The power of indistinguishable particles in quantum computation

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- Indistinguishable particles: quantum field theory
- Cluster states
- Measurement-based quantum computing
- Fermions are maximally entangled, but….
- Conclusions

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Indistinguishable particles

- All particles in the Universe come in two varieties: bosons (mostly mediators of forces) and fermions (mostly matter).

- Atoms are comprised of fermions, but (viewed from a distance) can be either bosonic (even # of constituent fermions) or fermionic (odd # of constituent fermions)

- Much work on making quantum degenerate atoms:

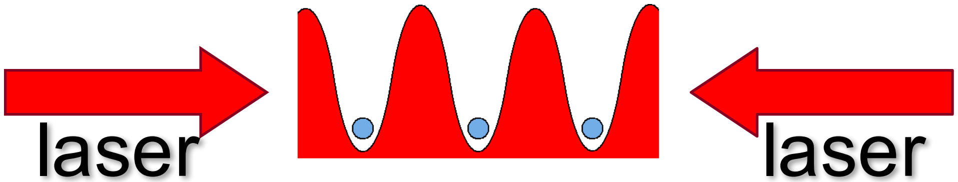
➢ Bosons: ^{1}H, ^{4}He*, ^{7}Li, ^{23}Na, ^{52}Cr, $^{85/87}$Rb, ^{133}Cs, etc

➢ Fermions: ^{6}Li, ^{40}K, ^{53}Cr, etc

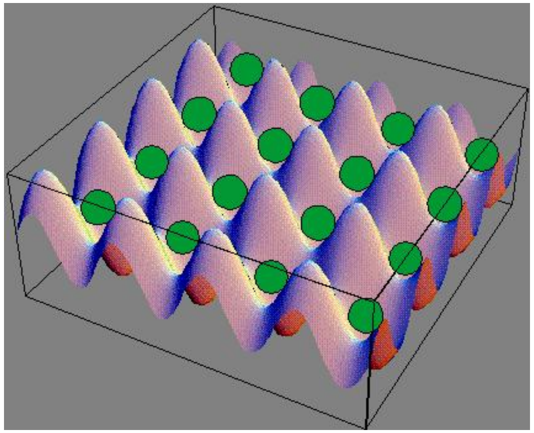
Indistinguishable particles

- Ultracold atoms can be confined in ‘optical lattices’

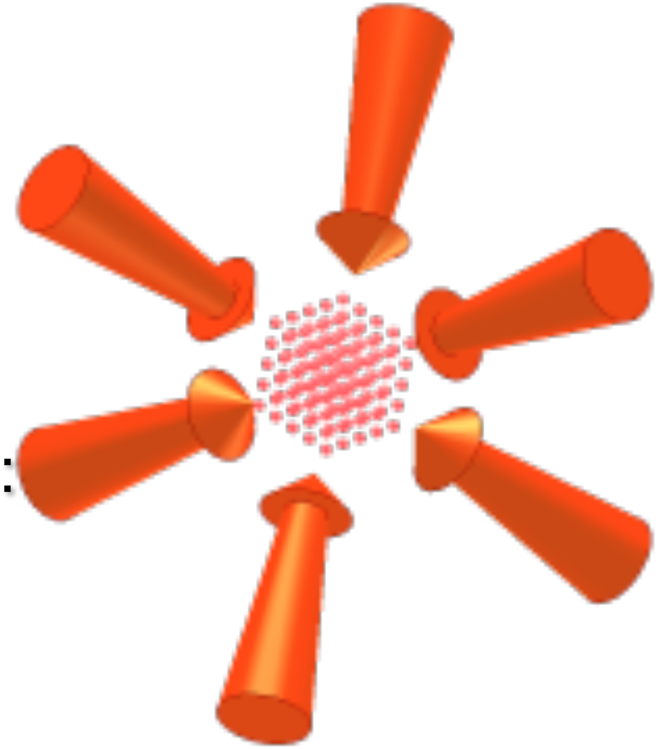
  - One-dimensional lattice:



  - 2D lattice:



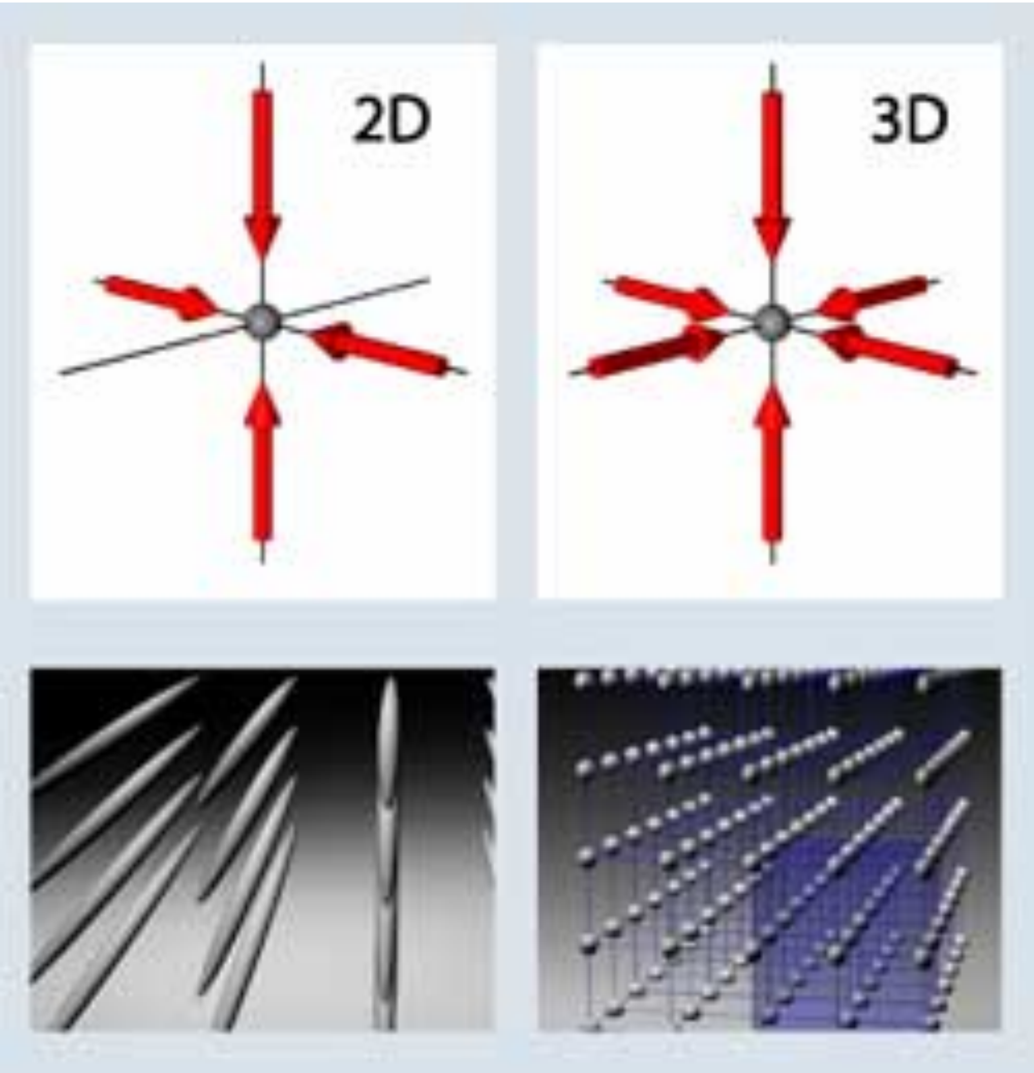
  - 3D lattice:



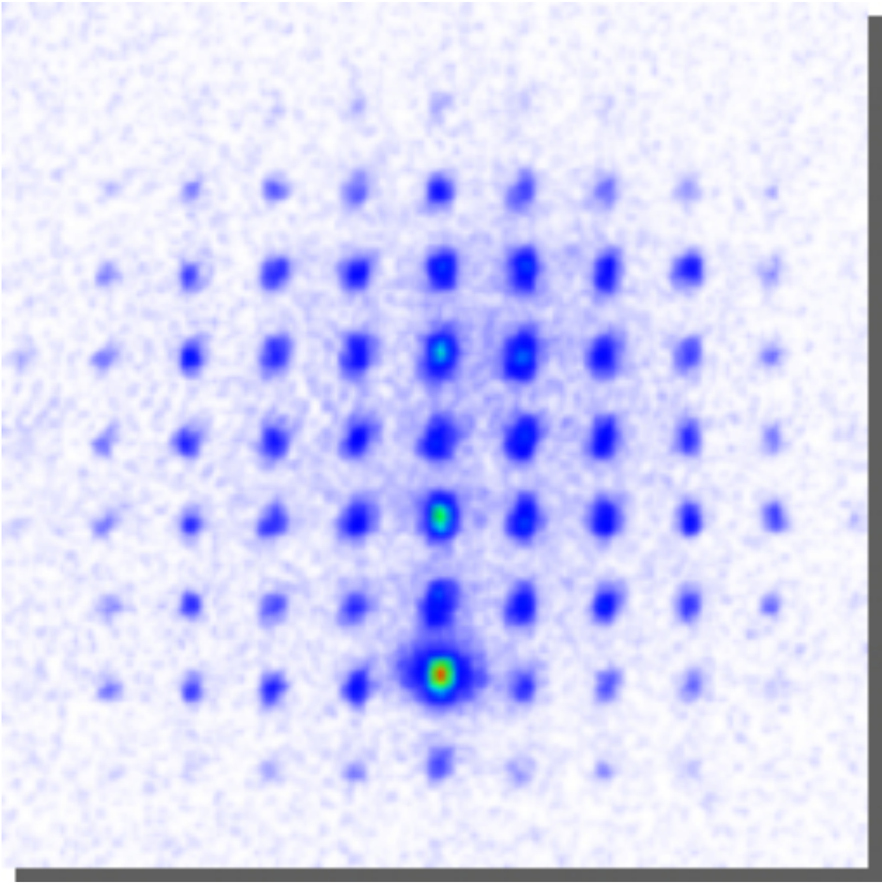
- Approximately 100 sites/dimension.

Indistinguishable particles

Can make effective 3D, 2D, or 1D optical lattices:



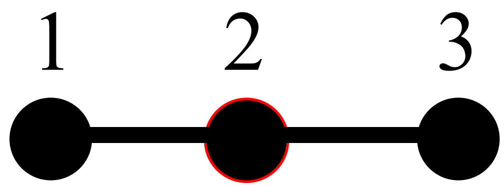
(Markus Greiner)



(Immanuel Bloch)

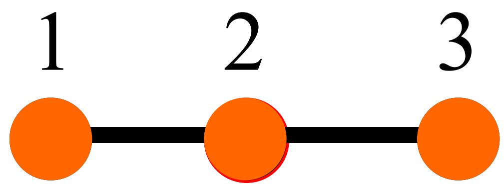
Indistinguishable particles

- Represent the sites of a lattice as a graph:



Indistinguishable particles

- Represent the sites of a lattice as a graph:



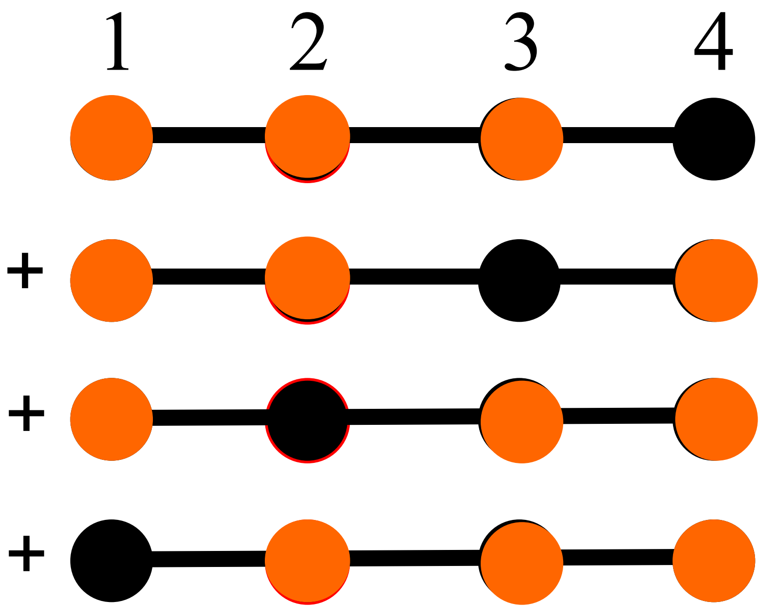
- Suppose that there are three fermions:

$$
\begin{aligned}
\psi(r_1, r_2, r_3) &= \phi_1(r_1)\left[\phi_2(r_2)\phi_3(r_3) - \phi_3(r_2)\phi_2(r_3)\right] \\
&- \phi_2(r_1)\left[\phi_1(r_2)\phi_3(r_3) - \phi_3(r_2)\phi_1(r_3)\right] \\
&+ \phi_3(r_1)\left[\phi_1(r_2)\phi_2(r_3) - \phi_2(r_2)\phi_1(r_3)\right] \\
&= \begin{vmatrix} \phi_1(r_1) & \phi_2(r_1) & \phi_3(r_1) \\ \phi_1(r_2) & \phi_2(r_2) & \phi_3(r_2) \\ \phi_1(r_3) & \phi_2(r_3) & \phi_3(r_3) \end{vmatrix}
\end{aligned}
$$

'Slater determinant' – accounts for fermionic antisymmetry

Indistinguishable particles

- Suppose that there are four sites instead:



- Too many Slater determinants – unwieldy notation

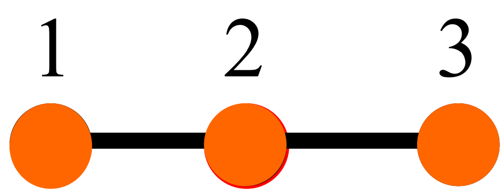
- With bosons, we need to use permanents instead; one also has more terms because of multiple occupancy of sites.

Indistinguishable particles

- Quantum field theory makes the description more efficient. Generic Hamiltonian is written in terms of quantum fields:

$$H = \sum_i \hat{\psi}^\dagger(\mathbf{x}_i)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\hat{\psi}(\mathbf{x}_i) + \sum_{ij}\hat{\psi}^\dagger(\mathbf{x}_i)\hat{\psi}^\dagger(\mathbf{x}_j)V(\mathbf{x}_i,\mathbf{x}_j)\hat{\psi}(\mathbf{x}_j)\hat{\psi}(\mathbf{x}_i)$$

- Expand quantum fields in suitable basis:



$$\hat{\psi}(\mathbf{x}_i) = \sum_{\mathbf{n}} \phi_{\mathbf{n}}(\mathbf{x}_i)\hat{c}_{\mathbf{n}}, \quad \hat{c}_{\mathbf{n}} = \begin{cases} \hat{f}_{\mathbf{n}}, & \text{fermions} \\ \hat{b}_{\mathbf{n}}, & \text{bosons} \end{cases}$$

$$\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}; \quad \left\{\hat{f}_i, \hat{f}_j^\dagger\right\} = \delta_{ij}.$$

Indistinguishable particles

- If M is number of sites and N is number of particles, then Hilbert space dimension is:

$$\Omega^{(\mathrm{F})} = \frac{M!}{N!(M-N)!}; \quad \Omega^{(\mathrm{B})} = \frac{(M+N-1)!}{N!(M-1)!}.$$

$$\sim \left(\frac{M}{N}\right)^{N} \qquad \sim \left(\frac{M}{N}+1\right)^{N}$$

- If $M >> N$ then the Hilbert space dimension grows exponentially in the number of particles.

