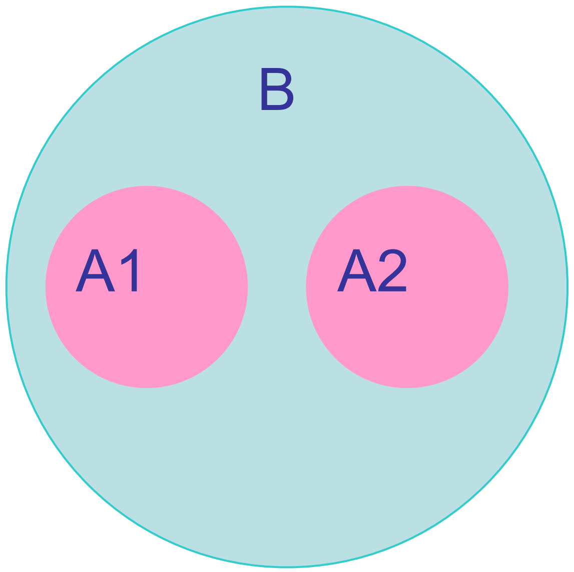
tripartite





- A1 and A2 can be either adjacent or disjointed.

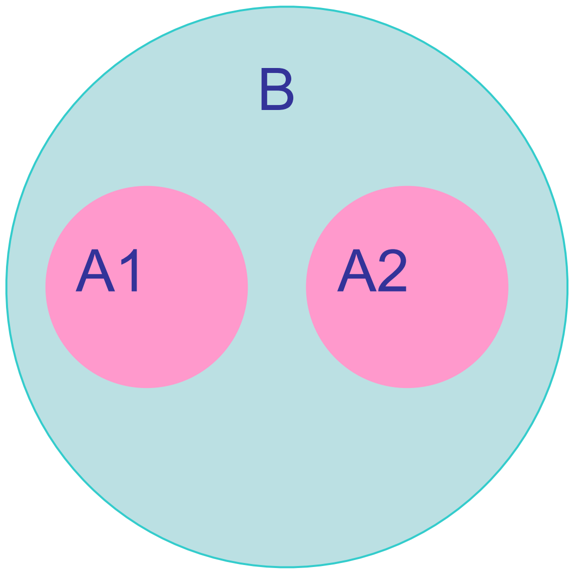
Entanglement negativity



Trace norm of a *partially transposed* reduced density matrix [Vidal-Werner 02]

1. $\rho_{A_1 \cup A_2} = \mathrm{tr}_B \rho$ is a mixed state

Entanglement negativity



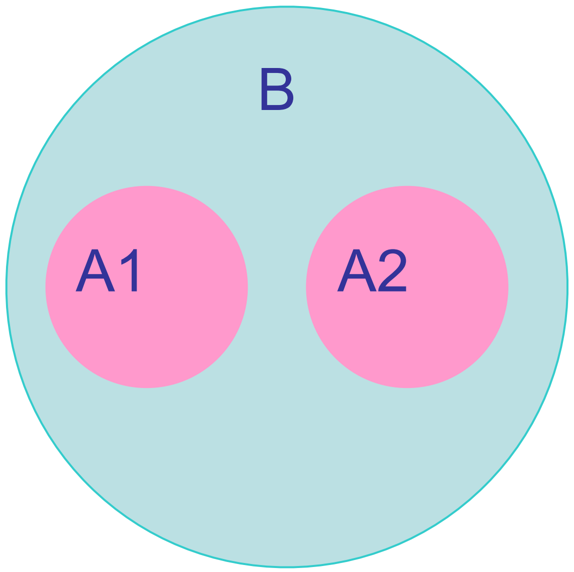
Trace norm of a *partially transposed* reduced density matrix [Vidal-Werner 02]

1. $\rho_{A_1\cup A_2} = \mathrm{tr}_B \rho$ is a mixed state
2. $\rho^{T_2}_{A_1\cup A_2}$ partial transpose w.r.t. A2

$$\boxed{\langle e_i^{(1)} e_j^{(2)} | \, \rho^{T_2} \, | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \, \rho \, | e_k^{(1)} e_j^{(2)} \rangle}$$

$|e_i^{(1)}\rangle$ and $|e_i^{(2)}\rangle$ are bases of $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$

Entanglement negativity



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$|e_i^{(1)}\rangle$ and $|e_i^{(2)}\rangle$ are bases of $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$

3. Trace norm: $||\rho^{T_2}|| = \mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i$

It measures 'how much' the eigenvalues of ρ^{T_2} are negative.

Logarithmic negativity: $\mathcal{E} \equiv \ln ||\rho^{T_2}|| = \ln \mathrm{Tr}|\rho^{T_2}|$

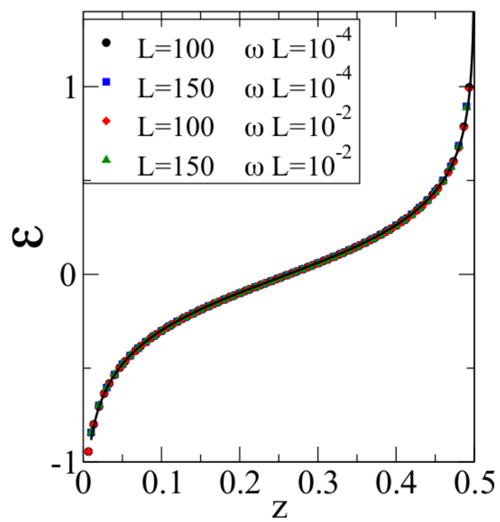
It is not easy to compute the entanglement negativity for a many body state!!!

- A replica trick + QFT (can be CFT or CS)
  [Calabrese, Cardy, Tonni, 12,13]
- Monte Carlo simulations
  [Chung, Alba, Bonnes, Chen, Lauchli,13]
- Tensor network (MPS)
  [Calabrese, Tagliacozzo, Tonni,13]
- An overlap matrix method (free fermions)
  [Chang, Wen,16]
- Representation theory (Valance bond solids)
  [Santos, Korepin,16]
- …

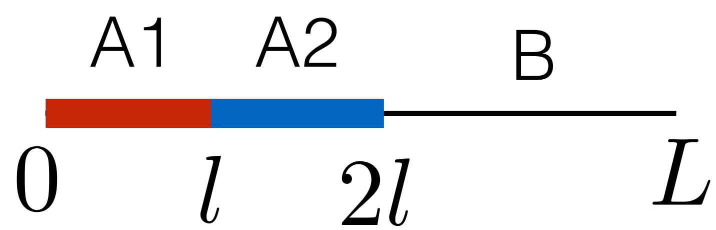
Some interesting results

1. Universal behavior for critical systems (CFT approaches)

[Calabrese, Cardy, Tonni, 12]



$$\mathcal{E} = \frac{1}{4}\ln[\tan(\pi z)] + \text{const.}$$



$$z = l/L$$

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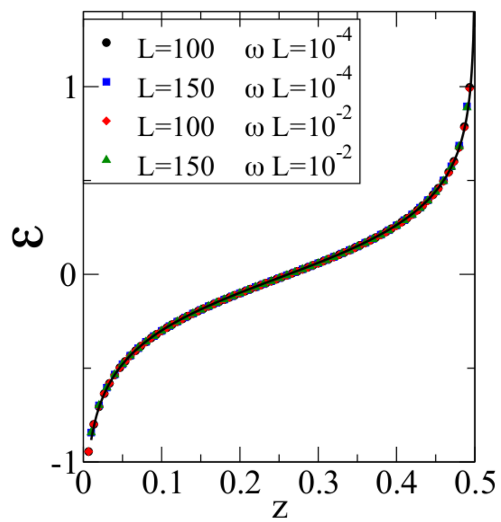

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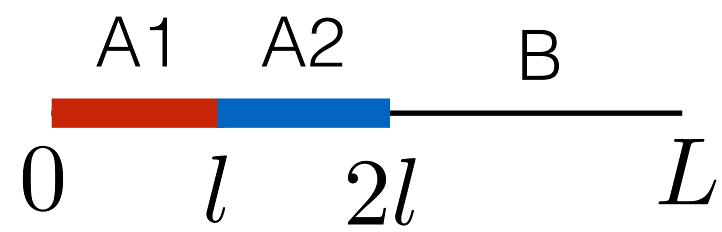
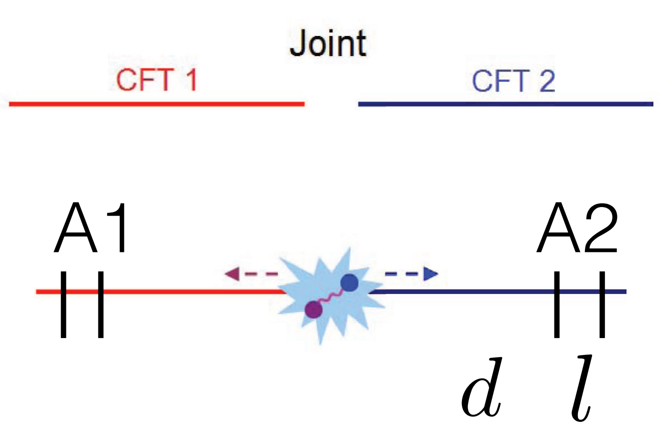
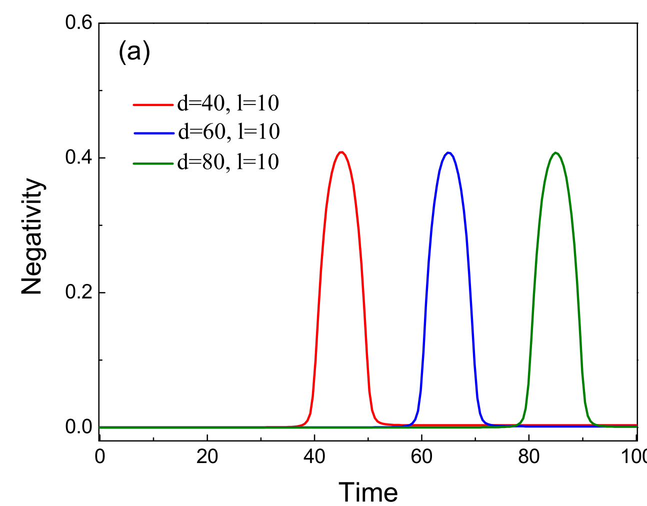


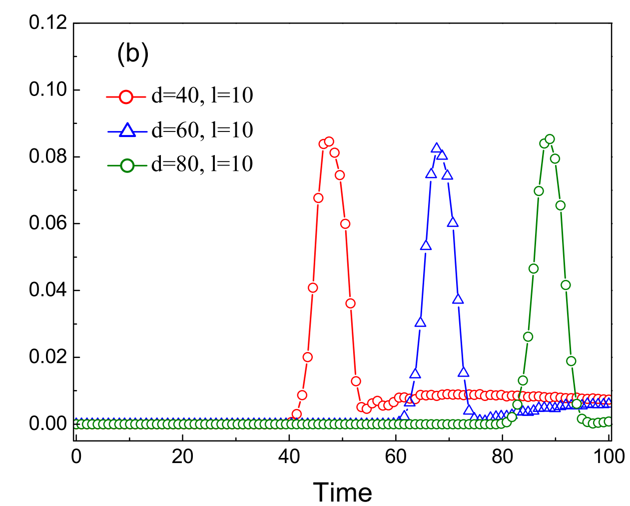
$$z = l/L$$

2. Light cone behaviors for quench studies

[Coser, Tonni, Calabrese,14]
[Wen, Chang, Ryu,15]







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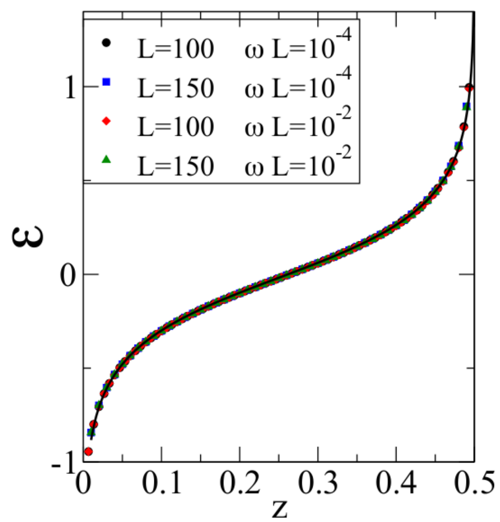

