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4-days conference:
"Holography & Quantum Information"

Snapshot entropy:
An alternative holographic entanglement entropy

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Purpose of this work

Purpose:

Entanglement, holography, RG, criticality, typicality, …

↕

SVD (singular value decomposition)

Strategy:

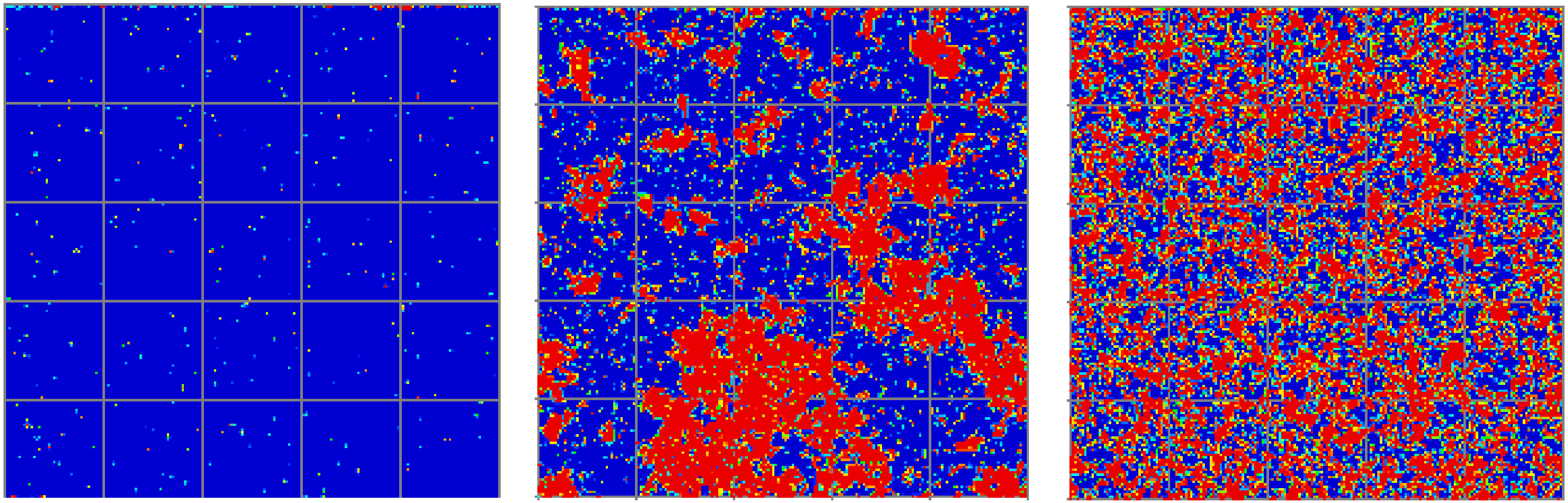
⇒ Take a spin configuration (Monte Carlo snapshot) for 2D classical Ising (& q=3 Potts) model
⇒ SVD for the snapshot matrix
⇒ Calculate the snapshot entropy S
⇒ Derive the scaling formula of S as a function of linear system size L

What does this scaling mean ?

Monte Carlo Simulation of the 2D Ising Model

Classical Ising Spin Model: $H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \qquad \sigma_i^z = \pm 1$

Snapshots at various temperatures



(a) $T = 1.52J$ (b) $T \approx T_c = 2.27J$ (c) $T = 3.02J$

$$L = 256 \qquad T_c = \frac{2J}{\log(1+\sqrt{2})} = 2.2692J$$

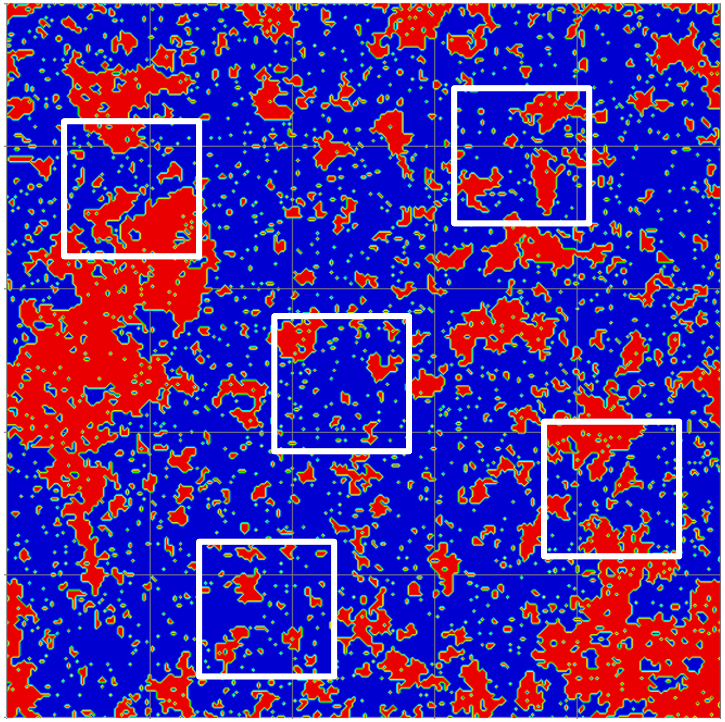
Criticality, Fractal, and amount of information

2D Ising model

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$$

Snapshot @Tc
→fractal-like spin structure

Typicality (?):
A set of partial systems roughly represent the information of all possible thermal fluctuation.