Indistinguishable particles

- Exponentially growing Hilbert space doesn't mean that simulating indistinguishable particles is classically inefficient.

- If particles are non-interacting, then all properties can be obtained from (time-evolution of) single-particle states:

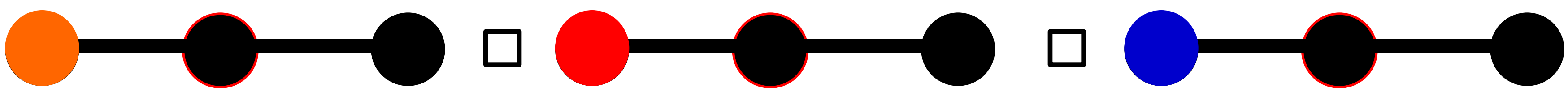
$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = i\hbar\frac{\partial}{\partial t}\sum_j \alpha_j\phi_j(r) = H\sum_j \alpha_j\phi_j(r)$$

$$\Rightarrow \psi(r,t) = \sum_j \alpha_j e^{-i\lambda_j t/\hbar}\phi_j(r).$$

- Need only know initial occupations $|\alpha_j|^2$

Indistinguishable particles

- Pretend that 3 bosons are actually distinguishable:



$$H_{\rm tot} = H \otimes I \otimes I \quad + \quad I \otimes H \otimes I \quad + \quad I \otimes I \otimes H$$

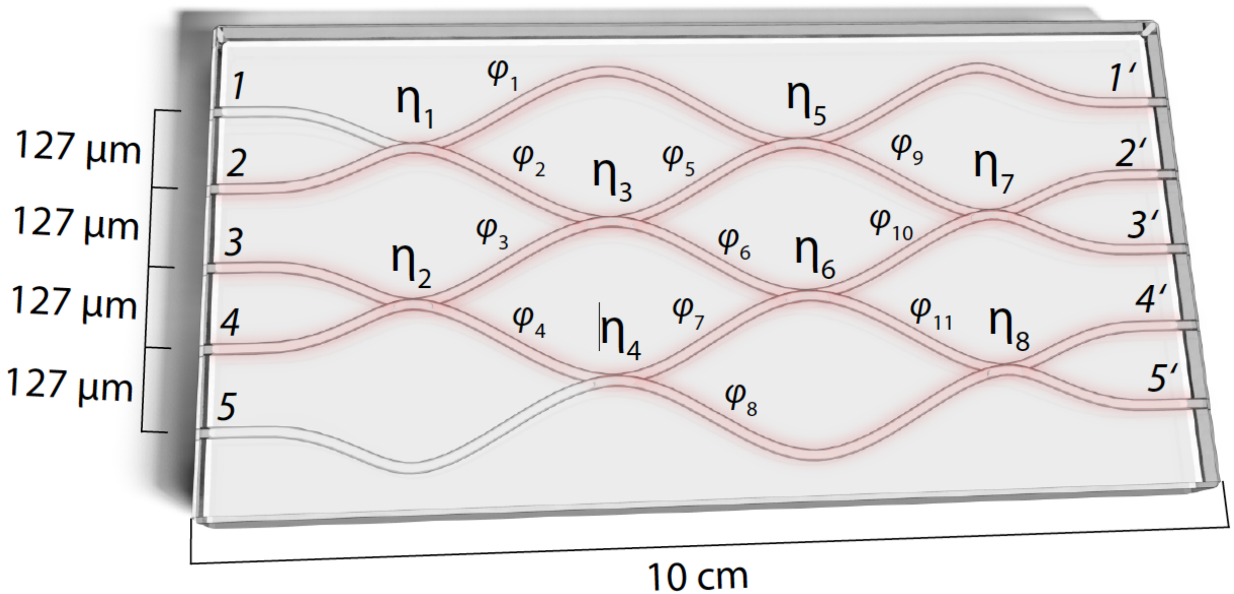
- Projecting into indistinguishable space requires repeating sums over identical labels: inefficient in principle.

- But don’t need to in practice for bosons: all observables are simply N-fold multiples of single-particle quantities!

Indistinguishable particles

- So why is *boson sampling* [Aaronson and Arkhipov, STOC 2011, p. 333] classically difficult?

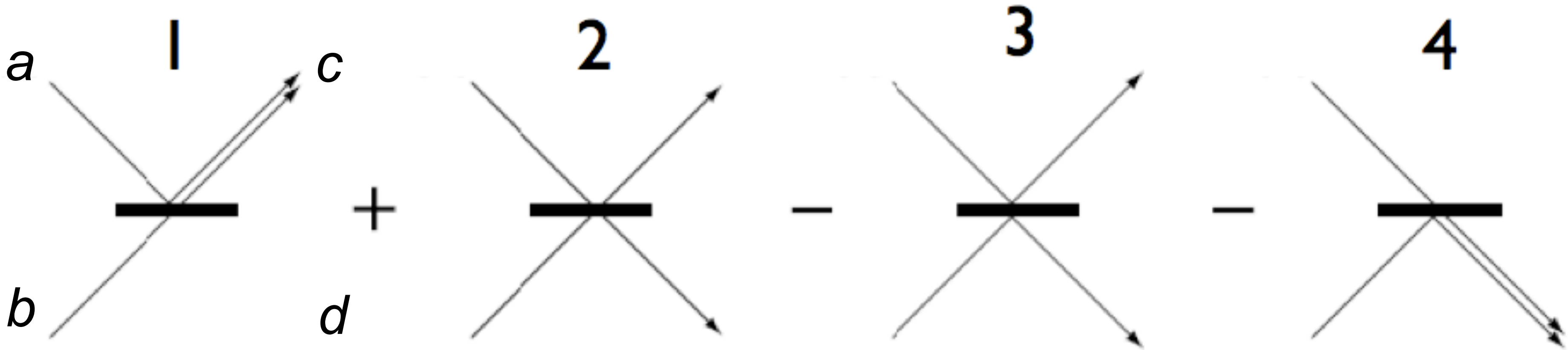
- Given some input to an optical circuit, what is the photon distribution at the output?



[Tillman et al, Nature Photonics **7**, 540–544 (2013)]

Indistinguishable particles

- Photons effectively interact! Hong-Ou-Mandel effect (photon bunching):



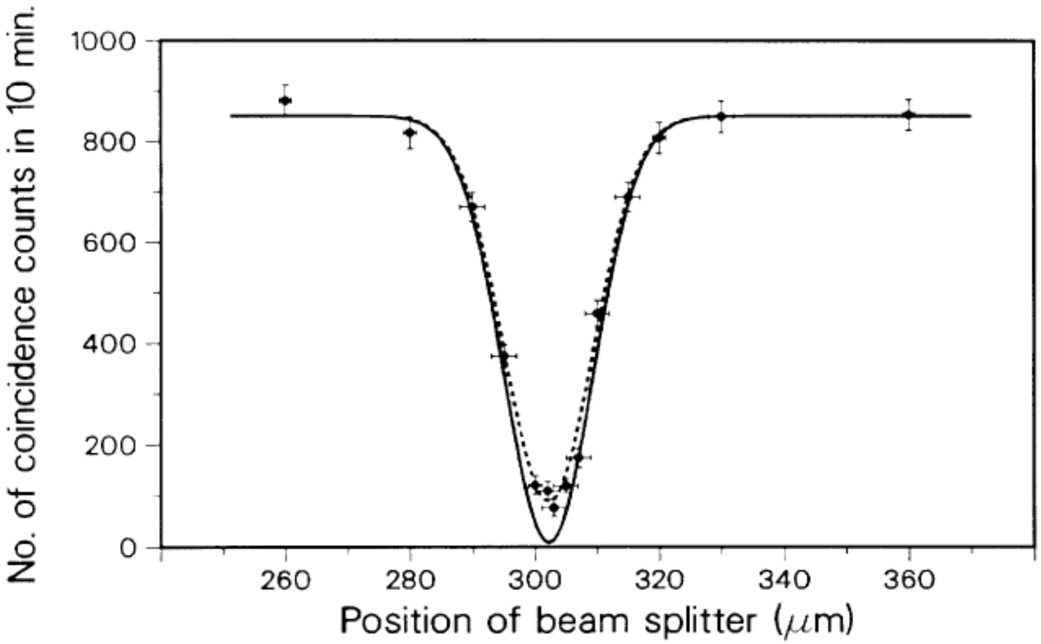
(reflection off the bottom gives a negative sign)

$$\hat{a}^\dagger \rightarrow \frac{\hat{c}^\dagger + \hat{d}^\dagger}{\sqrt{2}}; \quad \hat{b}^\dagger \rightarrow \frac{\hat{c}^\dagger - \hat{d}^\dagger}{\sqrt{2}}$$

$$|1,1\rangle_{a,b} = \hat{a}^\dagger \hat{b}^\dagger |\mathbf{0}\rangle \rightarrow \frac{1}{2}\left(\hat{c}^{\dagger\,2} - \hat{d}^{\dagger\,2}\right)|\mathbf{0}\rangle$$

Indistinguishable particles

- Photons effectively interact! Hong-Ou-Mandel effect (photon bunching):



[Hong, Ou, Mandel, PRL **59**, 2044 (1987)]

- Projecting into indistinguishable space is inefficient; no short cut because observables are **not** N-fold multiples of single-particle quantities: hard problem!

Indistinguishable particles

For bosons, need to evaluate 'Slater permanents', which is hard (Calculating permanents is #P-complete [Valiant, Theor. Comp. Sci. **8**, 189 (1979); also Aaronson, Proc. R. Soc. A **467**, 3393 (2011)])

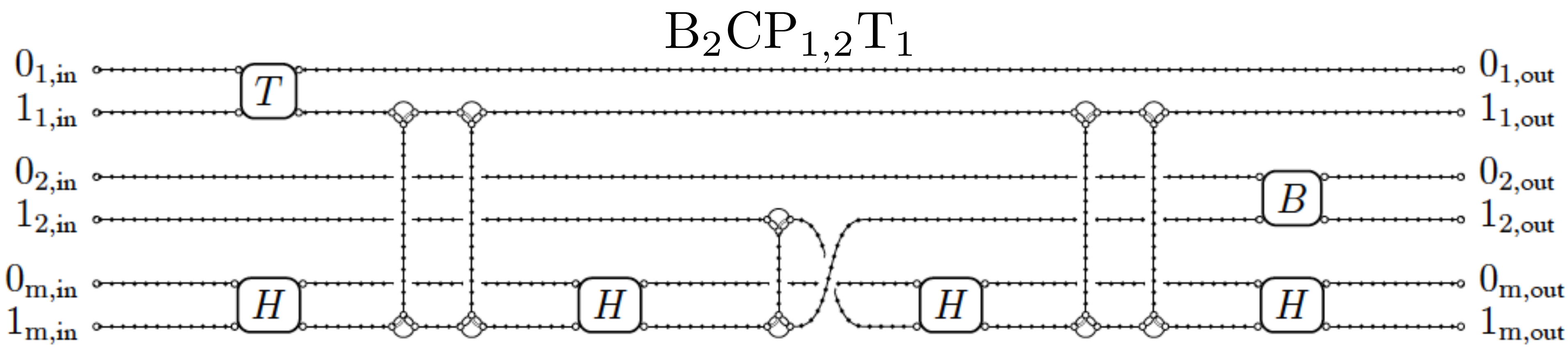
- ➢ NP example: Are there any subsets of a list of integers that add up to zero?
- ➢ #P example: How many subsets of a list of integers add up to zero?

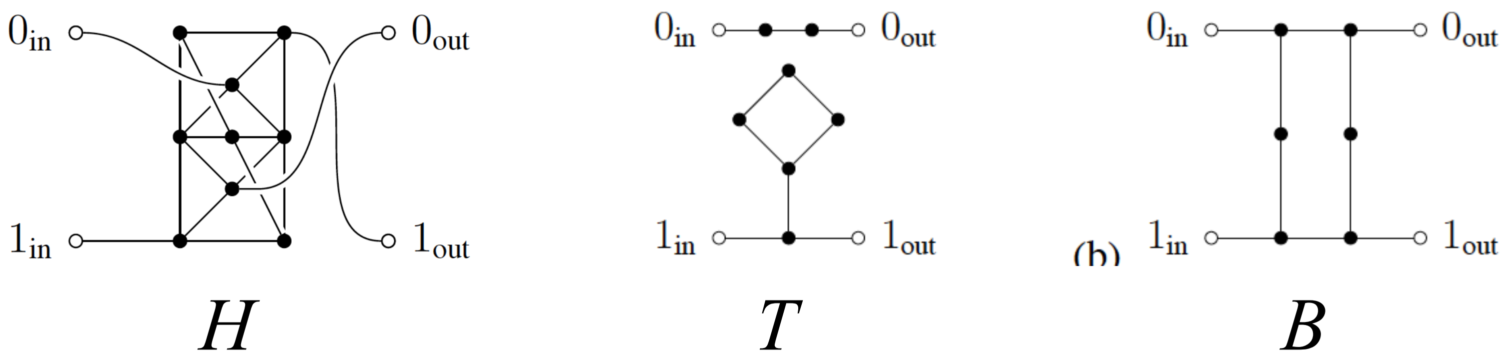
- Even though boson sampling is (likely) classically hard, can it be used to do anything interesting? Maybe not.

- But are interacting indistinguishable bosons powerful?

Indistinguishable particles

Quantum walks with *interacting* indistinguishable bosons can perform universal quantum computation (Childs, Gosset, and Webb, Science **339**, 791 (2013); also Underwood and Feder, Phys. Rev. A **85**, 052314 (2012)])





[Childs, Gosset, and Webb, Science **339**, 791 (2013)]

Indistinguishable particles

What about fermions?

- Perhaps surprisingly, non-interacting fermions are classically efficient to simulate! Calculating determinants is in P (Using Gaussian elimination the complexity scales with d like d^3).

- That said, interacting bosons are easy to *approximate* in quantum Monte Carlo, but interacting fermions are not (because of the sign problem).

- Of course, d is scaling exponentially with the number of particles N….

Indistinguishable particles

The behavior of non-interacting fermions can be simulated by *matchgates* acting on two spin-1/2 particles:

[Valiant, SIAM J. Comput. **31**, 1229 (2002); Terhal and DiVincenzo, Phys. Rev. A **65**, 032325 (2002); Brayvi, Contemp. Math. **482**, 179 (2009); Jozsa, Kraus, Miyake, Watrous, Proc. R. Soc. **466**, 809 (2010)]

$$G(A,B) = \begin{pmatrix} p & 0 & 0 & q \\ 0 & w & x & 0 \\ 0 & y & z & 0 \\ r & 0 & 0 & s \end{pmatrix}, \; A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \; B = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

$$\det(A) = \det(B) \text{ and } A, B \in \mathrm{U}(2) \text{ or } S\mathrm{U}(2)$$

• If matchgates only act on nearest-neighboring spins, the behavior can be efficiently simulated classically.

Indistinguishable particles

What is the relationship between matchgates and non-interacting fermions?