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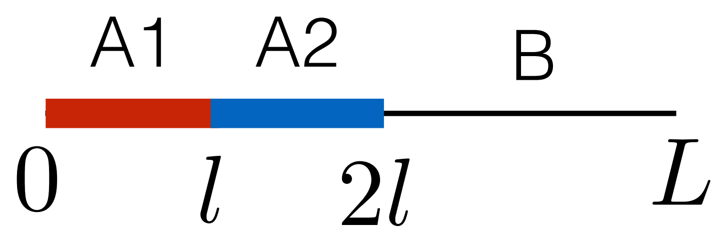

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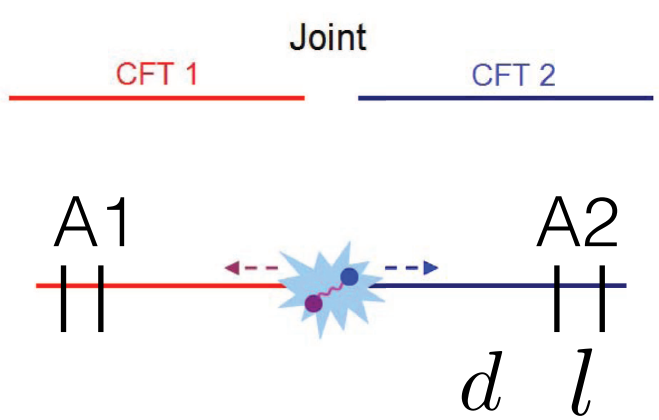
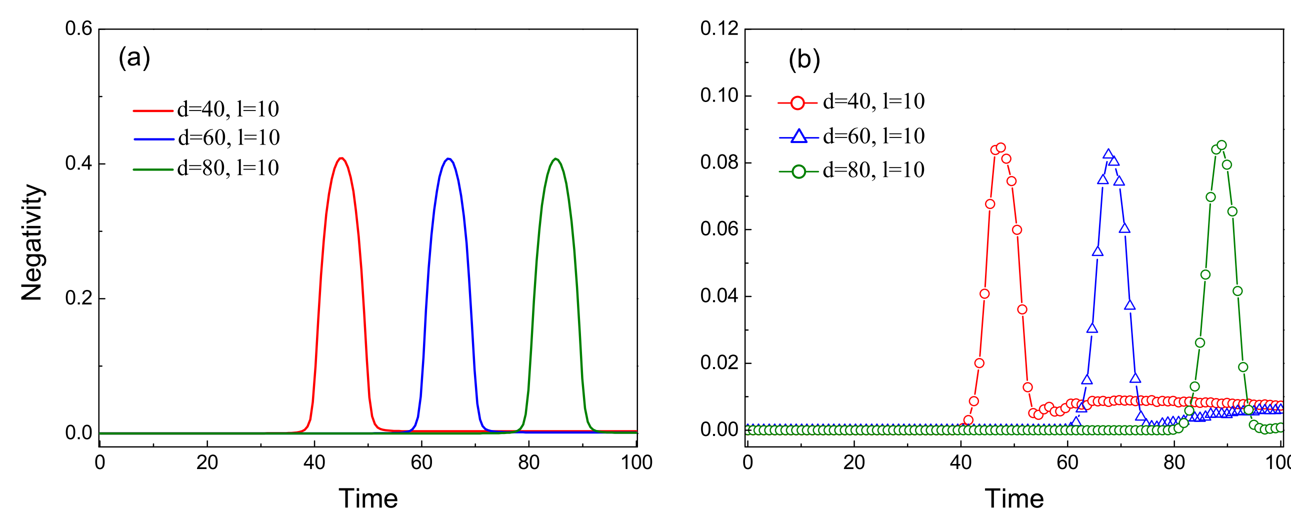
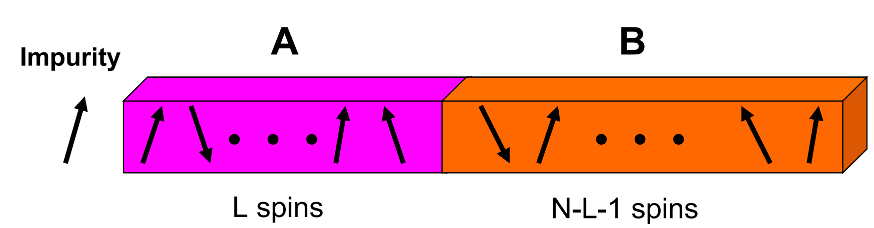




3. Estimate Kondo screening length

[Bayat, Sodano, Bose,10]

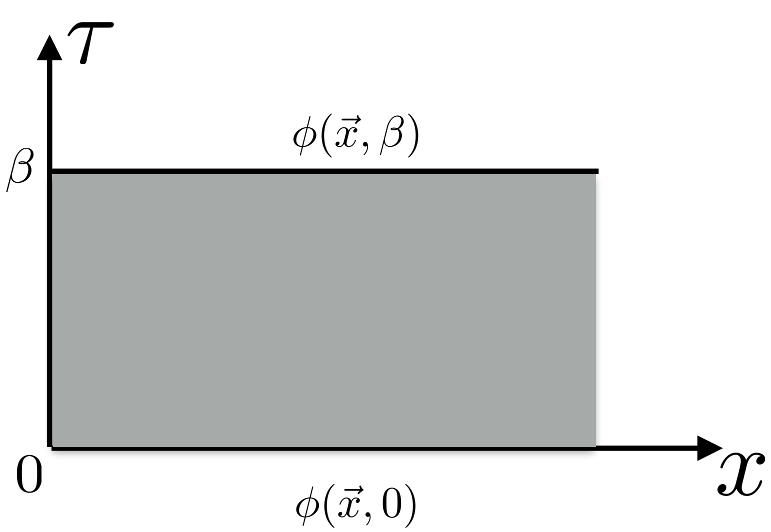


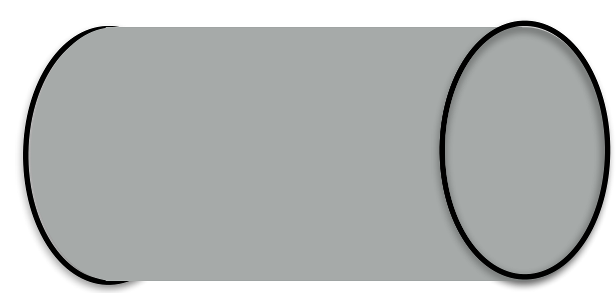
Chern-Simons theory—a surgery method approach

A path integral representation of a partially transposed reduced density matrix + a replica trick= computing a partition function on a 3-manifold

1. Density matrix

$$\rho\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big] = \frac{1}{Z(\beta)}\left\langle\{\varphi_0(\vec{x})\}|e^{-\beta H}|\{\varphi_\beta(\vec{x})\}\right\rangle$$
$$= \int \prod [d\phi(\vec{x},\tau)] e^{-S_E} \prod_{\vec{x}} \delta[\phi(\vec{x},0)-\varphi_0(\vec{x})]\delta[\phi(\vec{x},\beta)-\varphi_\beta(\vec{x})]$$

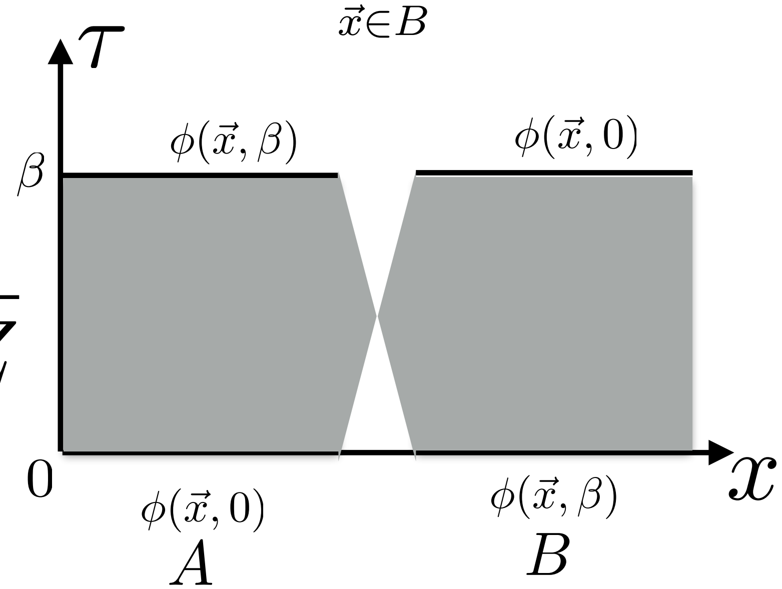
$$\rho = \frac{1}{Z}$$


$$Z =$$


2. Partially transposed density matrix

$$\rho^{T_B}\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big] = \int \prod_{\vec{x},\tau}[d\phi(\vec{x},\tau)]e^{-S_E}\prod_{\vec{x}\notin B}\delta[\phi(\vec{x},0)-\varphi_0(\vec{x})]\delta[\phi(\vec{x},\beta)-\varphi_\beta(\vec{x})]$$
$$\prod_{\vec{x}\in B}\delta[\phi(\vec{x},0)-\varphi_\beta(\vec{x})]\delta[\phi(\vec{x},\beta)-\varphi_0(\vec{x})].$$

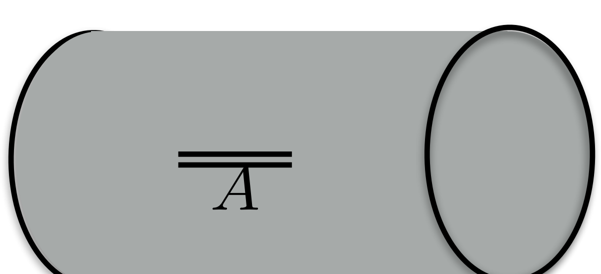
$$\rho^{T_B} = \frac{1}{Z}$$



3. Reduced density matrix

$$\rho_{A_1\cup A_2}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\middle|\vec{x}\in A_1\cup A_2\right]$$
$$=\int\left(\prod_{\vec{x}\in B}[d\varphi_0(\vec{x})d\varphi_\beta(\vec{x})]\delta[\varphi_0(\vec{x})-\varphi_\beta(\vec{x})]\right)\rho\Big[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\Big].$$

$$\rho_A = \frac{1}{Z}$$

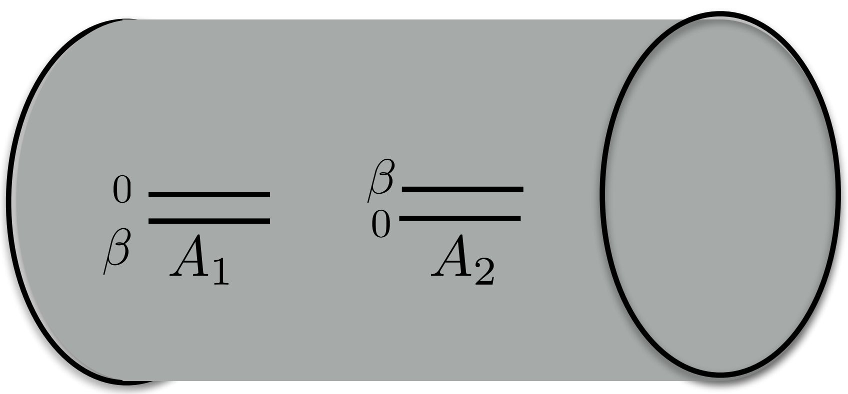


4. Partially transposed reduced density matrix

$$\rho_{A_1\cup A_2}^{T_{A_2}}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\middle|\vec{x}\in A_1\cup A_2\right]$$
$$=\int\left(\prod_{\vec{x}\in B}[d\varphi_0(\vec{x})d\varphi_\beta(\vec{x})]\delta[\varphi_0(\vec{x})-\varphi_\beta(\vec{x})]\right)\rho^{T_{A_2}}\left[\{\varphi_0(\vec{x})\},\{\varphi_\beta(\vec{x})\}\right].$$