- Matchgate group is generated by (XX, YY, IZ, ZI, XY, and YX), where i.e. $XX \equiv \sigma_x \otimes \sigma_x$

• Is there a relationship between fermions and spins? Fermions always anticommute (no matter what site they are on):

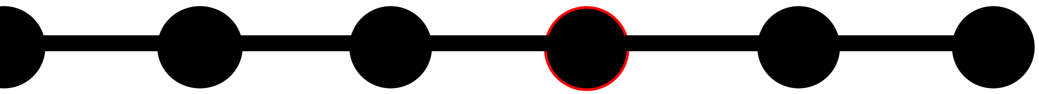
$$\hat{f}_i \hat{f}_j^\dagger = -\hat{f}_j^\dagger \hat{f}_i + \delta_{ij}; \quad \hat{f}_i \hat{f}_j = -\hat{f}_j \hat{f}_i; \quad \hat{f}_i^\dagger \hat{f}_j^\dagger = -\hat{f}_j^\dagger \hat{f}_i^\dagger$$

• Spins only anticommute if they are on the same site; they *commute* otherwise. For example:

$$X_i Z_i = -Z_i X_i; \quad X_i Z_j = Z_j X_i .$$

Indistinguishable particles

• In fact, spins and fermions are connected through the Jordan-Wigner transformation:



Indistinguishable particles

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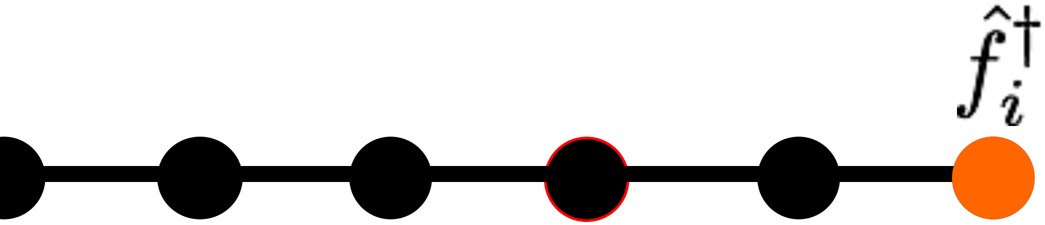

Indistinguishable particles

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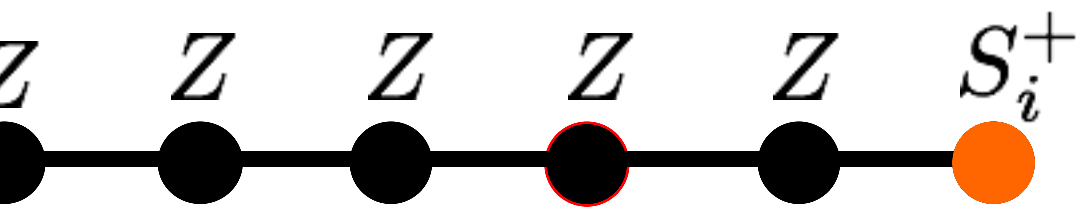


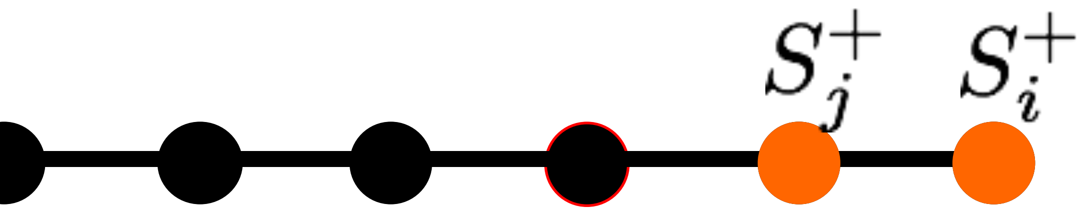
$$S_i^+ = X_i + iY_i$$

$$Z_i = 1 - 2\hat{f}_i^\dagger \hat{f}_i$$

• Note that in the spin representation, fermionic operators are strongly non-local!

Indistinguishable particles

• In fact, spins and fermions are connected through the Jordan-Wigner transformation:



$$S_i^+ = X_i + iY_i$$

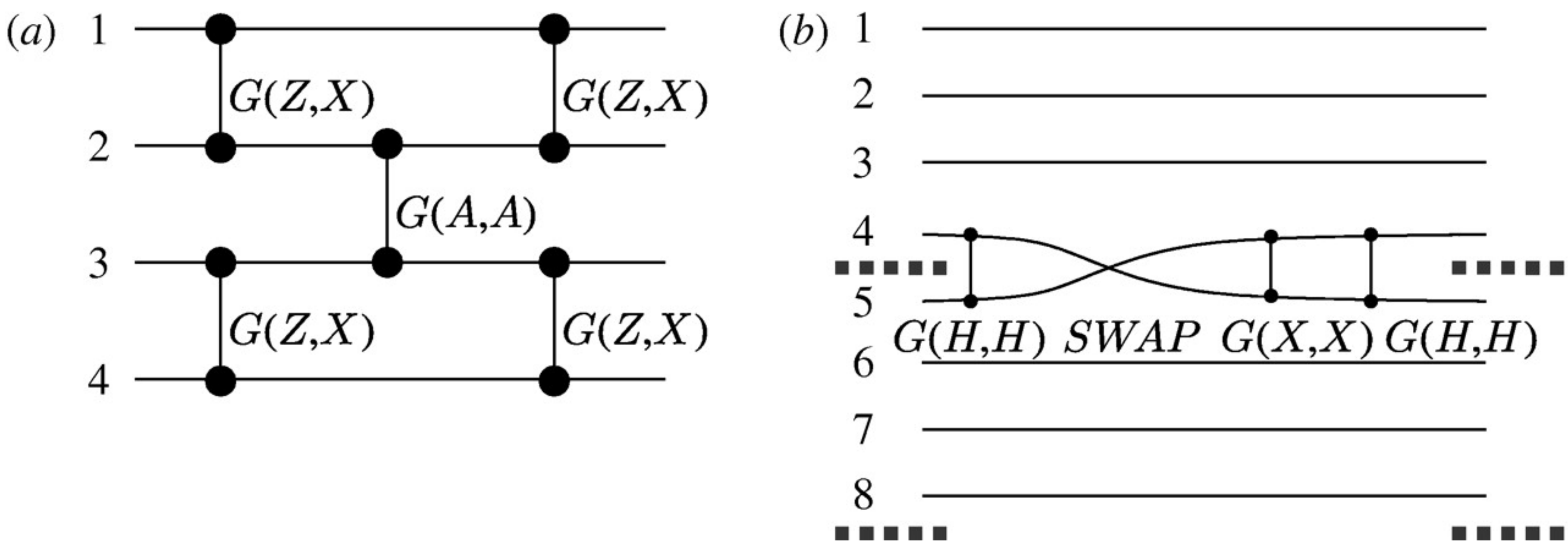
$$Z_i = 1 - 2\hat{f}_i^\dagger \hat{f}_i$$

• So, nearest-neighbor fermions are just like nearest-neighbor Pauli matrices of the type *XX*, *YY*, *XY*, and *YX*.

Indistinguishable particles

Amazingly, adding a SWAP operation to switch positions of fermions is enough to enable universal quantum computation!

[Jozsa and Miyake, Proc. R. Soc. **464**, 3089 (2008)]

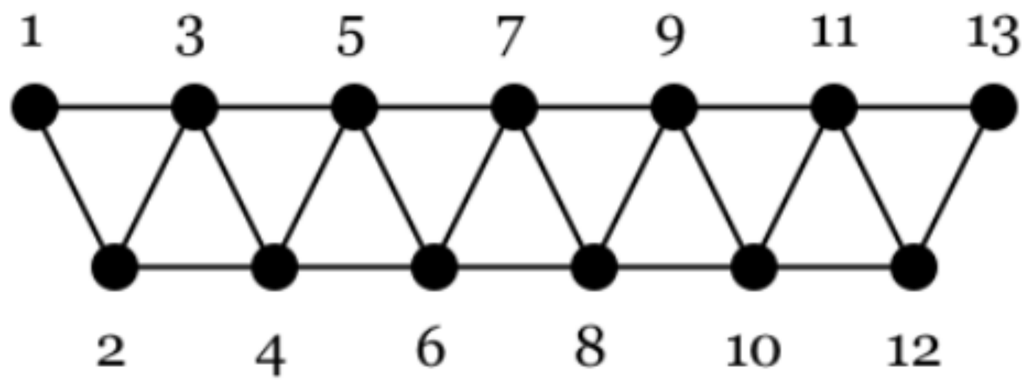


Note that $\mathrm{SWAP} = G(I, X) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is not a matchgate, because $\det(I) = -\det(X)$.

Indistinguishable particles

Even more amazingly, universal computing is possible with only matchgates for other geometries!

[Brod and Galvão, Physical Review A **86**, 052307 (2012)]



Indistinguishable particles

Important (open) question: Can the absence / presence of entanglement area laws enable us to infer the power of indistinguishable quantum systems for quantum computation?

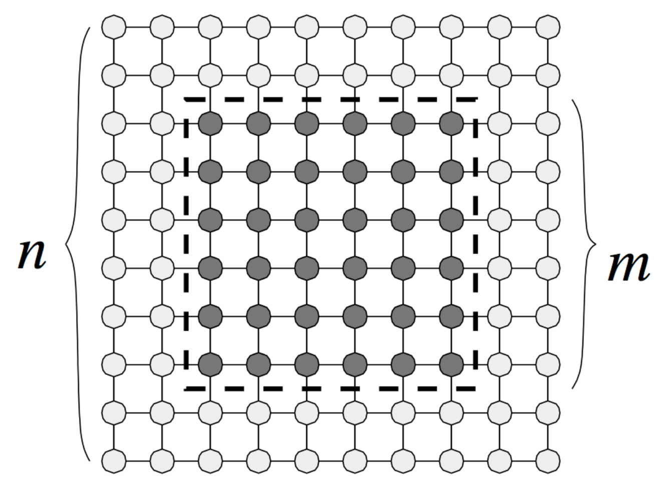
- The entanglement entropy for non-interacting bosons is proportional to the area (entanglement area law)

  [Plenio, Eisert, Dreißig, and Cramer, Phys. Rev. Lett. **94**, 060503 (2005)]:

von Neumann: $S = -\mathrm{tr}\rho \log \rho$

Rényi: $S = \dfrac{1}{1-\alpha} \log \mathrm{tr}\left(\rho^{\alpha}\right)$

$$S \sim L^{d-1}$$



Indistinguishable particles

Important (open) question: Can the absence / presence of entanglement area laws enable us to infer the power of indistinguishable quantum systems for quantum computation?

- Non-interacting fermions have ‘more entanglement’ than non-interacting bosons: the entanglement area law is violated logarithmically: [Wolf, Phys. Rev. Lett. **96**, 010404 (2006)]

$$S \sim L^{d-1} \log L$$

- Non-interacting fermions on a lattice are in a sense ‘critical:’

$$S \sim \frac{c}{3} \log(L/a)$$

- (critical bosons can still satisfy area laws)

Indistinguishable particles

Important (open) question: Can the absence / presence of entanglement area laws enable us to infer the power of indistinguishable quantum systems for quantum computation?

- For d=1 systems, the ground states of all gapped Hamiltonians satisfy entanglement area laws.
  [Brandão, Horodecki, Nature Physics **9**, 721 (2013)]

- All such models can be efficiently represented.

- Very recently it was proven that there exists an efficient algorithm to find the ground state.
  [Landau, Vazirani, Vidick, Nature Physics **11**, 566 (2015)]

- There are also efficient methods to approximate some d=1 gapless / critical models, though no formal proof exists.

These results suggest that gapped d=1 systems are not universal for quantum computation. Gapless case?

Indistinguishable particles

Important (open) question: Can the absence / presence of entanglement area laws enable us to infer the power of indistinguishable quantum systems for quantum computation?

- For d=2 or general d, much less is known / understood.
- The ground states of all gapped (gap ϵ) Hamiltonians have exponential correlation functions $\xi = O(1/\epsilon)$:
  [Hastings, Phys Rev B **69**, 104431 (2004)]

$$\left\langle \hat{O}\left(\mathbf{r}_i, \mathbf{r}_j\right)\right\rangle \sim e^{-|\mathbf{r}_i - \mathbf{r}_j|/\xi}$$

$$\hat{O} = \{\hat{b}^\dagger(\mathbf{r}_i)\hat{b}(\mathbf{r}_j), \hat{n}(\mathbf{r}_i)\hat{n}(\mathbf{r}_j), S^{(k)}(\mathbf{r}_i)S^{(k')}(\mathbf{r}_j), \ldots\}$$

- The ground states of all frustration-free Hamiltonians (including critical ones!) also have exponential correlation functions:
  [Gosset and Huang, Phys Rev Lett **116**, 097202 (2016)]:

$$\xi = O(1/\sqrt{\epsilon})$$

Indistinguishable particles

Important (open) question: Can the absence / presence of entanglement area laws enable us to infer the power of indistinguishable quantum systems for quantum computation?

- It is tempting to assume that systems with exponential correlations have efficient classical representations, but it isn't even known if all such systems satisfy area laws!
- In fact, it has been proven that there exist quantum states satisfying area laws that cannot be represented efficiently.

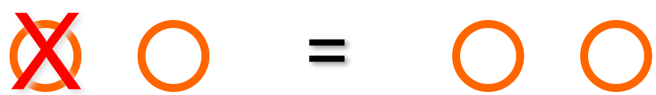
  [Ge and Eisert, arXiv:1411.2995]
- Cluster (stabilizer / quantum code) states are gapped spin states from local frustration-free Hamiltonians, satisfy entanglement area laws, are efficiently representable, and are universal for quantum computation via measurements*.

Cluster States

- Cluster states are highly entangled states that are resources for measurement-based quantum computation.

Suppose  is a qubit in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- Evidently, $X_1|+\rangle|+\rangle = |+\rangle|+\rangle$

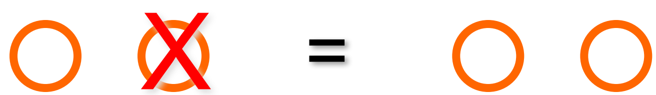


Cluster States

- Cluster states are highly entangled states that are resources for measurement-based quantum computation.

Suppose  is a qubit in the state $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

- Evidently, $X_2|+\rangle|+\rangle = |+\rangle|+\rangle \equiv |++\rangle$



- The stabilizer group for $|++\rangle$ is therefore $\{XI, IX, XX, II\}$

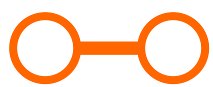
Cluster States

- Cluster states are highly entangled states that are resources for measurement-based quantum computation.

Suppose  is a qubit in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- Also, $CZ|++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

$$= \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)$$ (cluster / Bell state)

= 

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Cluster States

- With the commutation relation

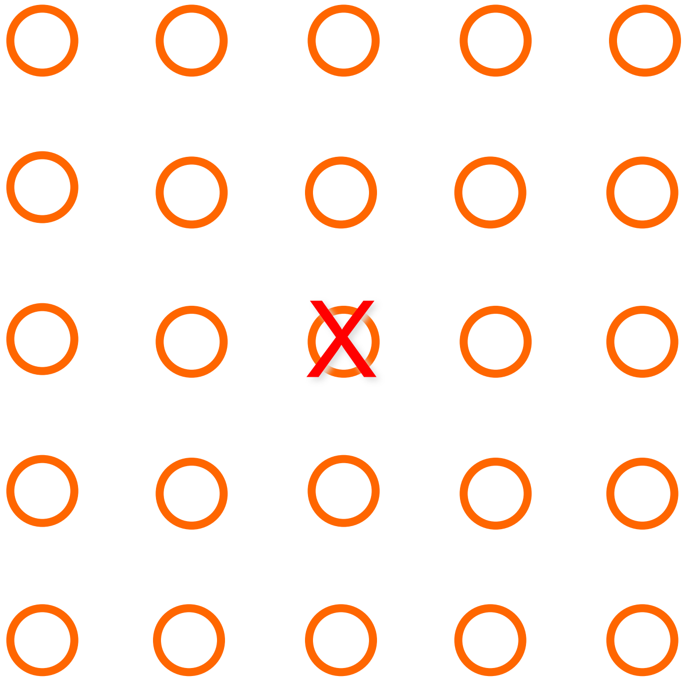
$$CZ|++\rangle = CZ(IX)|++\rangle = (ZX)CZ|++\rangle$$

- The stabilizer group for the two-qubit cluster state is $\{XZ, ZX, YY, II\}$

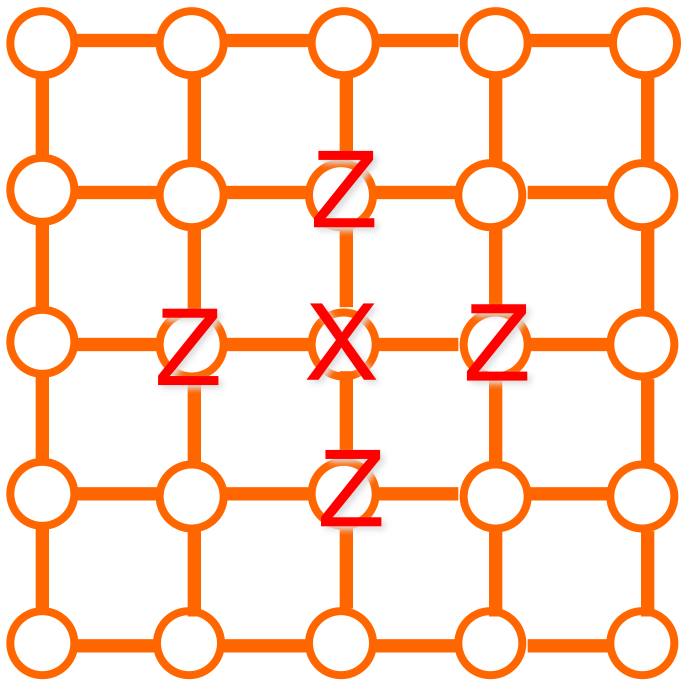
- All group elements commute.

- Recall matchgate / free fermion group is generated by (XX, YY, IZ, ZI, XY, and YX).

Cluster States



Cluster States



- The stabilizer generators for the cluster state are $X_i \prod_{j=\mathcal{N}(i)} Z_j$
- Every cut through a bond → one 'entropy unit'

Cluster States

- Choosing $H = -X_i \prod_{j=\mathcal{N}(i)} Z_j$ guarantees that the cluster state is the lowest energy eigenstate. 'Local' and gapped!

- This gives 3-body (5-body) Hamiltonian for 1D (2D) clusters.

- No (physical) two-body Hamiltonian can yield a (nondegenerate) ground state that is any cluster state [van den Nest, Luttmer, Dür, and Briegel, PRA **77**, 012301 (2008)].

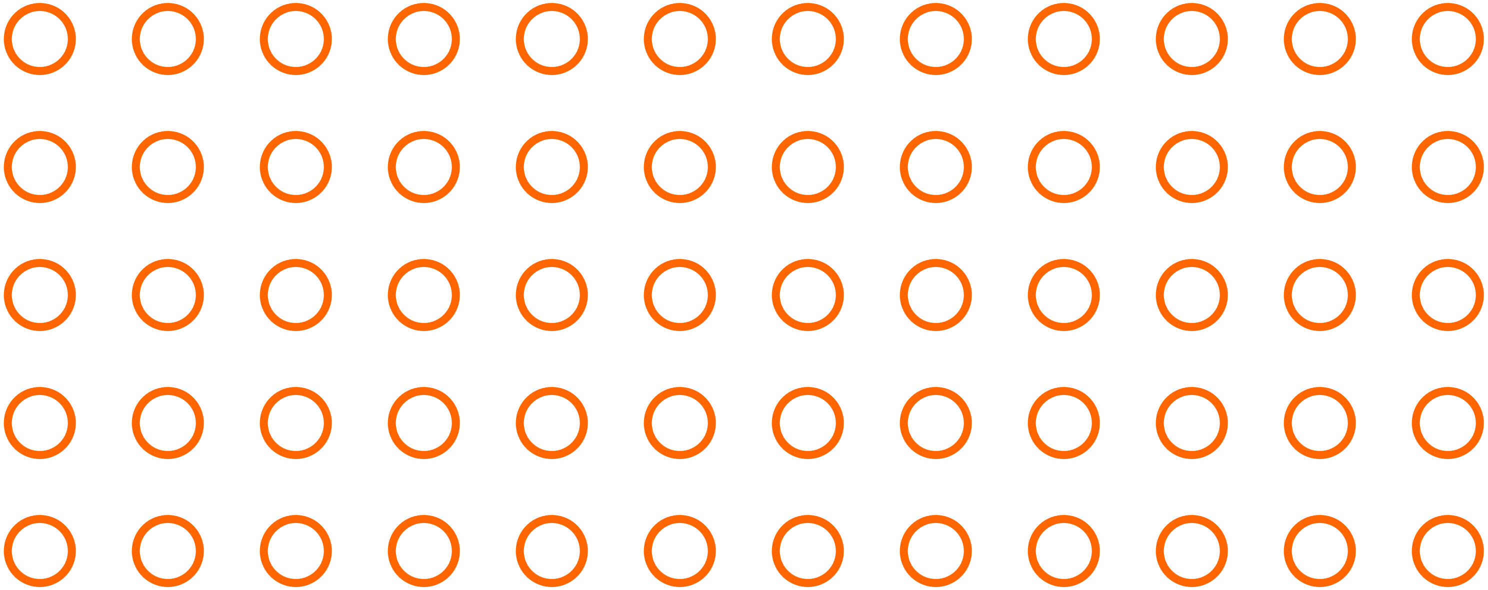
- It is impossible to find a physical Hamiltonian that yields a cluster state as the ground state, though one can get arbitrarily close [Darmawan and Bartlett, New Journal of Physics **16**, 073013 (2014)]

Measurement-Based Quantum Computing

- Why are cluster states interesting? Universal quantum computation is effected **solely** by making successive adaptive measurements [Raussendorf, Briegel, PRL **86**, 5188 (2001)].

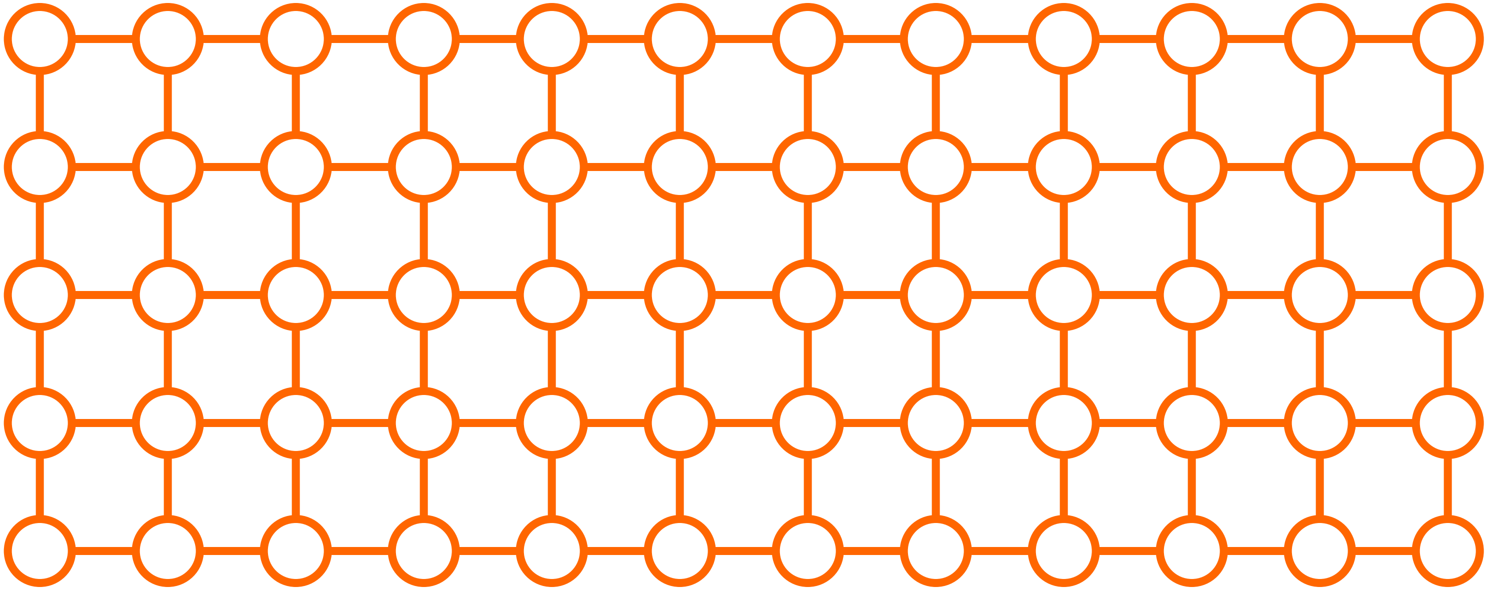
Measurement-Based Quantum Computing

1) Initialize all qubits in the state $|+\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$



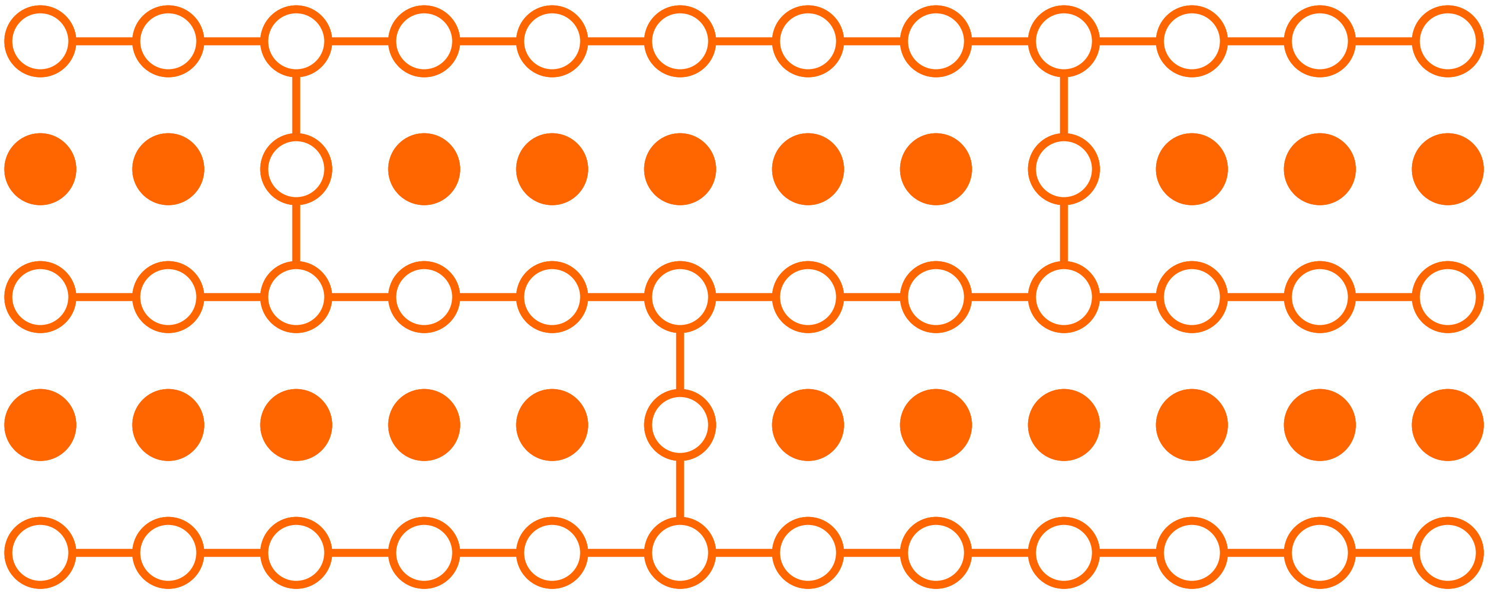
Measurement-Based Quantum Computing

2) Entangle qubits: Apply CZ gates between all nearest neighbours



Measurement-Based Quantum Computing

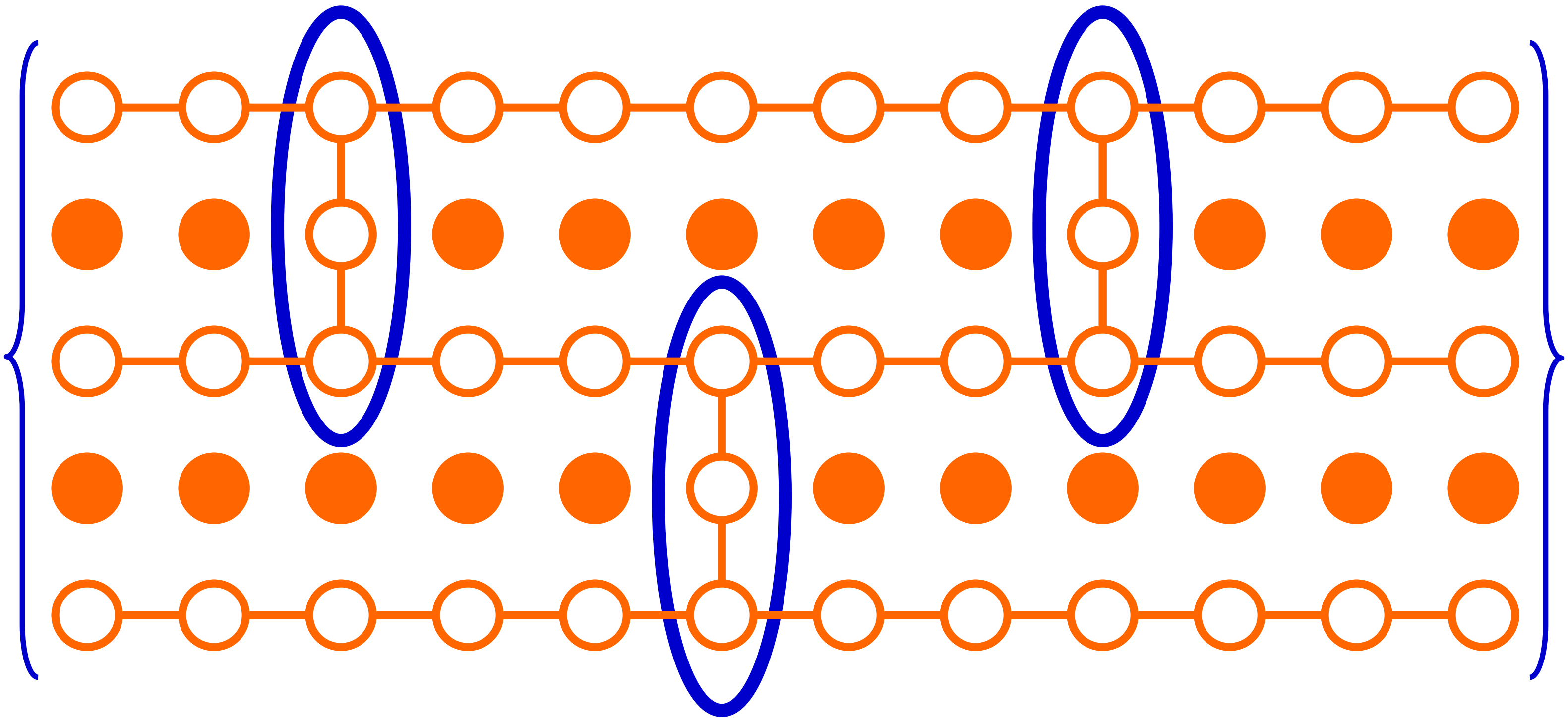
3) Remove unwanted qubits: Z-basis measurements