Not easy to compute

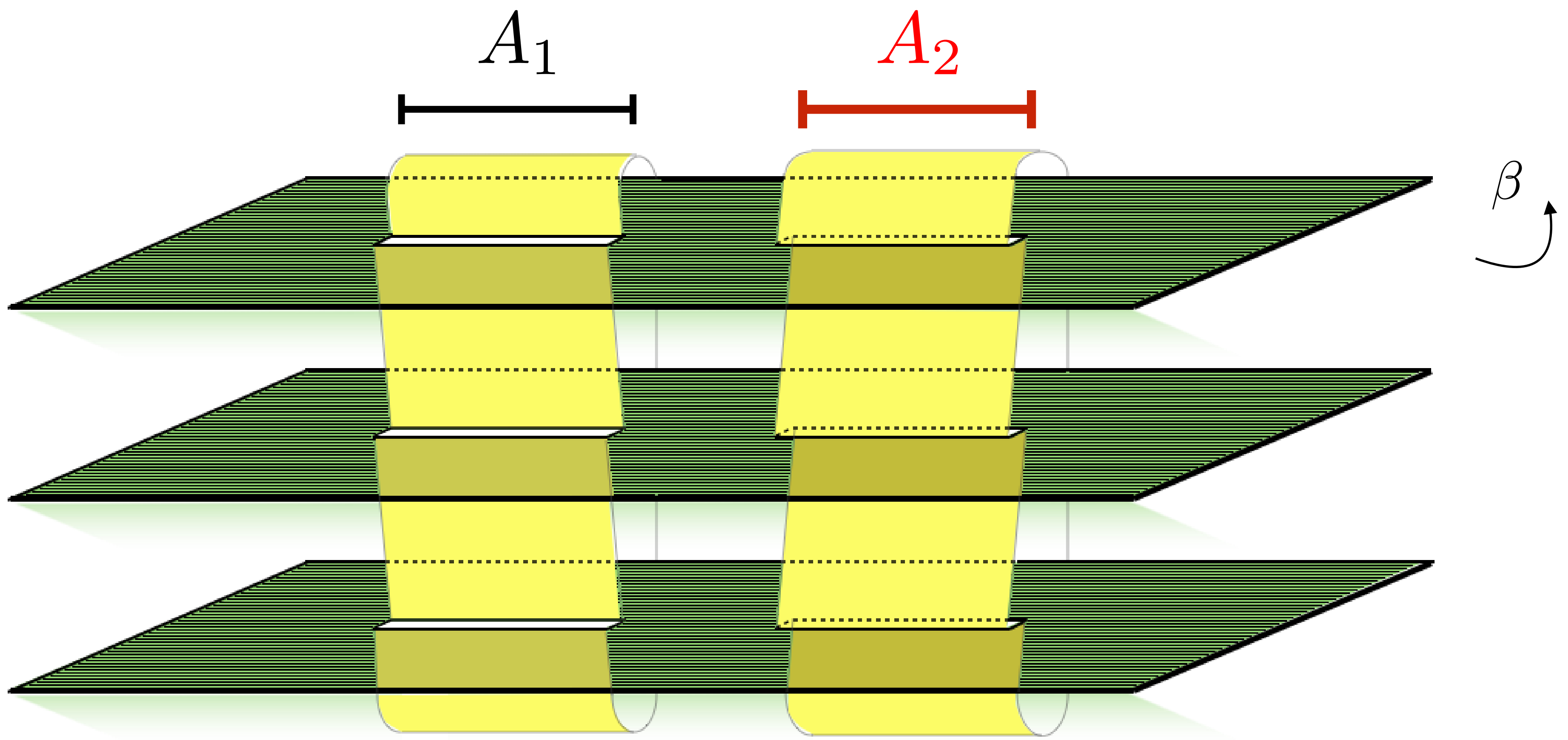
$$\rho_A^{T_{A_2}}=\frac{1}{Z}$$



5. Replica trick (n copies)

$$\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^n=\int\prod_{k=1}^{n}\left\{\prod_{\vec{x}}\left[d\varphi_0^{(k)}(\vec{x})d\varphi_\beta^{(k)}(\vec{x})\right]\prod_{\vec{x}\in B}\delta\left[\varphi_0^{(k)}(\vec{x})-\varphi_\beta^{(k)}(\vec{x})\right]\right.$$
$$\left.\prod_{\vec{x}\in A_1}\delta\left[\varphi_0^{(k)}(\vec{x})-\varphi_\beta^{(k+1)}(\vec{x})\right]\prod_{\vec{x}\in A_2}\delta\left[\varphi_\beta^{(k)}(\vec{x})-\varphi_0^{(k+1)}(\vec{x})\right]\rho\left[\{\varphi_0^{(k)}(\vec{x})\},\{\varphi_\beta^{(k)}(\vec{x})\}\right]\right\}.$$

e.g. $\mathrm{tr}(\rho_{\mathrm{A}_1\cup\mathrm{A}_2}^{\mathrm{T}_{\mathrm{A}_2}})^3$



[Calabrese, Cardy, Tonni, 12]

A trick of computing the entanglement negativity

1. Trace norm

$$\mathrm{tr}|\rho_{A_1\cup A_2}^{T_{A_2}}| = \sum_i |\lambda_i| = \sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|$$

2. Momenta of the partially transposed reduced density matrix

$$\mathrm{tr}(\rho_{A_1\cup A_2}^{T_{A_2}})^n = \sum_i \lambda_i^n = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$= \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

3. Entanglement negativity can be obtained by taking $n_e \to 1$

$$\mathcal{E}_{A_1 A_2} = \lim_{n_e\to 1} \ln \mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_2}\right)^{n_e}$$

Chern-Simons Theory

coupling constant (quantized)

1. CS theory
$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int_M \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

Manifold   connection of a gauge group

2. Partition function $Z(M) = \int [\mathcal{D}A] \mathrm{e}^{iS_{\mathrm{CS}}(A)}$

3. Wilson lines (links and knots) $W_R^{\mathcal{C}}(A) = \mathrm{tr}_R P \exp \int_{\mathcal{C}} A.$

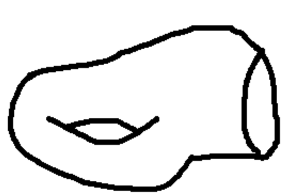
4. Correlators (partition function with links and knots)

$$Z(M, \hat{R}_1, \cdots, \hat{R}_N) = \langle W_{\hat{R}_1}^{\mathcal{C}_1} \cdots W_{\hat{R}_N}^{\mathcal{C}_N} \rangle = \int [\mathcal{D}A] \left(\prod_{i=1}^{N} W_{\hat{R}_i}^{\mathcal{C}_i}\right) \mathrm{e}^{iS_{\mathrm{CS}}}$$

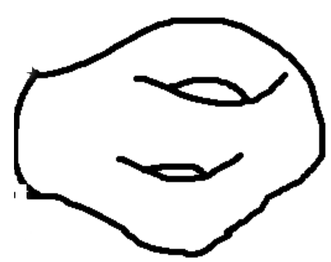
Witten told us how to compute the partition function of CS theories on any 3-manifold by using a surgery theory!

1. The partition function can be computed from the canonical quantization of a CS theory on a 3-manifold with boundary.

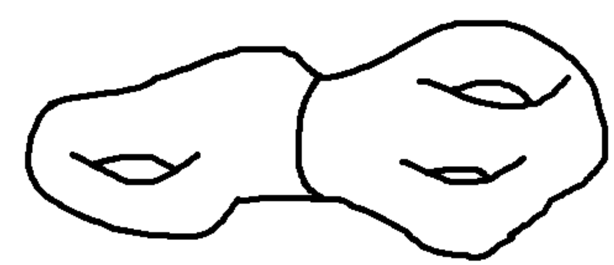
$|\Psi_{M_1}\rangle$



$\langle\Psi_{M_2}|$



$$Z(M) = \langle\Psi_{M_2}|U_f|\Psi_{M_1}\rangle$$



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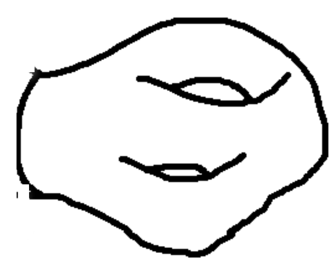

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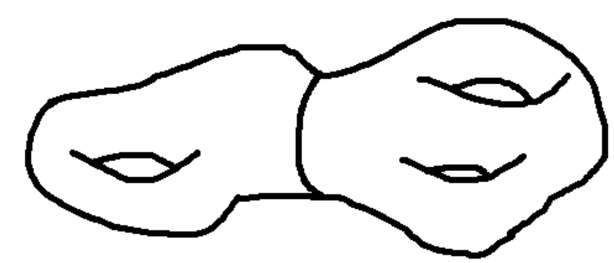
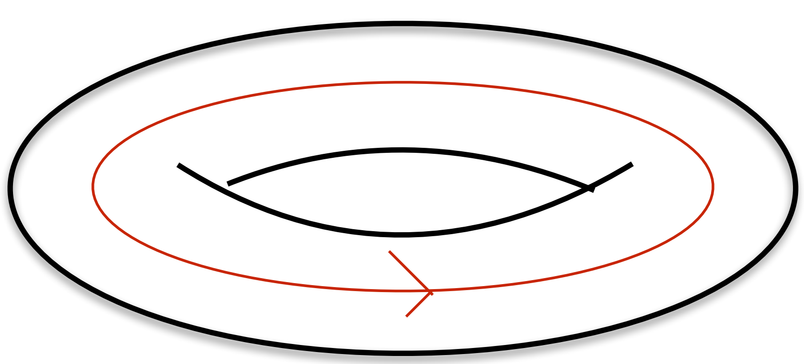


$\mathbf{T} = D \times S^1$



$$|\Psi_{\mathbf{T},\hat{R}_i}\rangle = |\hat{R}_i\rangle$$

Vacuum state: no Wilson line $|0\rangle$

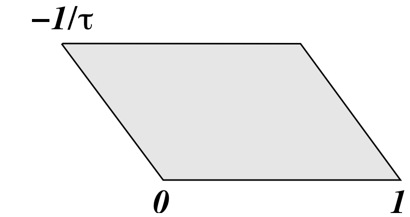
Gluing two slide toruses—> $S^2 \times S^1$

$$Z(S^2 \times S^1) = \langle \hat{0} | \hat{0} \rangle = 1.$$

Gluing two slide toruses with modular transformation—> S^3

$$Z(S^3) = \langle \hat{0} | S | \hat{0} \rangle = \mathcal{S}_{00}$$

$S : \tau \to -1/\tau$

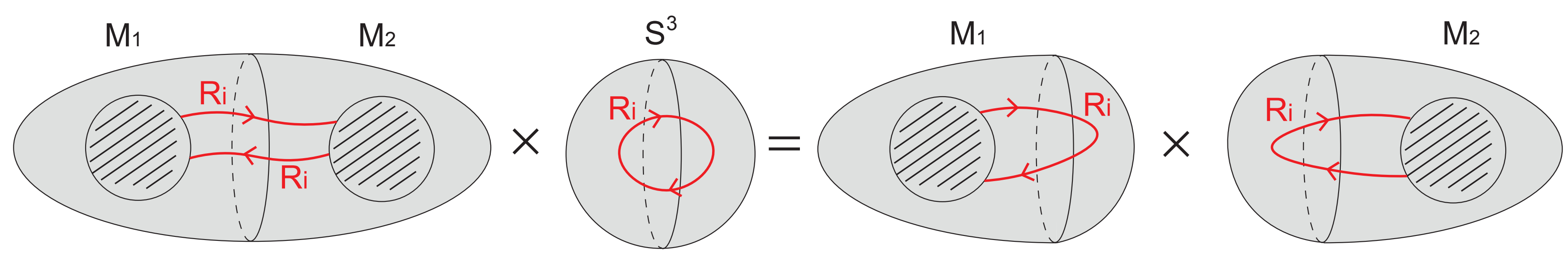


In the presence of Wilson lines

$$Z(S^2 \times S^1, \hat{R}_i, \hat{R}_j) = \langle \hat{R}_i | \hat{R}_j \rangle = \delta_{i,j}.$$

$$Z(S^3, \hat{R}_i, \hat{R}_j) = \langle \hat{R}_i | S | \hat{R}_j \rangle = \mathcal{S}_{ij}.$$

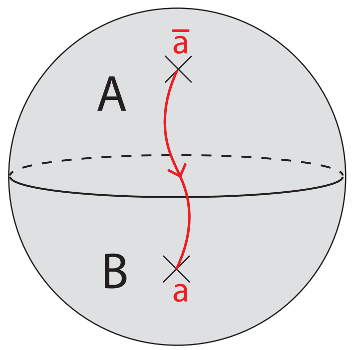
Factorability



$$Z(M, [\blacksquare_1, \blacksquare_2, \hat{R}_i, \hat{\overline{R}}_i]_{\mathcal{C}}) \cdot Z(S^3, \hat{R}_i) = Z(M_1, [\blacksquare_1, \hat{R}_i]_{\mathcal{C}_1}) \cdot Z(M_2, [\blacksquare_2, \hat{R}_i]_{\mathcal{C}_1})$$

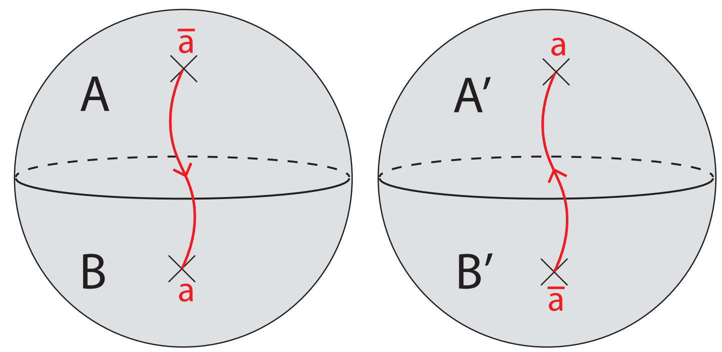
Now let us compute the entanglement negativity in various cases

Ex1



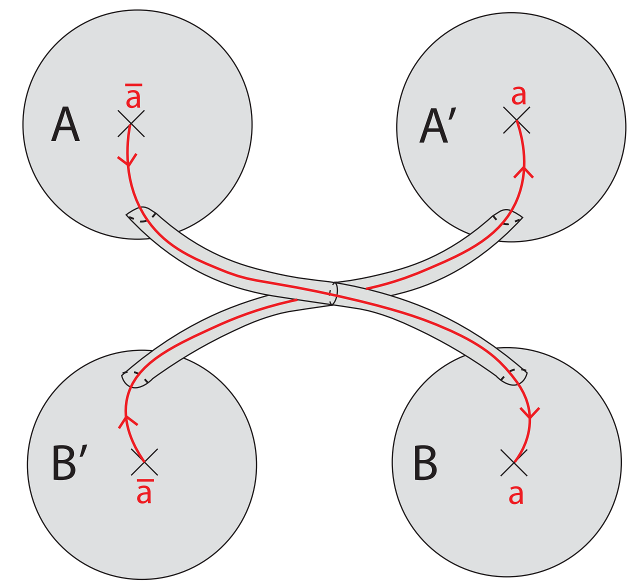
(a)

$|\Psi\rangle$



(b)

$\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$

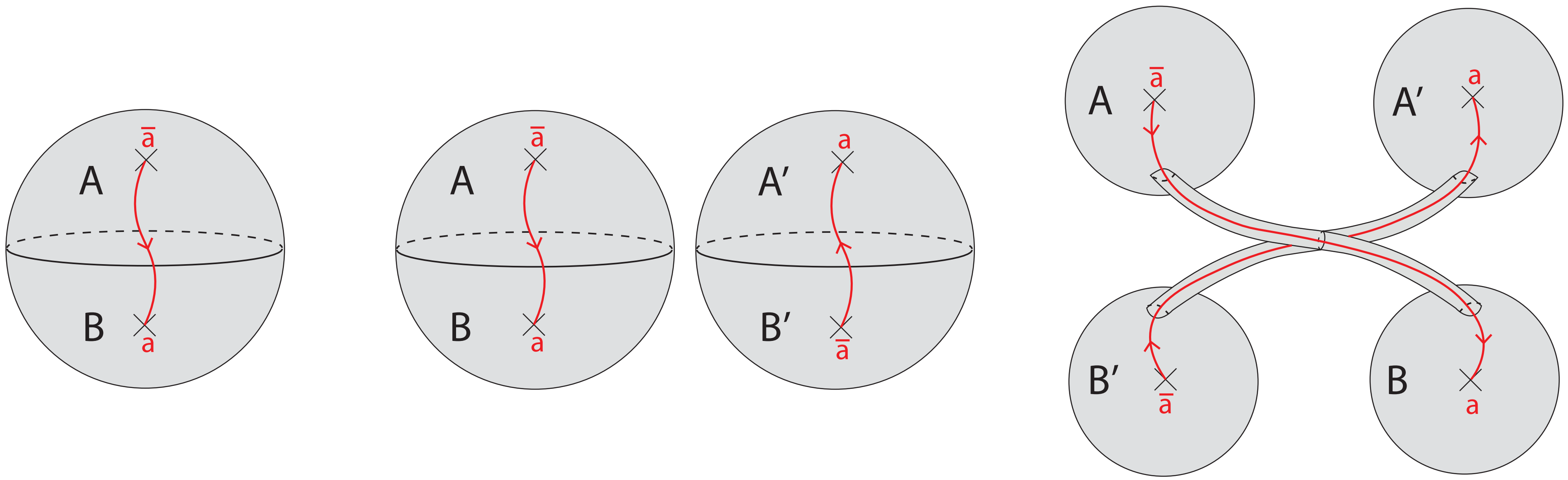


(c)

$\rho_{A\cup B}^{T_B}$

Now let us compute the entanglement negativity in various cases

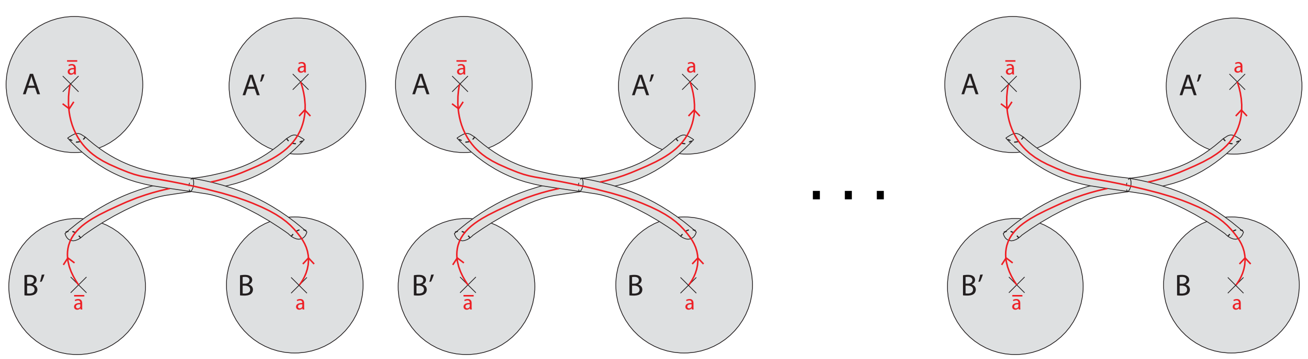
Ex1



(a) $|\Psi\rangle$

(b) $\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$

(c) $\rho^{T_B}_{A\cup B}$

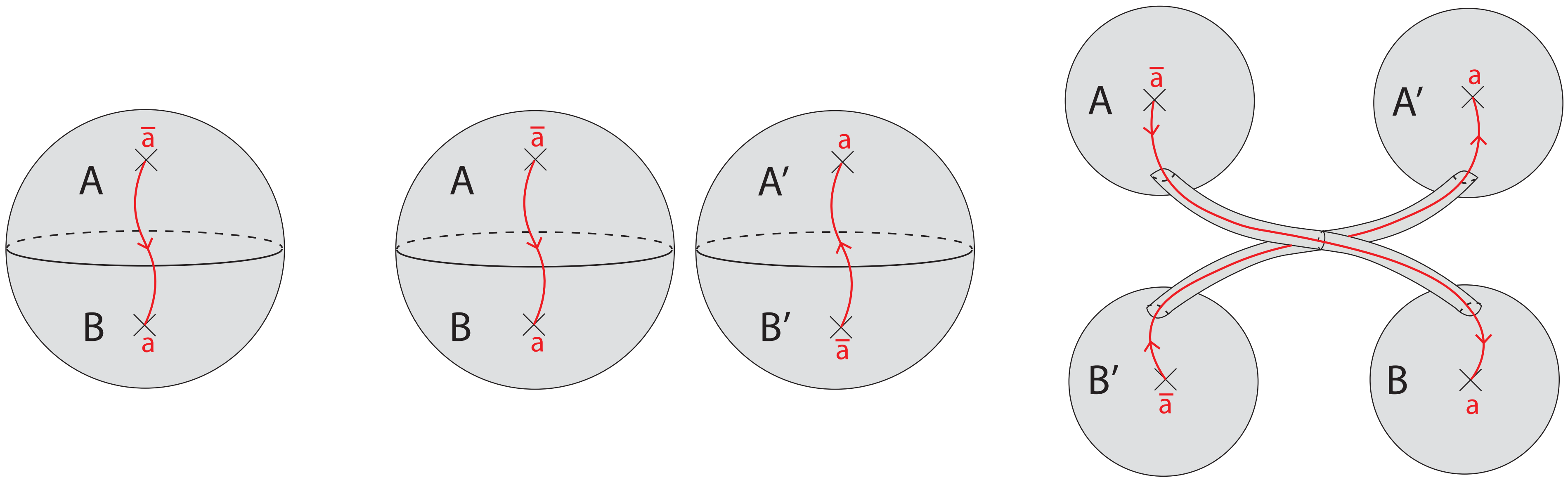


$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_o}}{(\mathrm{tr}\rho^{T_B})^{n_o}} = \frac{Z(S^3,\hat{R}_a)}{Z(S^3,\hat{R}_a)^{n_o}} = Z(S^3,\hat{R}_a)^{1-n_o} = (\mathcal{S}_{0a})^{1-n_o}$$

$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^{n_e}} = Z(S^3,\hat{R}_a)^{2-n_e} = (\mathcal{S}_{0a})^{2-n_e}$$

Now let us compute the entanglement negativity in various cases

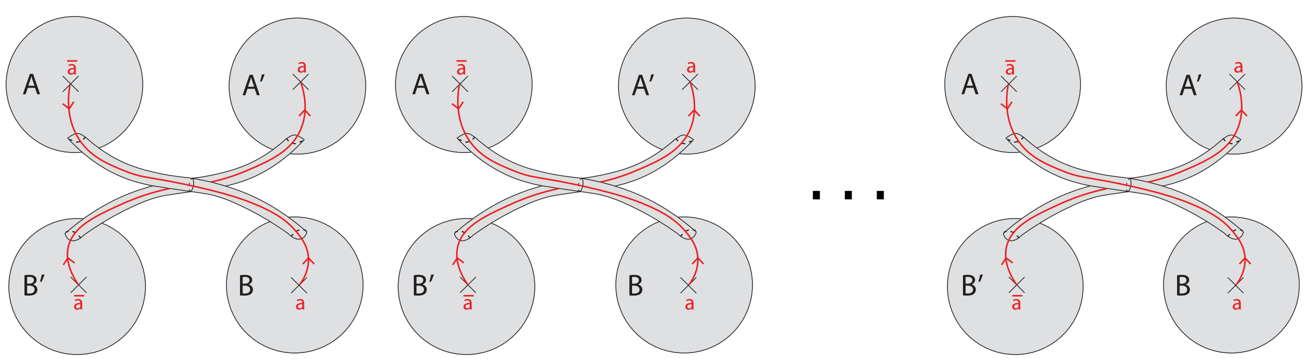
Ex1



(a) $|\Psi\rangle$

(b) $\rho_{A\cup B} = |\Psi\rangle\langle\Psi|$

(c) $\rho_{A\cup B}^{T_B}$



$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_o}}{(\mathrm{tr}\rho^{T_B})^{n_o}} = \frac{Z(S^3,\hat{R}_a)}{Z(S^3,\hat{R}_a)^{n_o}} = Z(S^3,\hat{R}_a)^{1-n_o} = (\mathcal{S}_{0a})^{1-n_o}$$