→ *"real-space quantum circuit"*

Measurement-Based Quantum Computing

4) Computation via measurements in the X and Y bases:



horizontal chains = logical qubits.

vertical links = 2-qubit gates

Measurement-Based Quantum Computing

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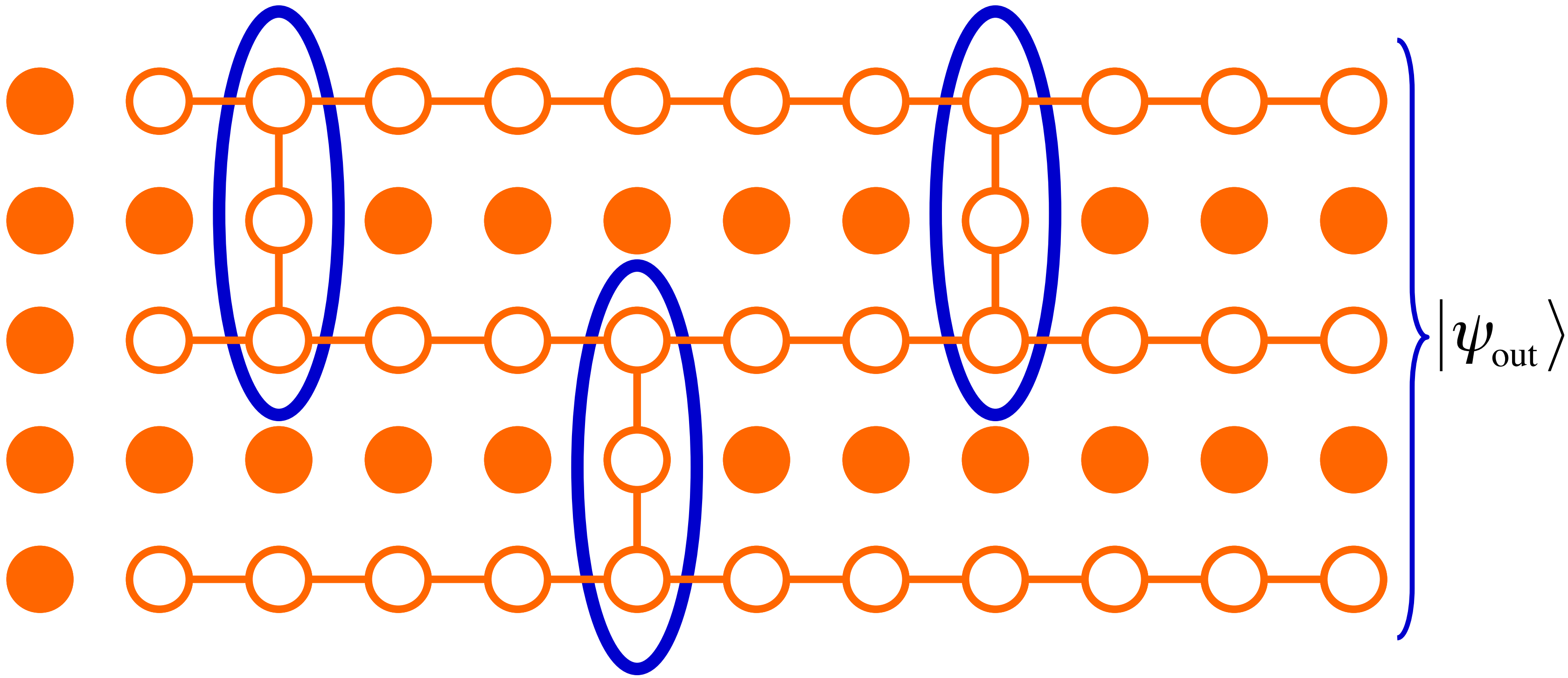

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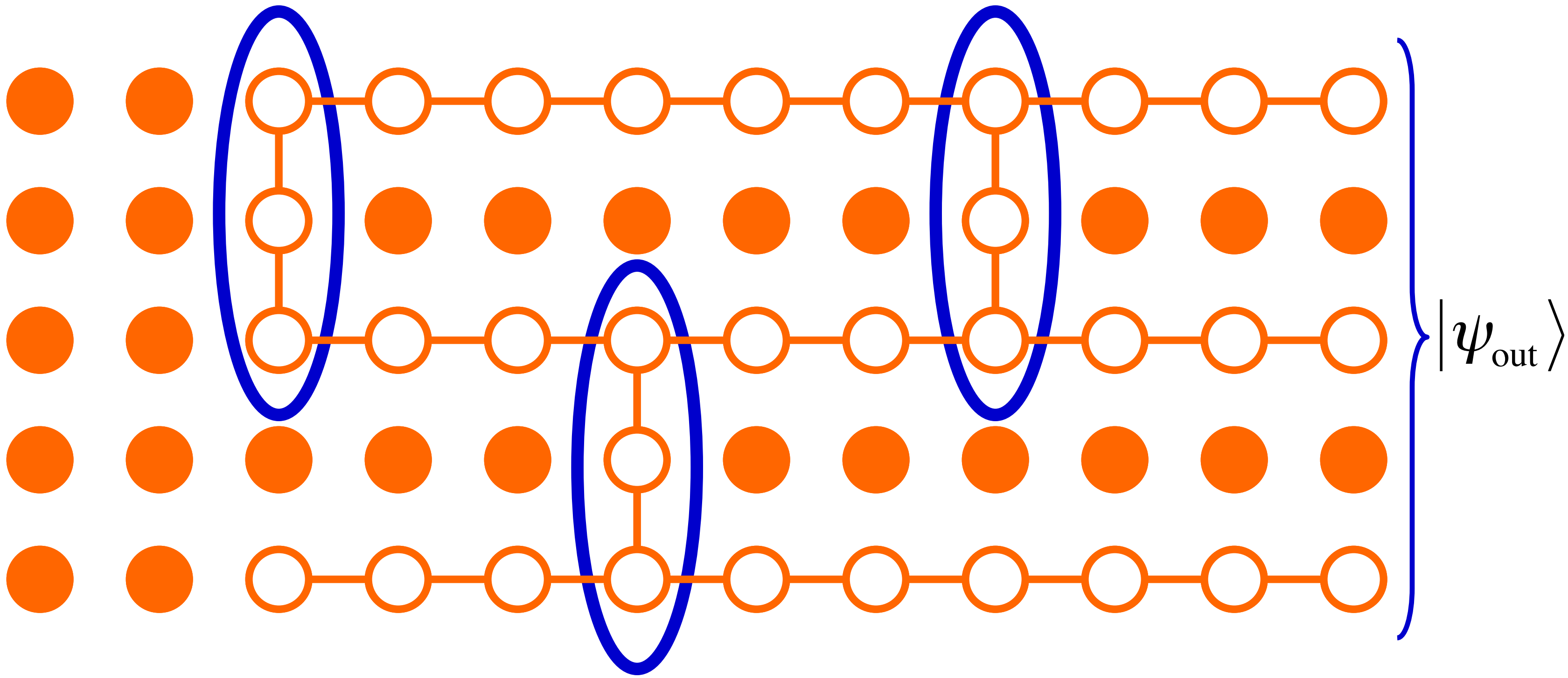

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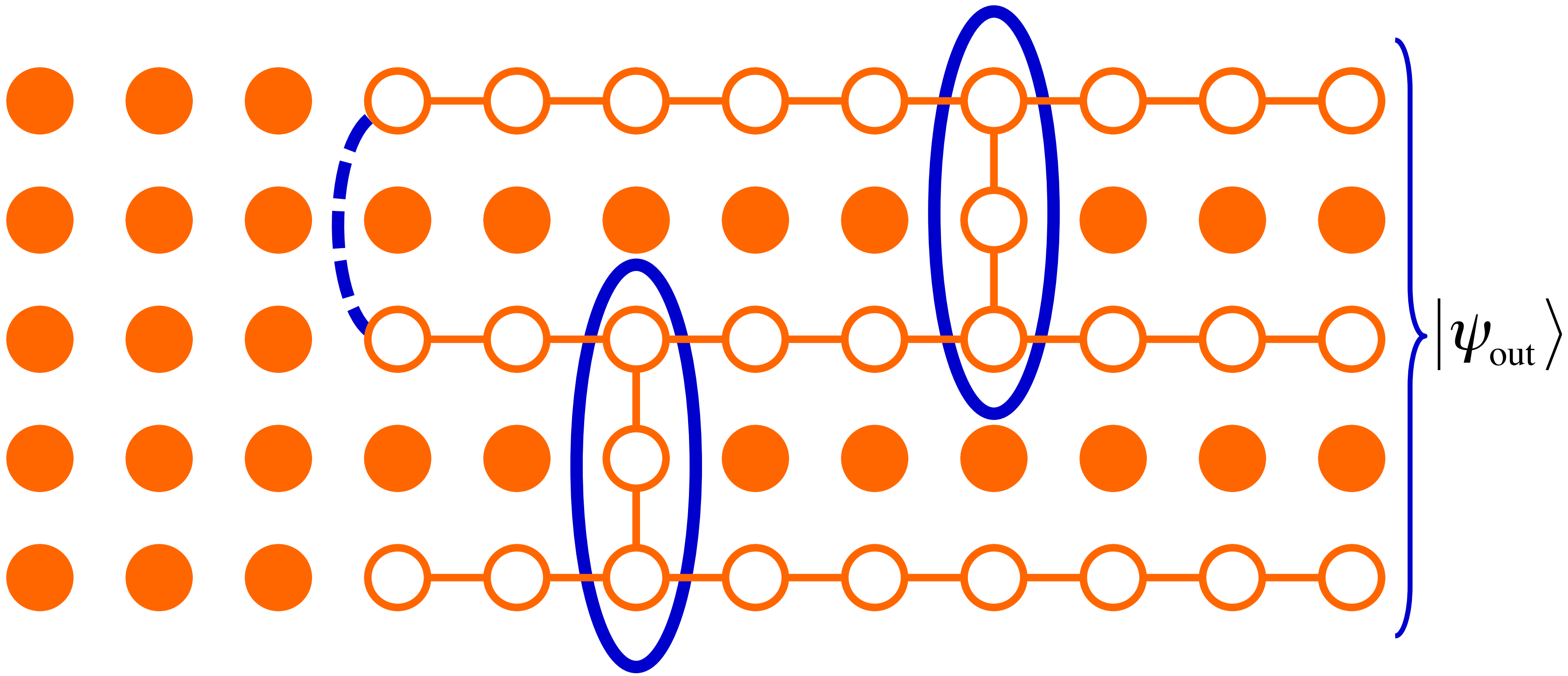

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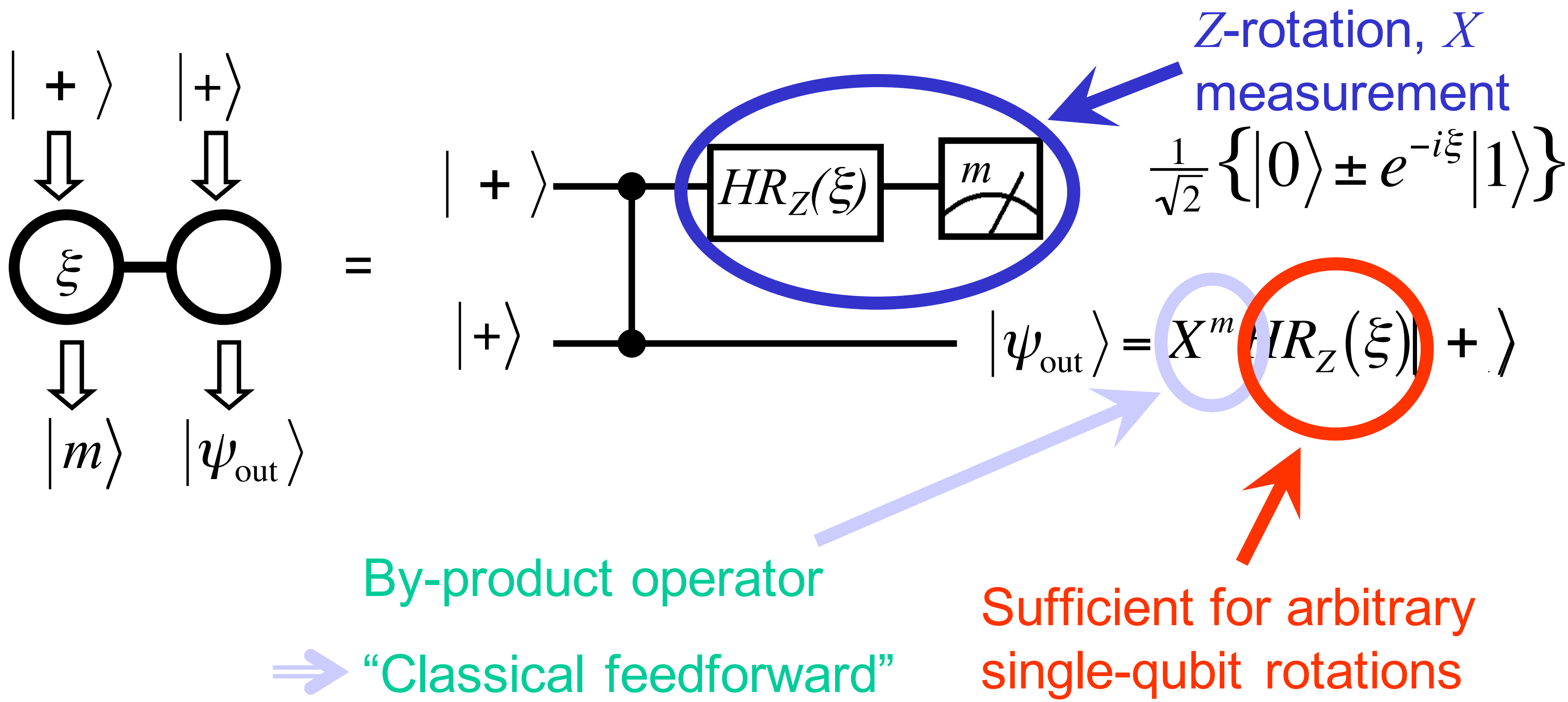
Measurement-Based Quantum Computing