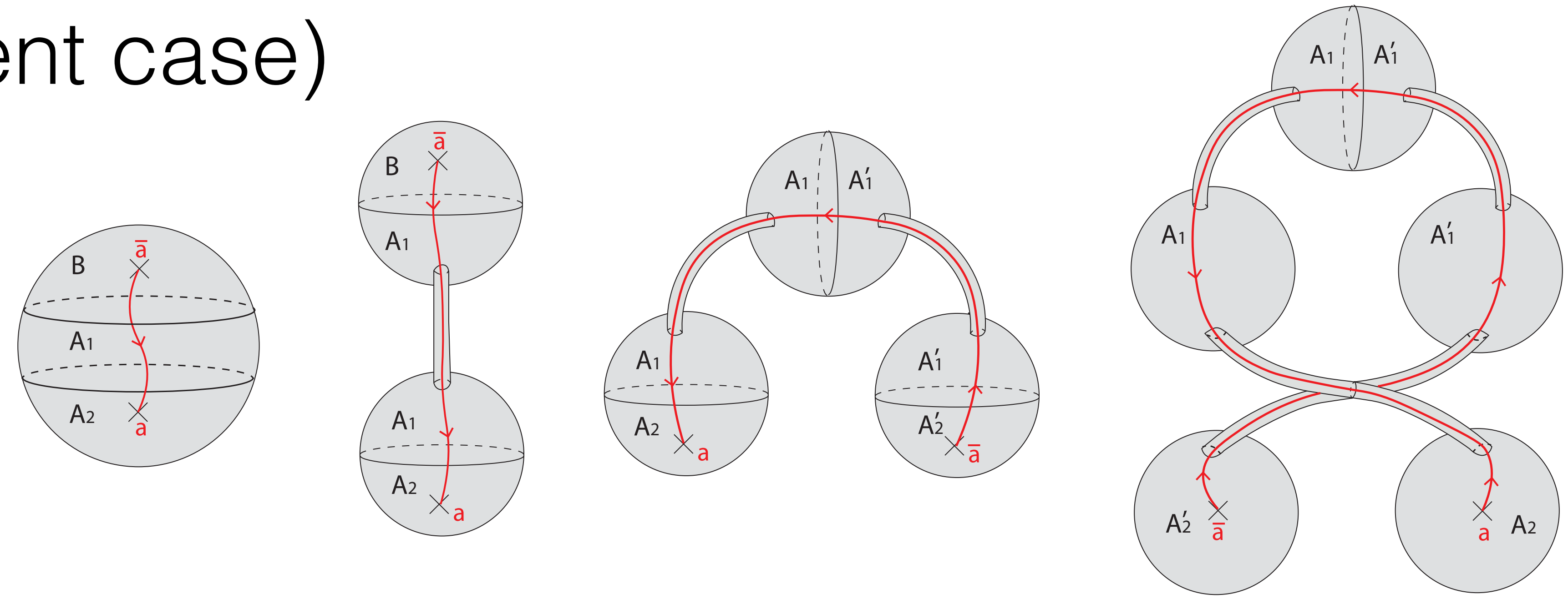
$$\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^{n_e}} = Z(S^3,\hat{R}_a)^{2-n_e} = (\mathcal{S}_{0a})^{2-n_e}$$

$$\mathcal{E}_{AB} = \lim_{n_e\to 1} \ln \frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \ln \mathcal{S}_{0a} = \ln d_a - \ln \mathcal{D}.$$

quantum dimension

$$d_a = \frac{\mathcal{S}_{0a}}{\mathcal{S}_{00}}. \qquad (\mathcal{S}_{00})^{-1} = \sqrt{\sum_i |d_i|^2} =: \mathcal{D}.$$

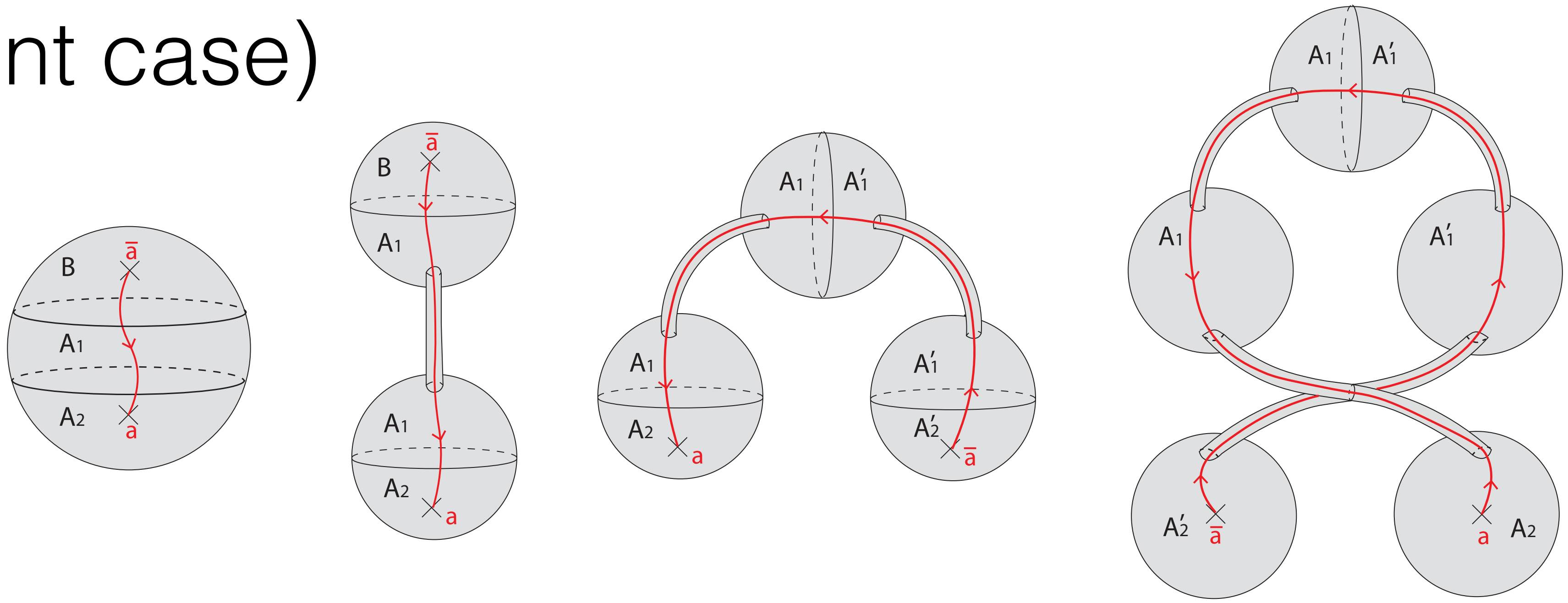
Ex2 (adjacent case)



(a) (b) (c) (d)

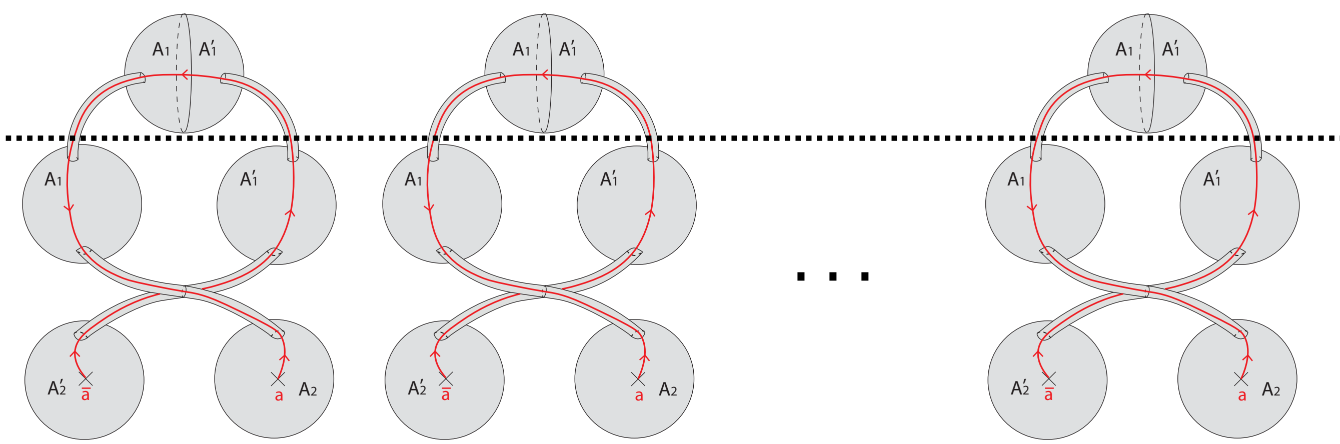
$|\Psi\rangle$ $\qquad \rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ $\qquad \rho^{T_{A_2}}_{A_1\cup A_2}$

Ex2 (adjacent case)



(a) (b) (c) (d)

$|\Psi\rangle$ $\quad$ $\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ $\quad$ $\rho_{A_1\cup A_2}^{T_{A_2}}$

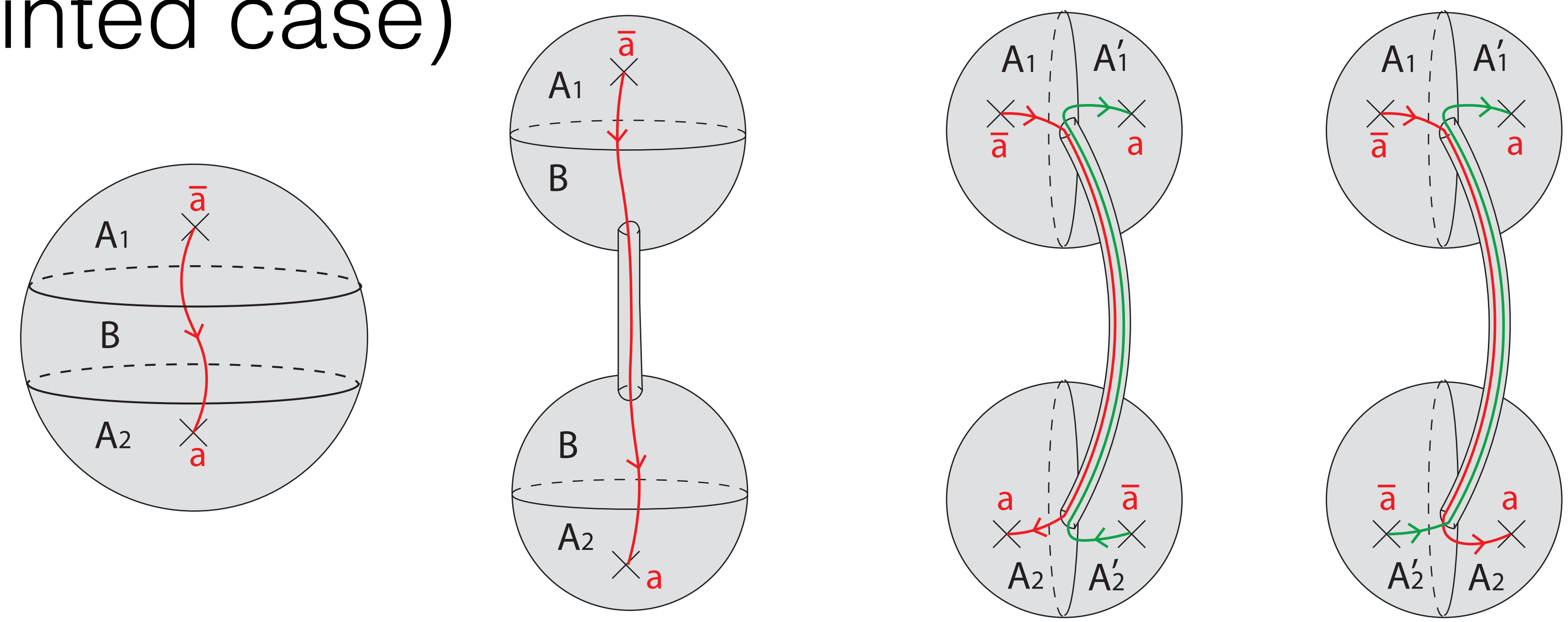


$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}} = \frac{1}{Z(S^3,\hat{R}_a)^{n_o}}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^{n_o}} = Z(S^3,\hat{R}_a)^{2-2n_o} = (\mathcal{S}_{0a})^{2-2n_o}$$