- The key is single-qubit gate teleportation:



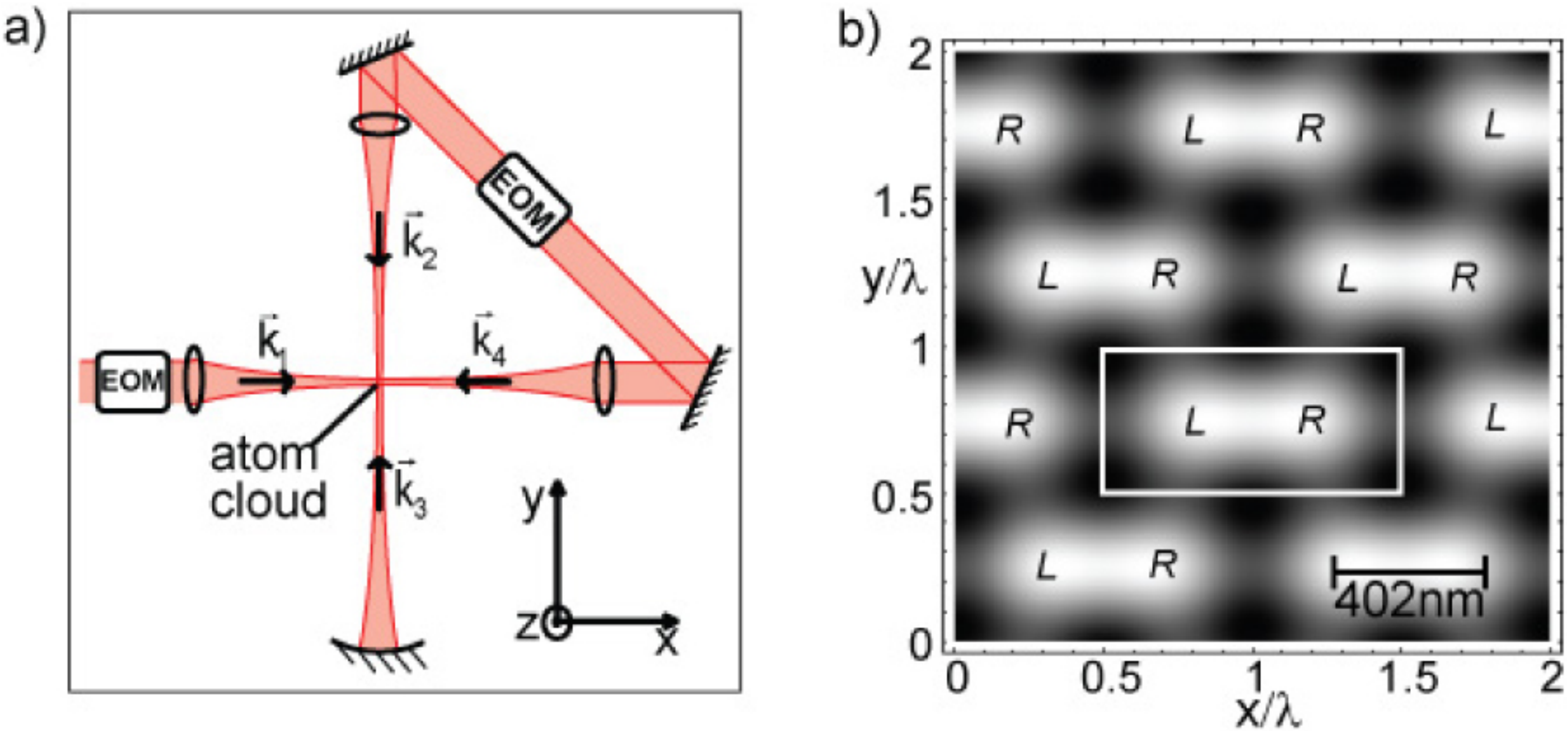
Measurement-Based Quantum Computing

What about area laws?

- The quantum information always resides on the ‘surface’ of the state, so entanglement area laws are always strictly satisfied.
- A similar situation exists for MBQC on symmetry-protected / Haldane-phase states, which have exponential correlations.
  [Verstraete, Wolf, Perez-Garcia, Cirac, PRL **96**, 220601 (2006);
  Wei, Affleck, Raussendorf, PRA **86**, 032328 (2012); Wei, Raussendorf, PRA **92**, 012310 (2015)]

Fermions in double-well arrays

• Consider (ultracold) fermions in independent double-well lattices:

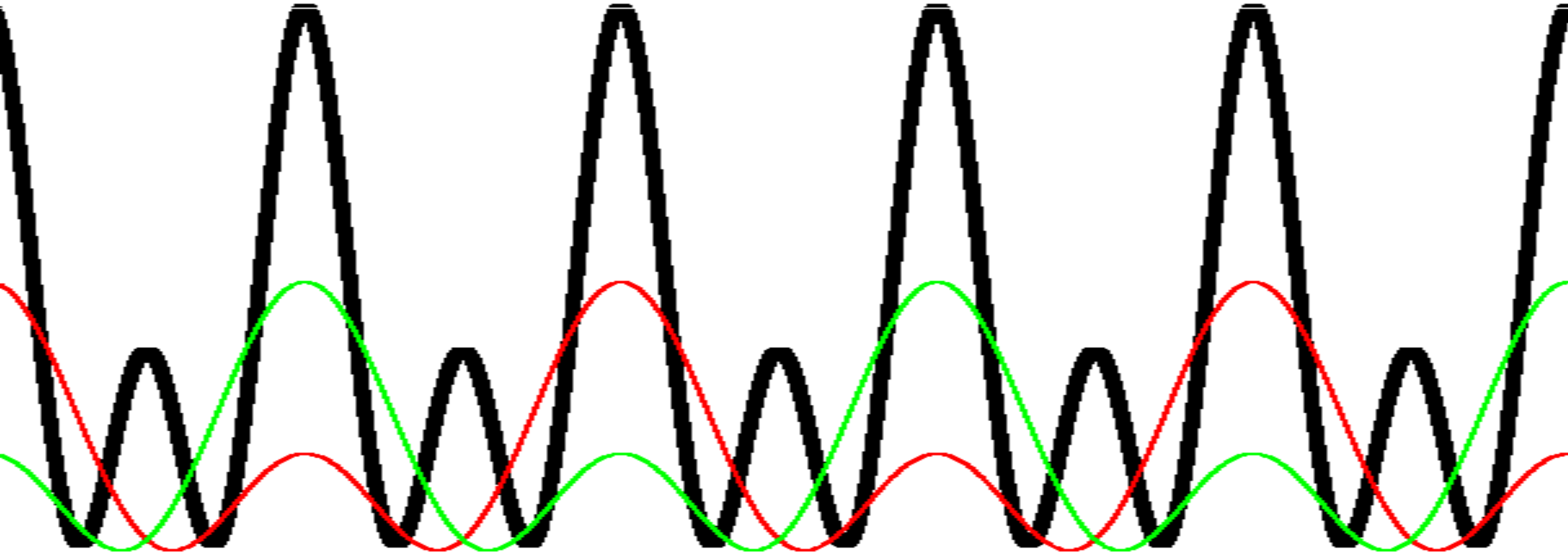


[Lee et al. (Trey Porto), PRL **99**, 020402 (2007)].

• Spatial qubits if there is one particle in each double-well: Left is $|0\rangle$ and right is $|1\rangle$

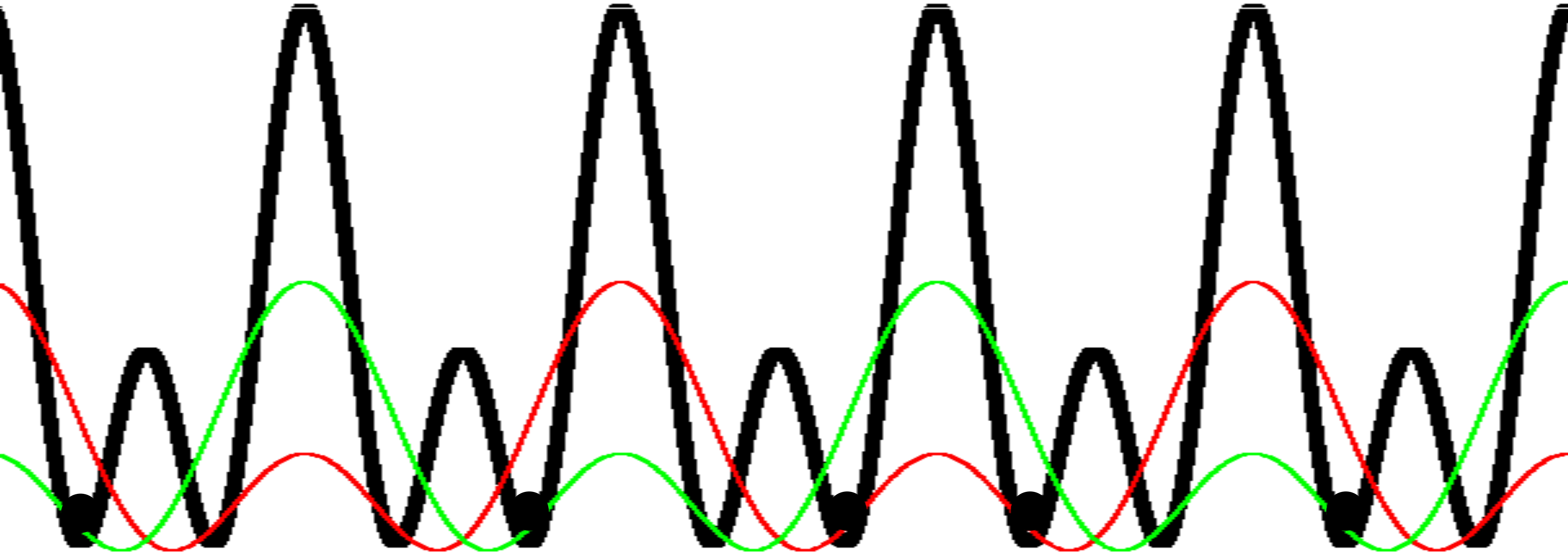
Fermions in double-well arrays

- Suppose we have a series of interconnected two-site lattices:



Fermions in double-well arrays

• Suppose that there is exactly one fermion in each double-well:



Fermions in double-well arrays

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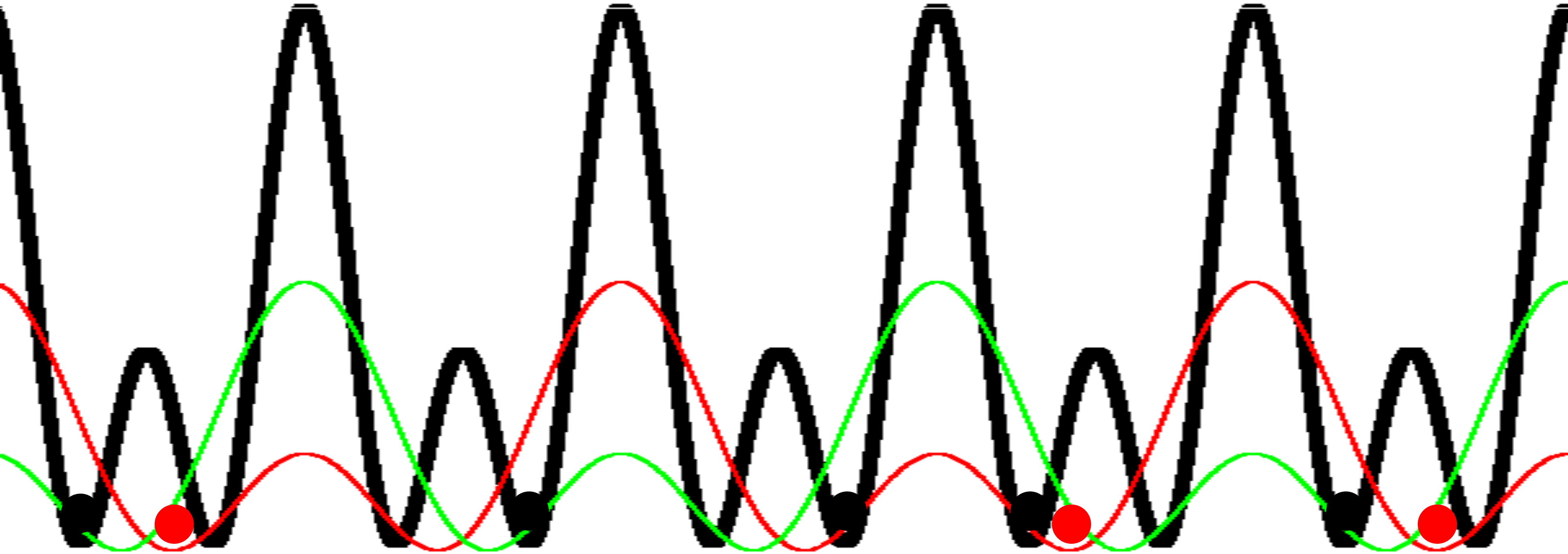

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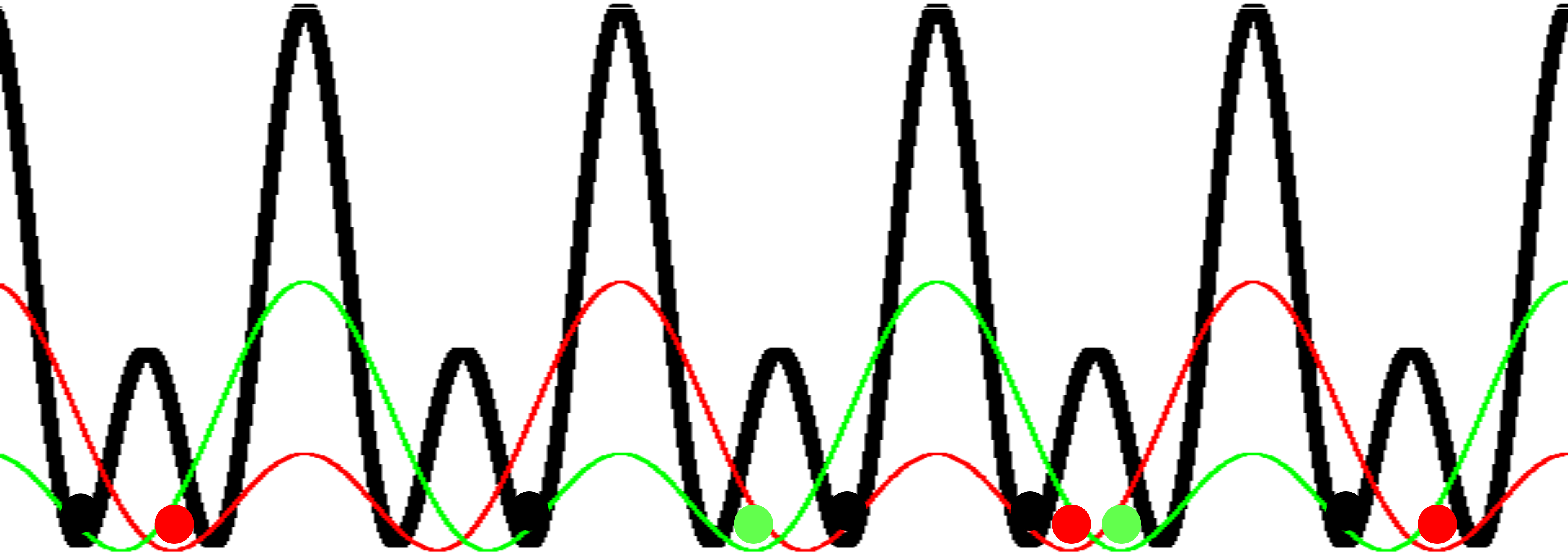