$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}} = \frac{1}{Z(S^3,\hat{R}_a)^{n_e}}\cdot\frac{Z(S^3,\hat{R}_a)^3}{Z(S^3,\hat{R}_a)^{n_e}} = Z(S^3,\hat{R}_a)^{3-2n_e} = (\mathcal{S}_{0a})^{3-2n_e}$$

$$\mathcal{E}_{A_1A_2}(B\neq\emptyset) = \mathcal{E}_{A_1A_2}(B=\emptyset).$$

Ex3 (disjointed case)



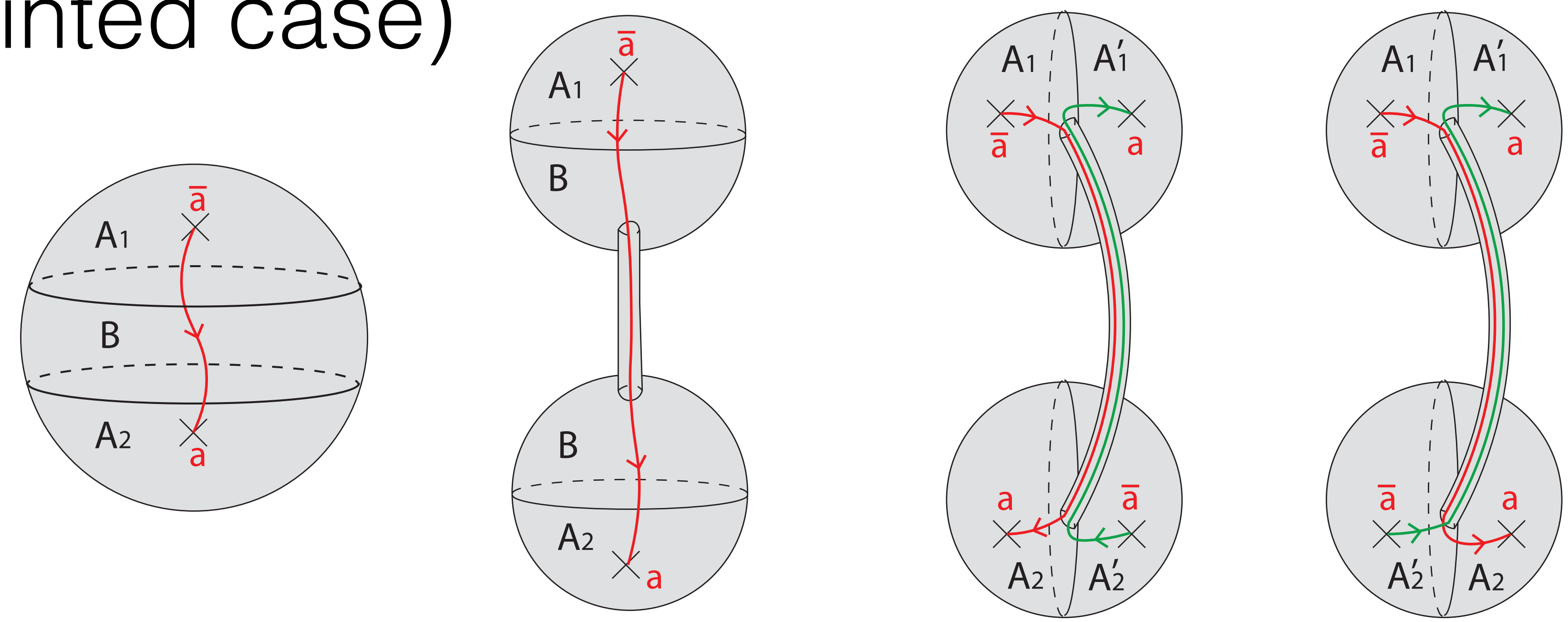
(a) (b) (c) (d)

$|\Psi\rangle$

$\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

$\rho^{T_{A_2}}_{A_1\cup A_2}$

Ex3 (disjointed case)



(a) (b) (c) (d)

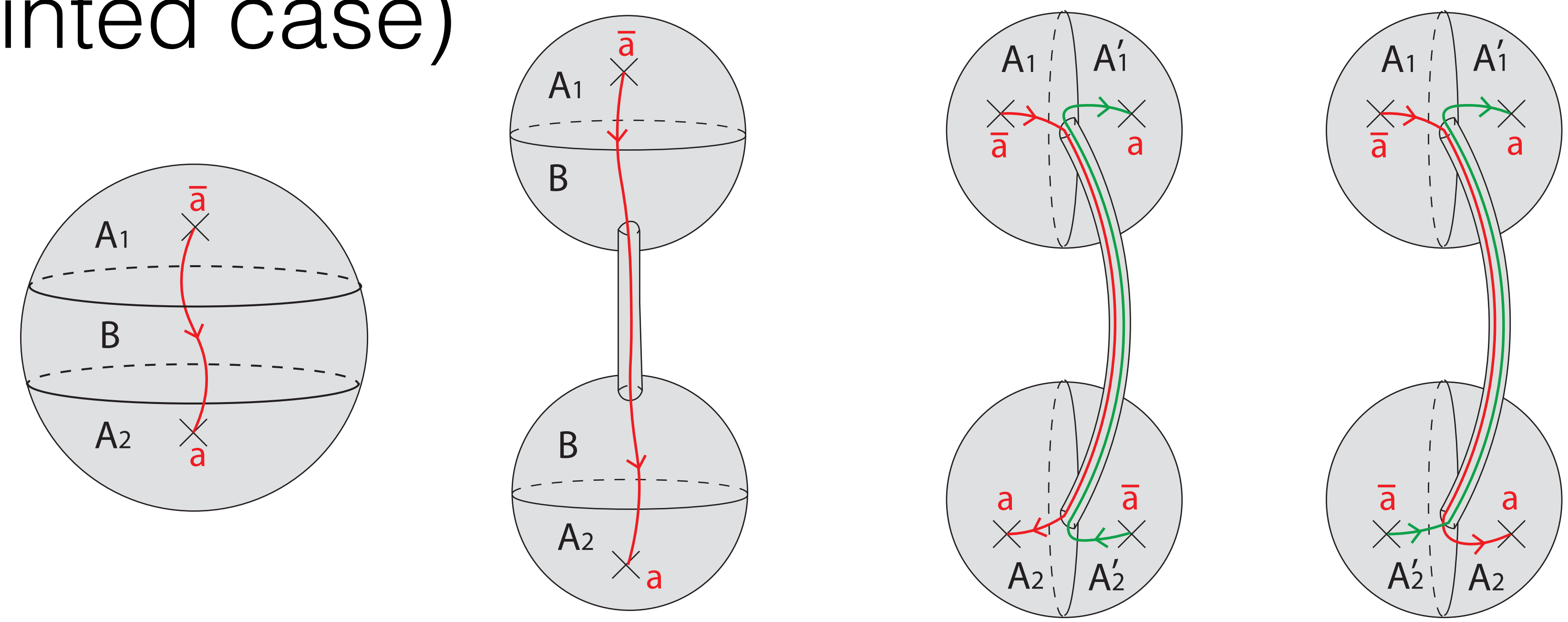
$|\Psi\rangle$

$\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

$\rho_{A_1\cup A_2}^{T_{A_2}}$

$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^n}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^n} = \frac{1}{Z(S^3,\hat{R}_a)^n}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^n} = Z(S^3,\hat{R}_a)^{2-2n} = (\mathcal{S}_{0a})^{2-2n}$$

Ex3 (disjointed case)



(a) $|\Psi\rangle$

(b)

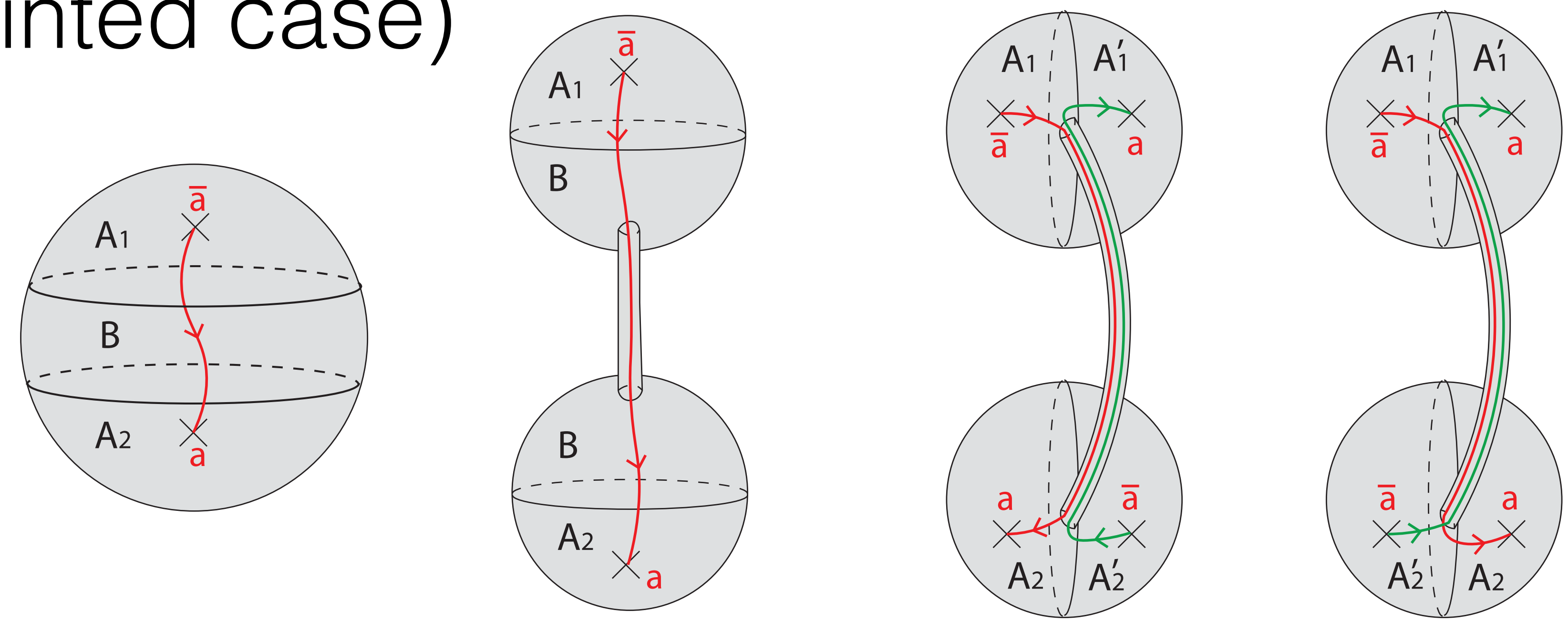
(c) $\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