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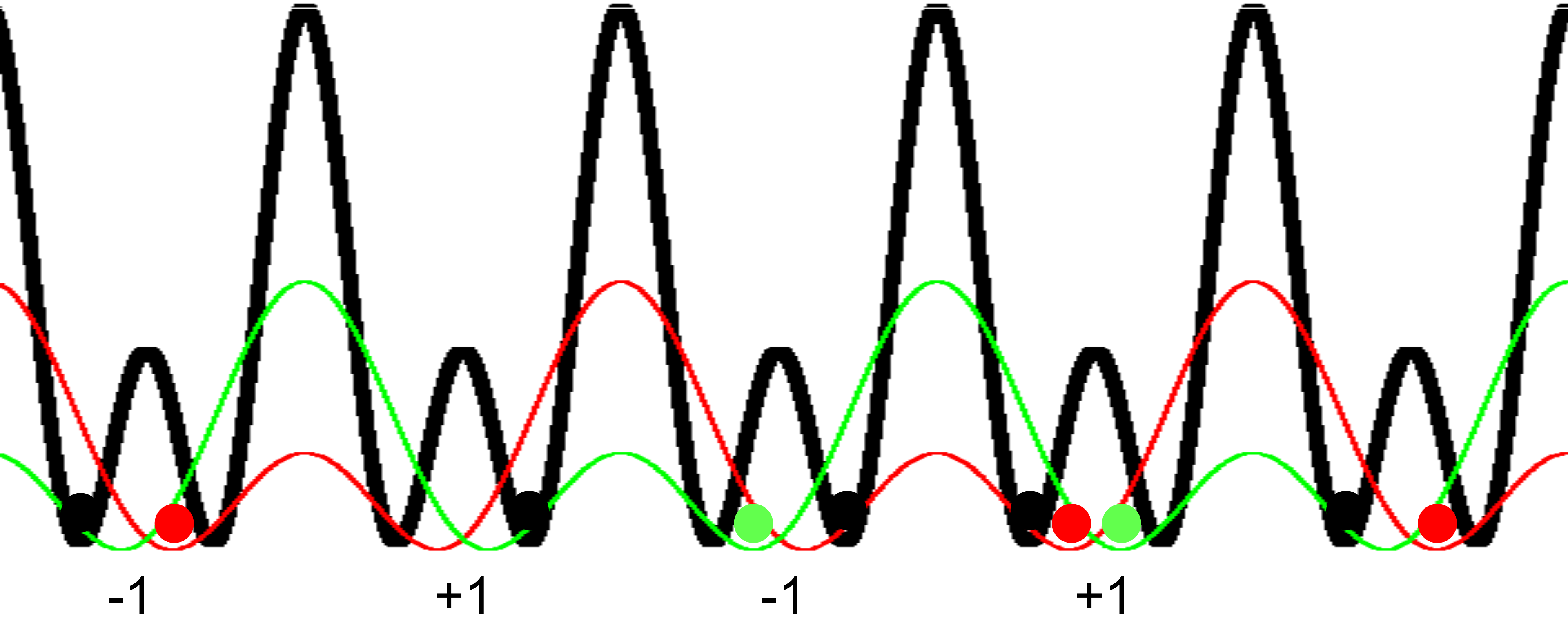
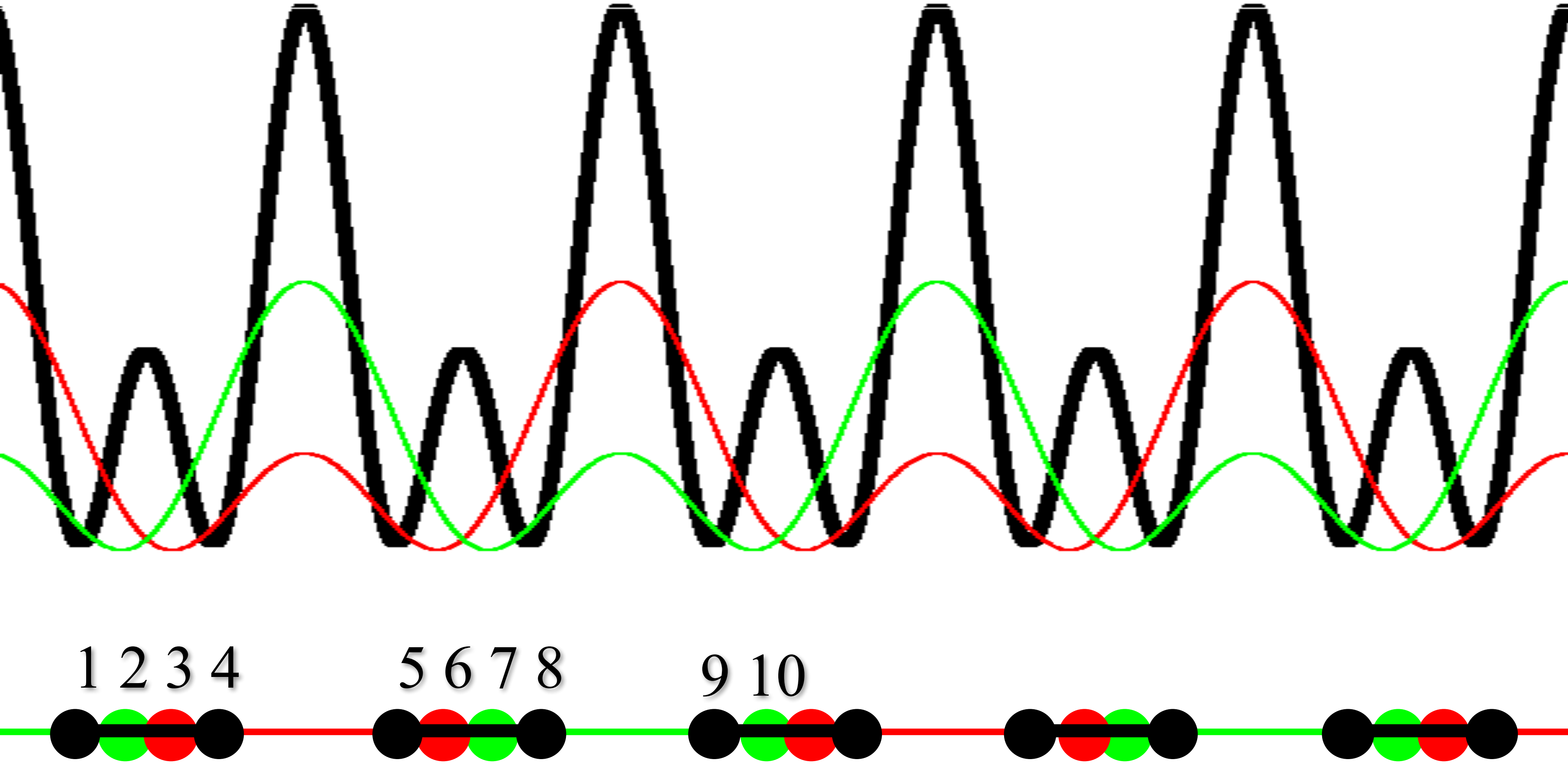


• These phases are the same as the ones you get by applying a maximally entangling CZ gate on qubits!

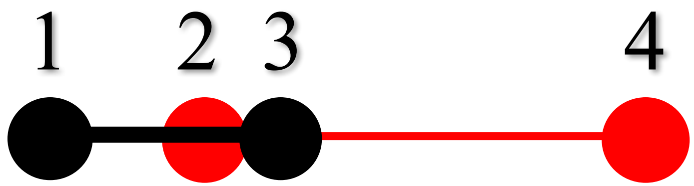
Fermions in double-well arrays

• We have a series of interconnected two-site lattices:



Fermions are maximally entangled

- Consider only two interlocking links:



- The basis corresponds to the states:

$$|0\rangle = f_1^+ f_2^+ |\Phi\rangle;$$
$$|1\rangle = f_1^+ f_4^+ |\Phi\rangle;$$
$$|2\rangle = f_2^+ f_3^+ |\Phi\rangle;$$
$$|3\rangle = f_3^+ f_4^+ |\Phi\rangle.$$

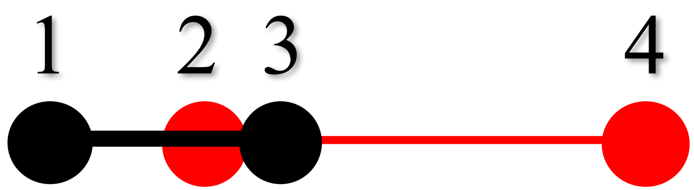
- The Hamiltonian $H = -\tau\left(f_1^+ f_3 + f_3^+ f_1 + f_2^+ f_4 + f_4^+ f_2\right)$ is then:

$$H = -\tau \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = -\tau\left(Z \otimes X - X \otimes Z\right)$$

Cluster state stabilizer!

Fermions are maximally entangled

- Consider only two interlocking links:

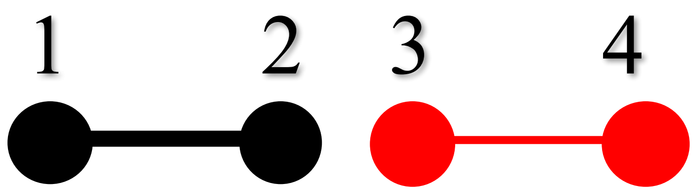


- The ground state is the superposition of occupying both sites of each link:

$$
\begin{aligned}
|g.s.\rangle &= \frac{1}{\sqrt{2}}\left(f_1^+ + f_3^+\right)\frac{1}{\sqrt{2}}\left(f_2^+ + f_4^+\right)|\Phi\rangle \\
&= \frac{1}{2}\left(f_1^+ f_2^+ + f_1^+ f_4^+ + f_3^+ f_2^+ + f_3^+ f_4^+\right)|\Phi\rangle \\
&= \frac{1}{2}\left(f_1^+ f_2^+ + f_1^+ f_4^+ - f_2^+ f_3^+ + f_3^+ f_4^+\right)|\Phi\rangle
\end{aligned}
$$

Fermions are maximally entangled

• This is very different from two non-interlocking links:

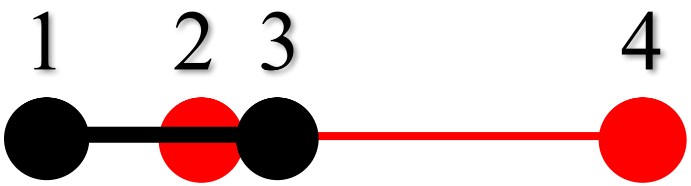


• The ground state is the superposition of occupying both sites of each link:

$$\left|g.s.\right\rangle = \frac{1}{\sqrt{2}}\left(f_1^+ + f_2^+\right)\frac{1}{\sqrt{2}}\left(f_3^+ + f_4^+\right)\left|\Phi\right\rangle$$

$$= \frac{1}{2}\left(f_1^+ f_3^+ + f_1^+ f_4^+ + f_2^+ f_3^+ + f_2^+ f_4^+\right)\left|\Phi\right\rangle$$

Fermions are maximally entangled



- Compare the fermion ground state:

$$\left|g.s.\right\rangle = \frac{1}{2}\left(f_1^+ f_2^+ + f_1^+ f_4^+ - f_2^+ f_3^+ + f_3^+ f_4^+\right)\left|\Phi\right\rangle$$

to the modified two-qubit cluster state:

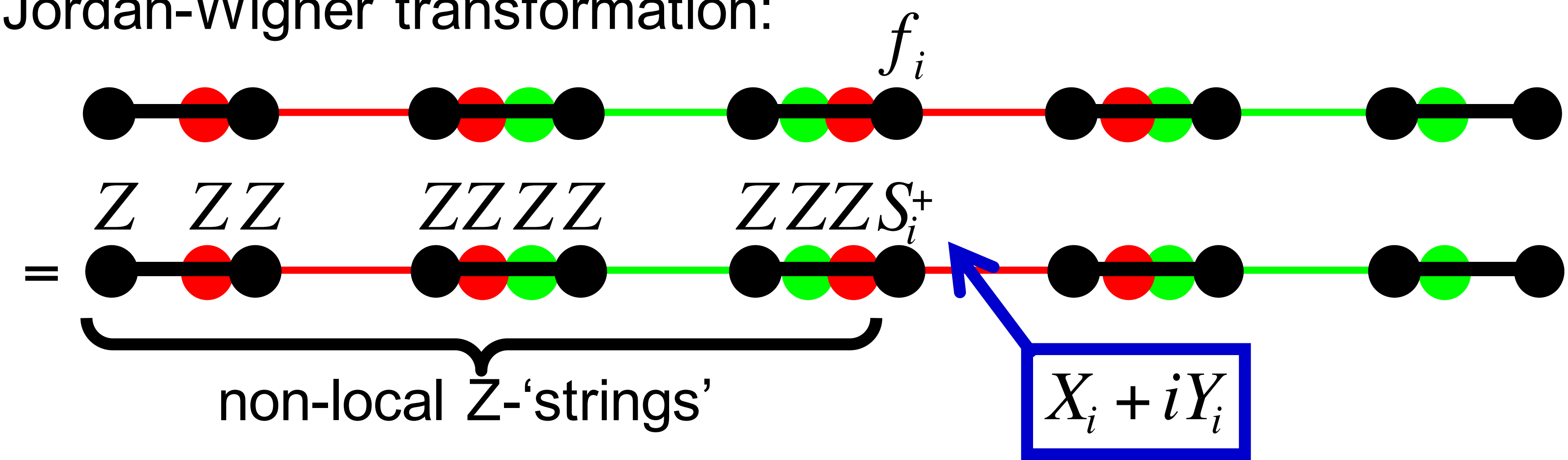
$$Z_1 CZ\left|++\right\rangle = \frac{1}{2}\left(\left|00\right\rangle + \left|01\right\rangle - \left|10\right\rangle + \left|11\right\rangle\right).$$

- These are the same if: $f_1^+ f_2^+\left|\Phi\right\rangle \Leftrightarrow \left|00\right\rangle;\ f_1^+ f_4^+\left|\Phi\right\rangle \Leftrightarrow \left|01\right\rangle;$

$$f_2^+ f_3^+\left|\Phi\right\rangle \Leftrightarrow \left|10\right\rangle;\ f_3^+ f_4^+\left|\Phi\right\rangle \Leftrightarrow \left|11\right\rangle.$$