(d) $\rho^{T_{A_2}}_{A_1\cup A_2}$

$$\frac{\mathrm{tr}\left(\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n}{\left(\mathrm{tr}\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n} = \frac{1}{Z(S^3,\hat{R}_a)^n}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^n} = Z(S^3,\hat{R}_a)^{2-2n} = (\mathcal{S}_{0a})^{2-2n}$$

$$\mathcal{E}_{A_1A_2} = \lim_{n_e\to 1}\ln\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \ln(\mathcal{S}_{0a})^0 = 0.$$

Ex3 (disjointed case)



(a) (b) (c) (d)

$|\Psi\rangle$

$\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$

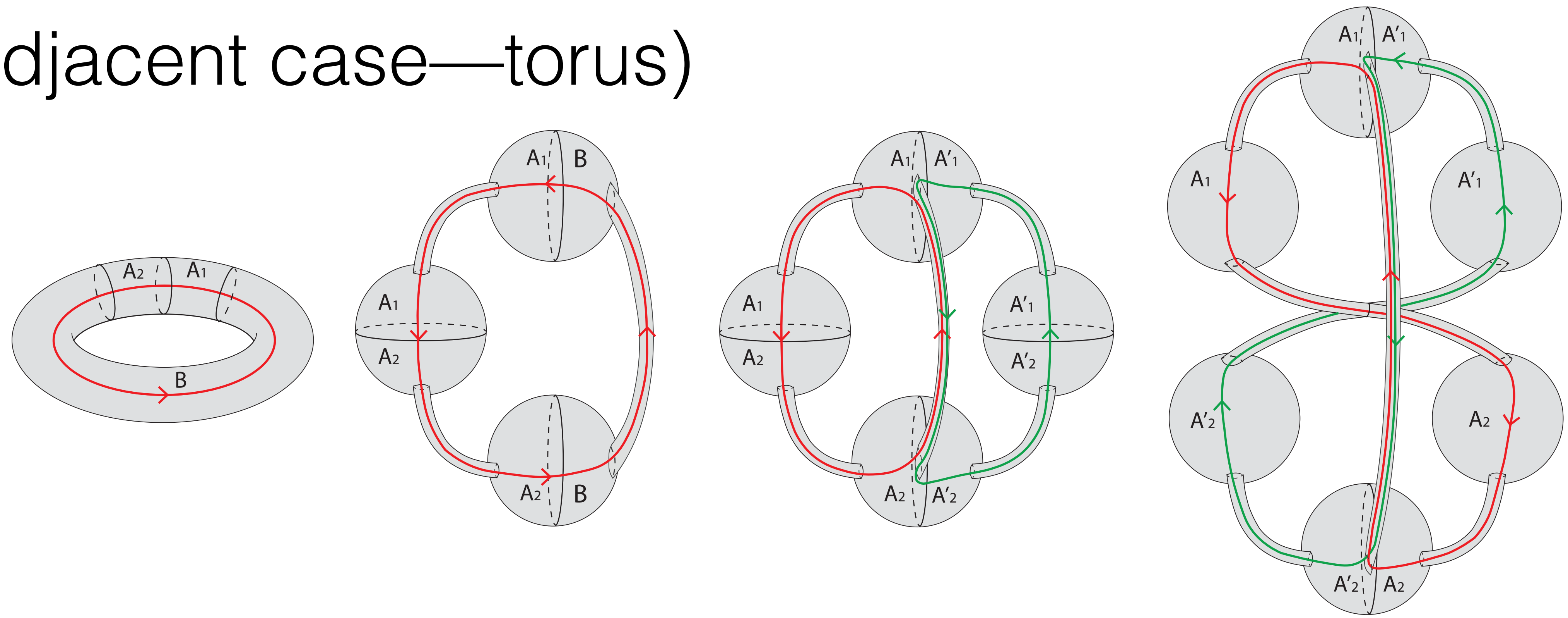
$\rho^{T_{A_2}}_{A_1\cup A_2}$

$$\frac{\mathrm{tr}\left(\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n}{\left(\mathrm{tr}\rho^{T_{A_2}}_{A_1\cup A_2}\right)^n} = \frac{1}{Z(S^3,\hat{R}_a)^n}\cdot\frac{Z(S^3,\hat{R}_a)^2}{Z(S^3,\hat{R}_a)^n} = Z(S^3,\hat{R}_a)^{2-2n} = (\mathcal{S}_{0a})^{2-2n}$$

$$\mathcal{E}_{A_1A_2} = \lim_{n_e\to 1}\ln\frac{\mathrm{tr}\left(\rho^{T_B}\right)^{n_e}}{(\mathrm{tr}\rho^{T_B})^{n_e}} = \ln(\mathcal{S}_{0a})^0 = 0.$$

No entanglement if A1 and A2 do not have interfaces!

Ex4 (adjacent case—torus)



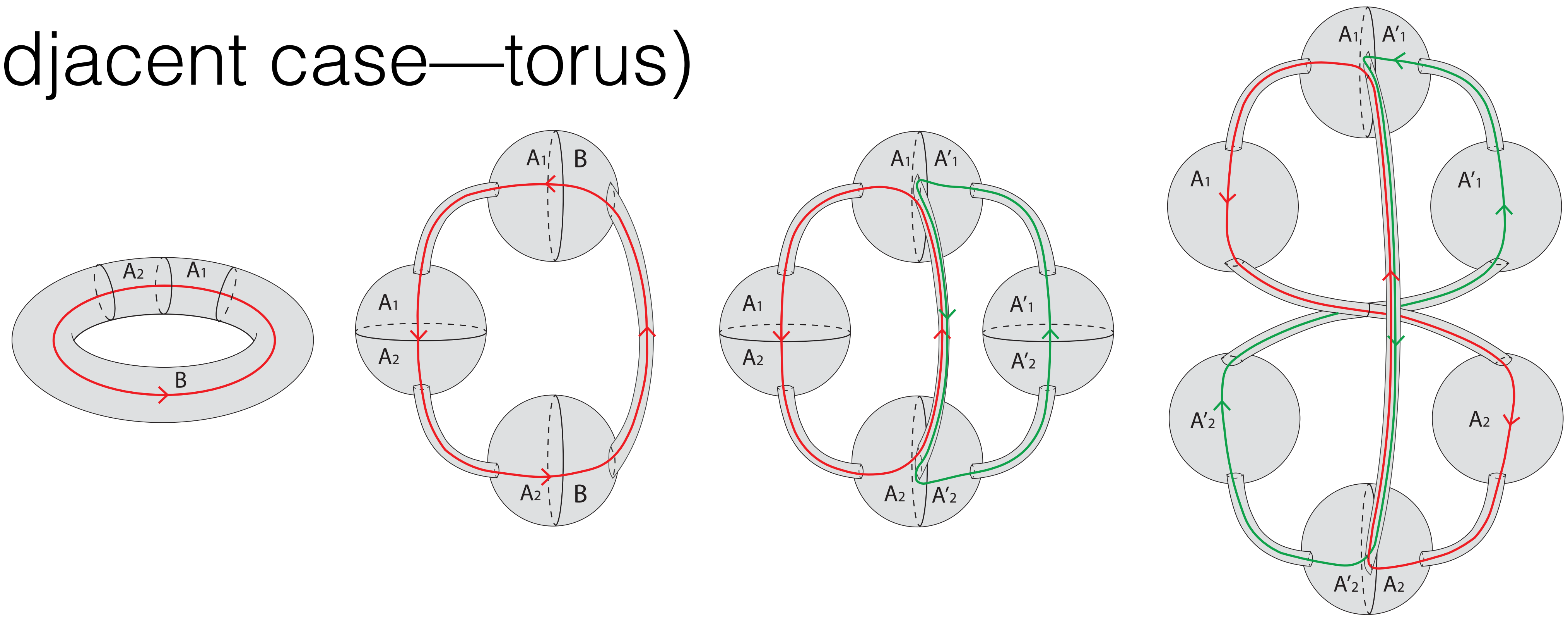
(a) (b) (c) (d)

$|\Psi\rangle$

$\rho_{A_1 \cup A_2} = \mathrm{tr}_B |\Psi\rangle\langle\Psi|$

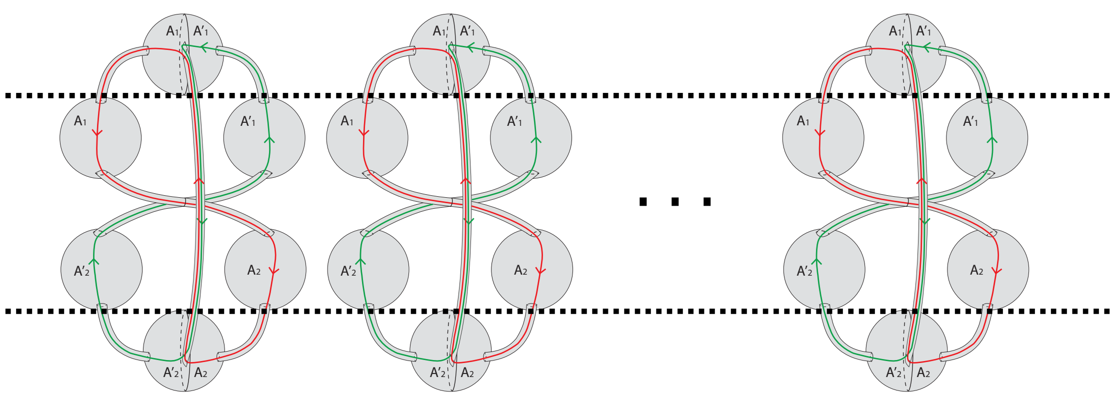
$\rho_{A_1 \cup A_2}^{T_{A_2}}$

Ex4 (adjacent case—torus)



(a) (b) (c) (d)

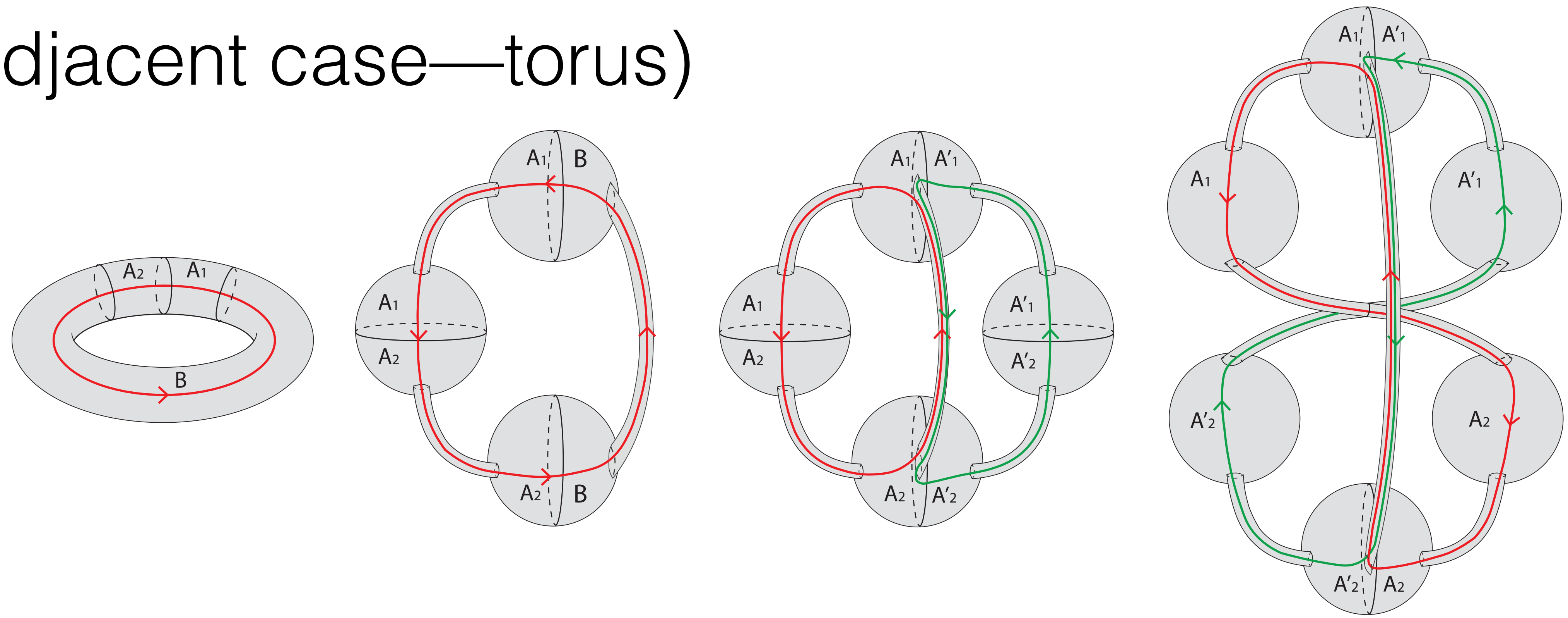
$|\Psi\rangle$ $\qquad \rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ $\qquad \rho_{A_1\cup A_2}^{T_{A_2}}$



$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}} = \frac{1}{Z(S^2\times S^1,\hat{R}_a,\hat{\overline{R}}_a)^{n_o}}\cdot\frac{Z(S^3,\hat{R}_a)^3}{Z(S^3,\hat{R}_a)^{3n_o}} = Z(S^3,\hat{R}_a)^{3-3n_o} = (\mathcal{S}_{0a})^{3-3n_o}$$

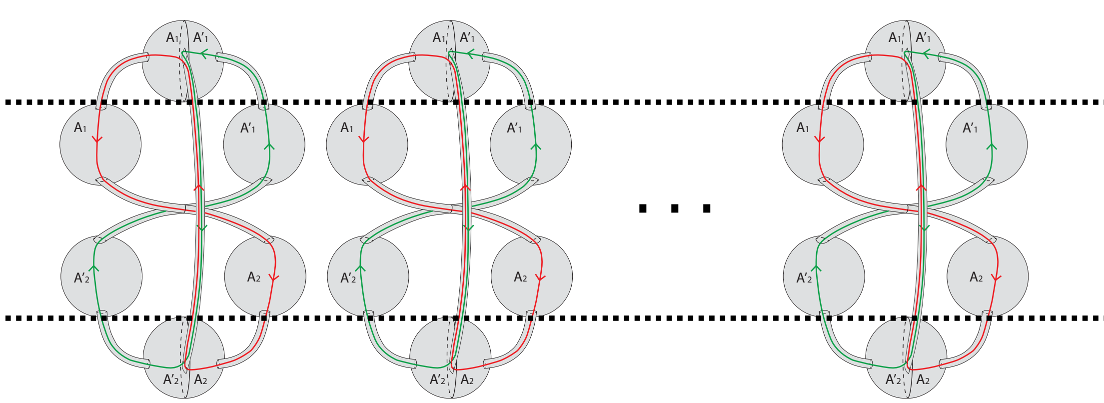
$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}} = \frac{1}{Z(S^2\times S^1,\hat{R}_a,\hat{\overline{R}}_a)^{n_e}}\cdot\frac{Z(S^3,\hat{R}_a)^4}{Z(S^3,\hat{R}_a)^{3n_e}} = Z(S^3,\hat{R}_a)^{4-3n_e} = (\mathcal{S}_{0a})^{4-3n_e}$$

Ex4 (adjacent case—torus)



(a) (b) (c) (d)

$|\Psi\rangle$ $\qquad$ $\rho_{A_1\cup A_2} = \mathrm{tr}_B|\Psi\rangle\langle\Psi|$ $\qquad$ $\rho_{A_1\cup A_2}^{T_{A_2}}$



$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}} = \frac{1}{Z(S^2\times S^1,\hat{R}_a,\hat{\overline{R}}_a)^{n_o}}\cdot\frac{Z(S^3,\hat{R}_a)^3}{Z(S^3,\hat{R}_a)^{3n_o}} = Z(S^3,\hat{R}_a)^{3-3n_o} = (\mathcal{S}_{0a})^{3-3n_o}$$

$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}} = \frac{1}{Z(S^2\times S^1,\hat{R}_a,\hat{\overline{R}}_a)^{n_e}}\cdot\frac{Z(S^3,\hat{R}_a)^4}{Z(S^3,\hat{R}_a)^{3n_e}} = Z(S^3,\hat{R}_a)^{4-3n_e} = (\mathcal{S}_{0a})^{4-3n_e}$$

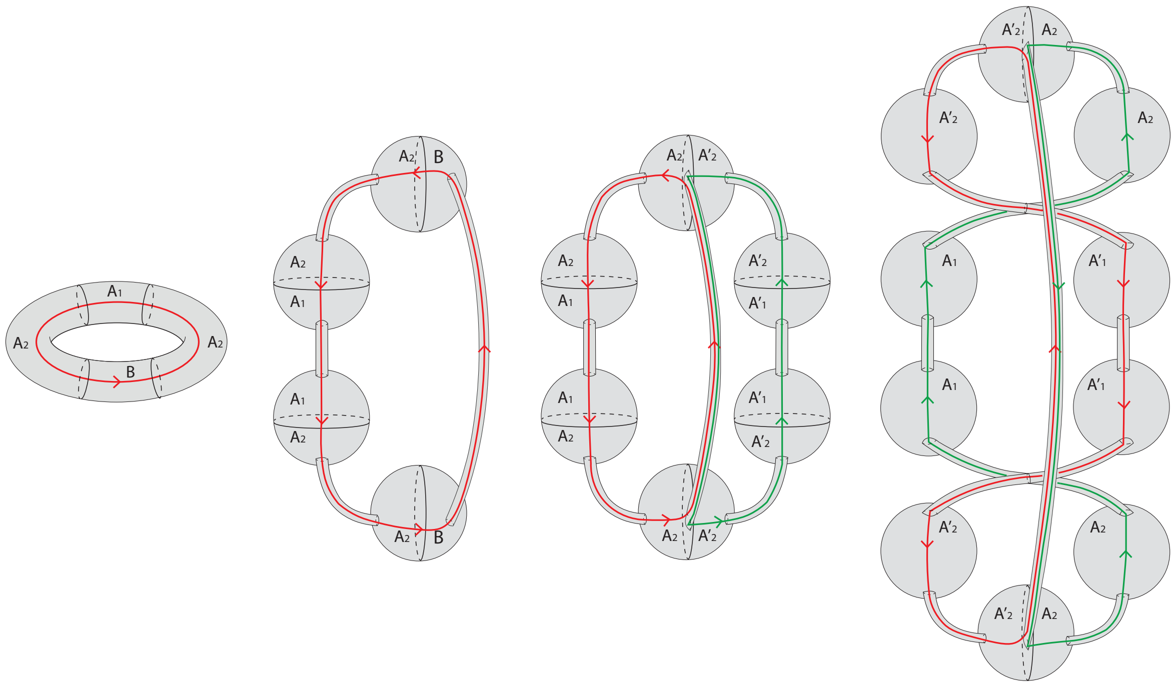
A general state

$$|\psi\rangle = \sum_j \psi_j|\hat{R}_j\rangle$$

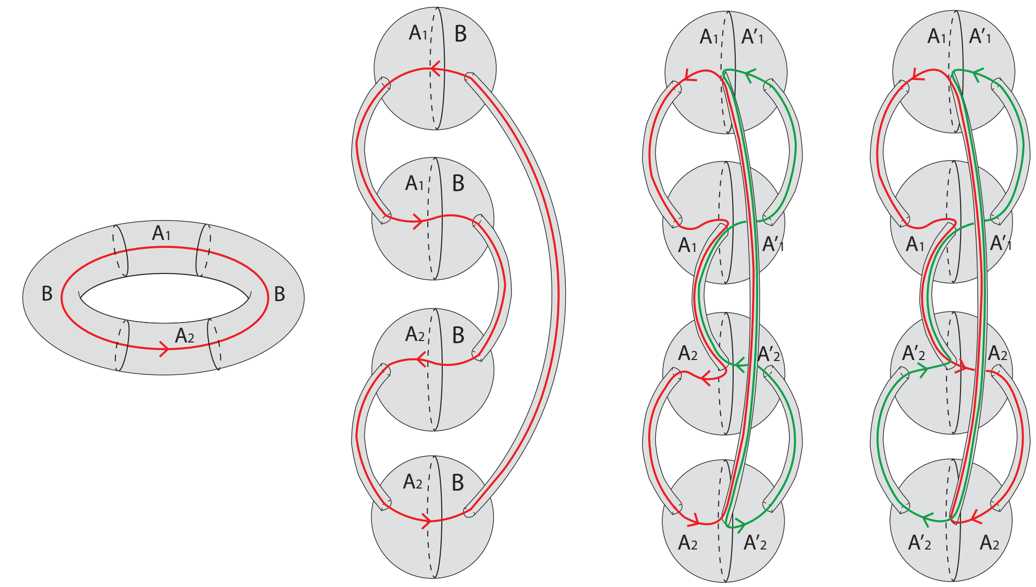
$$\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_o}} = \frac{\sum_j|\psi_j|^{2n_o}\left(\mathcal{S}_{0j}\right)^{3-3n_o}}{\left(\sum_j|\psi_j|^2\right)^{n_o}},\quad \frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}} = \frac{\sum_j|\psi_j|^{2n_e}\left(\mathcal{S}_{0j}\right)^{4-3n_e}}{\left(\sum_j|\psi_j|^2\right)^{n_e}}.$$

$$\mathcal{E}_{A_1A_2} = \lim_{n_e\to 1}\ln\mathrm{tr}\frac{\mathrm{tr}\left(\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}}{\left(\mathrm{tr}\rho_{A_1\cup A_2}^{T_{A_2}}\right)^{n_e}} = \ln\left(\sum_j|\psi_j|^2\mathcal{S}_{0j}\right) - \ln\sum_j|\psi_j|^2.$$

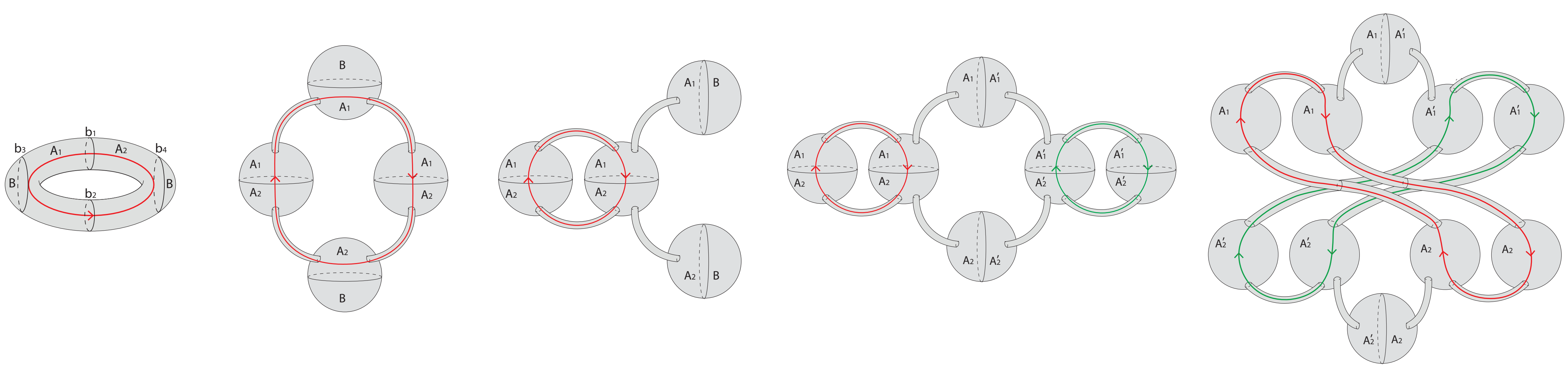
More cases



(a) (b) (c) (d)



(a) (b) (c) (d)



Conclusion:

- Entanglement negativity is always zero for disjointed intervals.
- Entanglement negativity depends on the number of interfaces between A1 and A2.
- Entanglement negativity depends on the choice of ground state. — can distinguish Abelian and non-Abelian theories.

Questions:

- Generalization for higher dimensions?
- Non-chiral topological field theories?