Fermions are maximally entangled

• Recall that spins and fermions are connected through the Jordan-Wigner transformation:



$$Z_i \equiv 1 - 2 f_i^+ f_i$$

Fermions are maximally entangled

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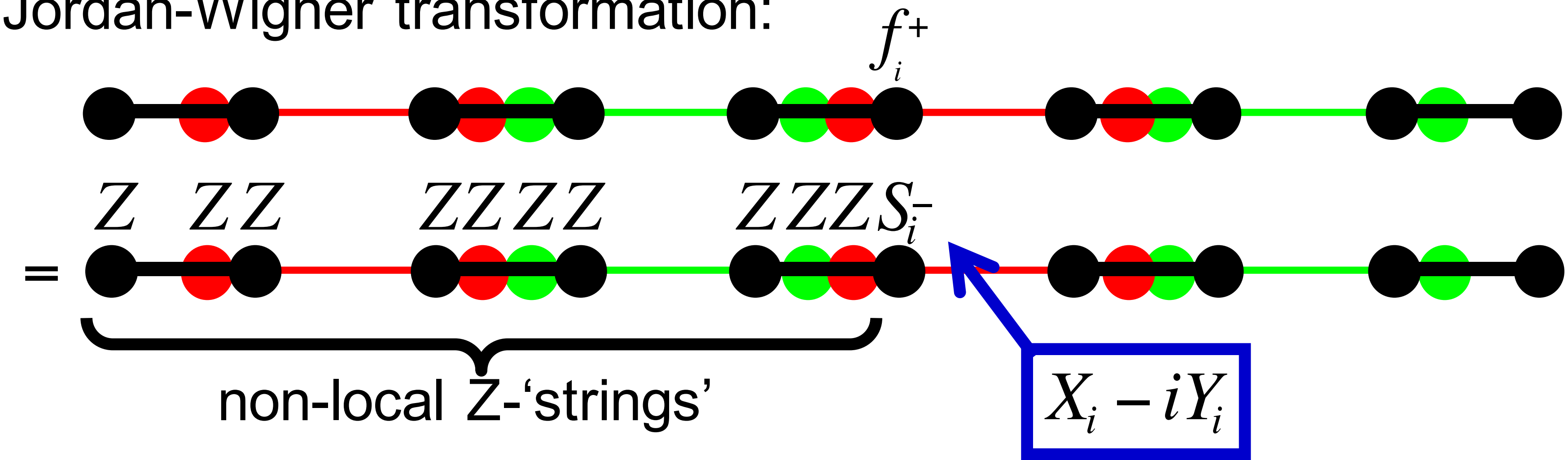

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Fermions are maximally entangled

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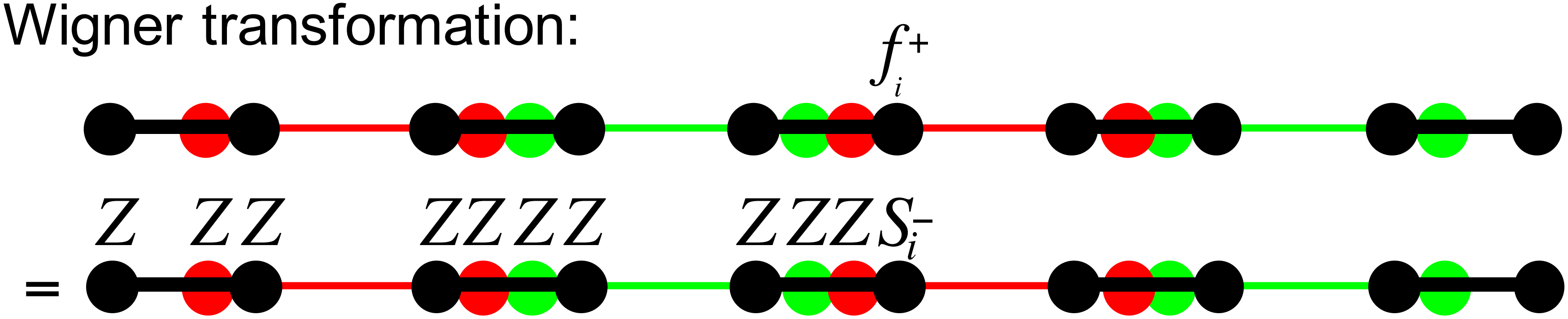
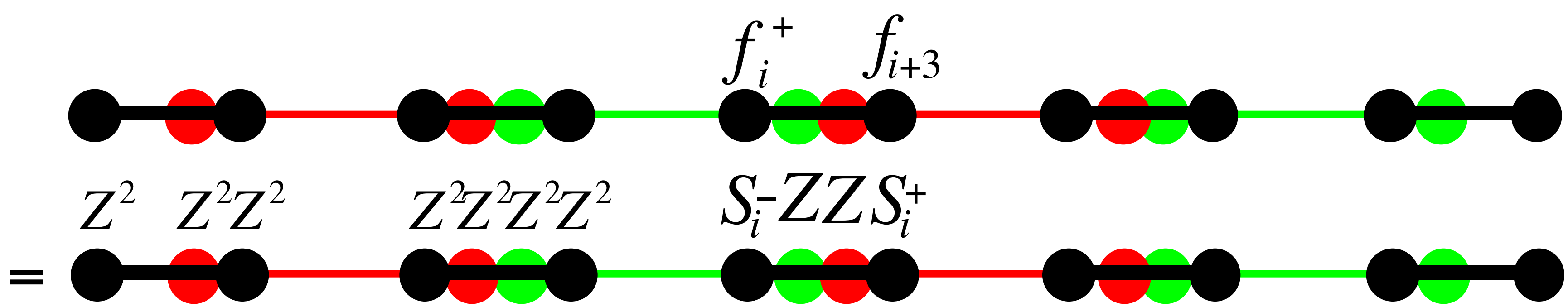


• Consider the hopping of a fermion in one of the double-wells:



Fermions are maximally entangled

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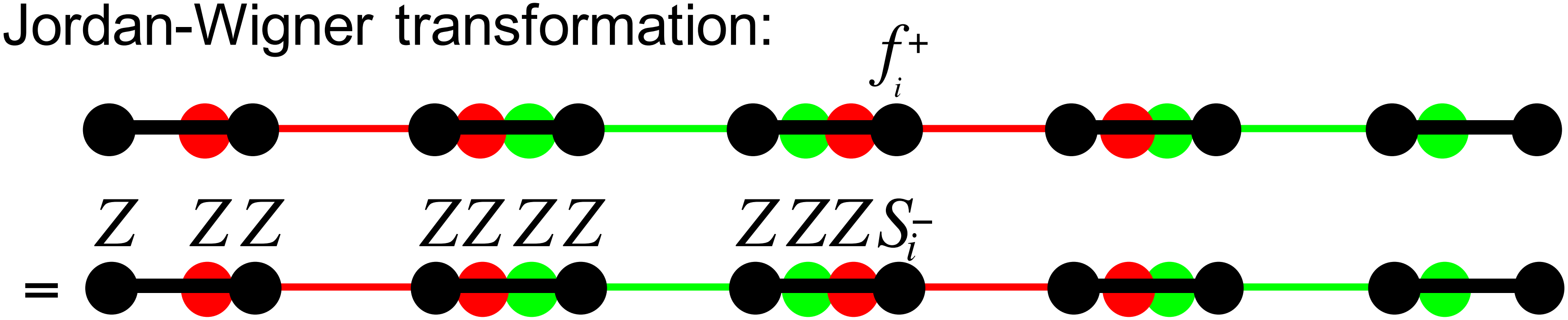

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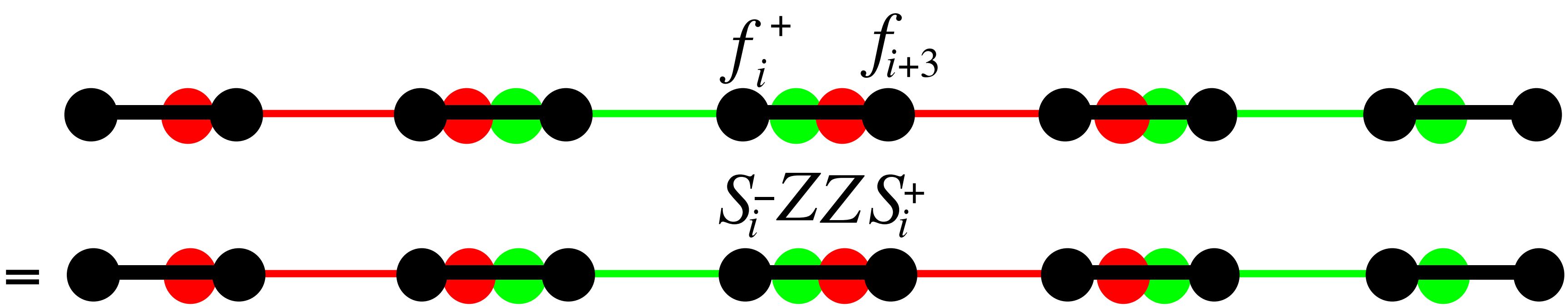


- This term involves four spin operators – Hamiltonian has **effective four-body interactions.**

Fermions are maximally entangled

- The fermion Hamiltonian in spin form becomes:

$$H = -\frac{\tau}{2}\sum_{j=0}^{N-1} Z_{2j+2}Z_{2j+3}\left(X_{2j+1}X_{2j+4} + Y_{2j+1}Y_{2j+4}\right).$$

- Introduce an encoded basis $\left|\underline{0}_j\right\rangle \equiv \left|1_{2j+1}0_{2j+4}\right\rangle = f^{\dagger}_{2j+1}|\mathcal{O}\rangle;$

$$\left|\underline{1}_j\right\rangle \equiv \left|0_{2j+1}1_{2j+4}\right\rangle = f^{\dagger}_{2j+4}|\mathcal{O}\rangle$$

- Define $\underline{X}_j \equiv \frac{1}{2}\left(X_{2j+1}X_{2j+4} + Y_{2j+1}Y_{2j+4}\right)$ and

$$\underline{Z}_j \equiv I_{2j+1}Z_{2j+4} = -Z_{2j+1}I_{2j+4}$$

Fermions are maximally entangled

• In the encoded basis the Hamiltonian becomes

$$H = \tau \sum_{j=1}^{N-1} \underline{Z}_{j-1}\underline{X}_j\underline{Z}_{j+1} - \tau \underline{Z}_{N-1}\underline{X}_N$$

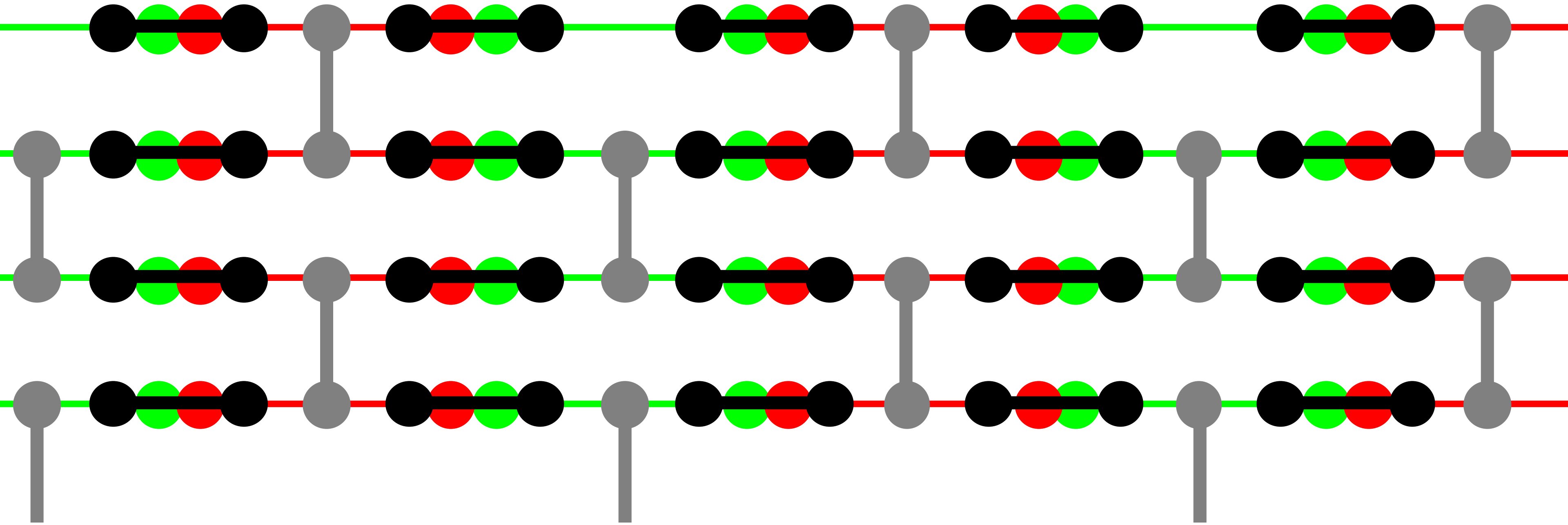
• This is locally equivalent to the 1D cluster-state Hamiltonian! (conjugate sites 1 through N-1 by $\underline{Z}_j$):

$$H = -\tau \sum_{j=1}^{N} \underline{Z}_{j-1}\underline{X}_j\underline{Z}_{j+1}$$

• The fermionic ground state is therefore gapped, independent of size (excitations cost energy 2τ).

Fermions are entangled

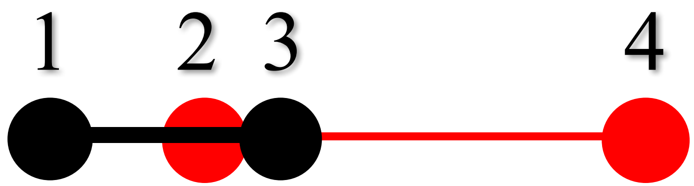
• Likewise, a two-dimensional encoded cluster state can be constructed by non-interacting fermions hopping on this structure:



• So the ground-state of non-interacting fermions hopping on overlapping lattices is universal for measurement-based quantum computaton??

Fermions are maximally entangled, but…

• Of course, there is a catch! Return to the two-qubit case:



• Quantum teleportation requires X-basis measurements, so first one must transform the first qubit by a Hadamard:

$$\underline{H}_1 = \frac{1}{\sqrt{2}} \left[1 - 2n_1 + (1 - 2n_2) \left(f_3^\dagger f_1 + f_1^\dagger f_3 \right) \right]$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Hopping amplitude (sign) depends on occupation of second site. Need quartic term = particle interactions.

Fermions are maximally entangled, but…

• So, even though the ground state is maximally entangled, one cannot perform local operations unless the fermions interact!

• In fact, performing a local (encoded) unitary operation $\underline{U}_1$ instead yields $\underline{CZ}_{12}\underline{U}_1\underline{CZ}_{12}$ which is a matchgate (modulo local operations).

• For example, performing $\underline{H}$ yields $G(H,H)$.

• This is the measurement-based analog of universal matchgate computing.

• In practice, we need to implement $\underline{CZ}$ to counteract the induced ones; this requires real interactions. Using this we can construct $\underline{\mathrm{SWAP}}$.

Review

- ‘Non-interacting’ bosons are computationally non-trivial

- Non-interacting fermions are efficiently simulatable

- In 1D, all ground states of gapped / frustration-free Hamiltonians are efficiently simulatable. Not so for gapless Hamiltonians

- In 2D or higher, not too much is known! Seemingly trivial extensions of non-interacting fermions are not classically simulatable.

- Relationships among frustration-free/frustrated, gapped/gapless, area law satisfied/violated, ground-state representation efficient/inefficient, ground-state finding efficient/inefficient…?