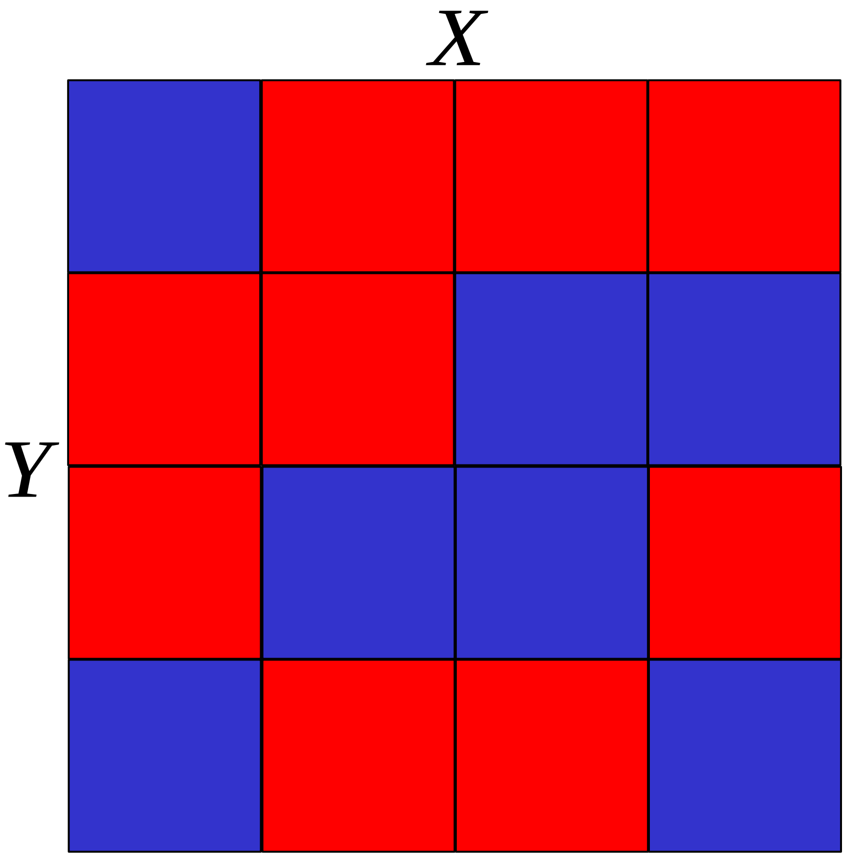


A typical snapshot of the Ising model
256x256, T=2.26J

Single snapshot @Tc⇔Information of partition function

Density matrix of a snapshot

A snapshot determined by Monte Carlo simulation



$$
= \overset{\psi}{\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}}
$$

$$
\rho_Y = \psi \psi^* \quad \rho_X = \psi^* \psi
$$

Matrix product → trace over partial degree of freedom

Singular Value Decomposition (SVD)

Singular Value Decomposition (SVD) of matrix Ψ (Snapshot Data)

$$\psi(x,y) = \sum_l U_l(x)\sqrt{\Lambda_l}\,V_l(y)$$

Λ_l : singular value (non-negative, uniquely determined)

$U_l(x), V_l(y)$: (unitary matrices, various choices)

$$\rho_X(x,x') = \sum_y \psi(x,y)\psi^*(x',y) = \sum_l U_l(x)\,\Lambda_l\,U_l^*(x')$$

$$\rho_Y(y,y') = \sum_x \psi(x,y)\psi^*(x,y') = \sum_l V_l(y)\,\Lambda_l\,V_l^*(y')$$

Snapshot Entropy → boundary law (not extensive)

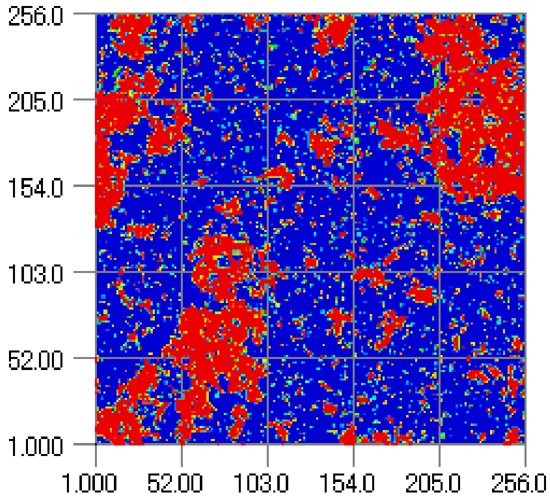
$$S_X = -\sum_l \lambda_l \log \lambda_l = S_Y \qquad \lambda_l = \Lambda_l / \sum_l \Lambda_l$$

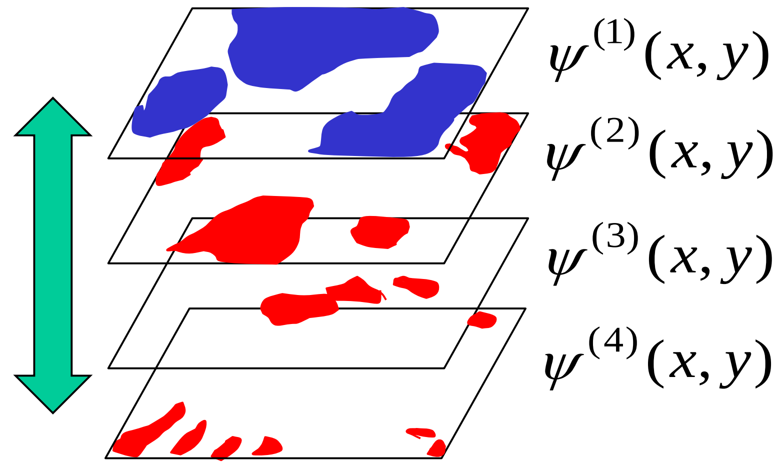
Hyperbolic structure hidden in our SVD method

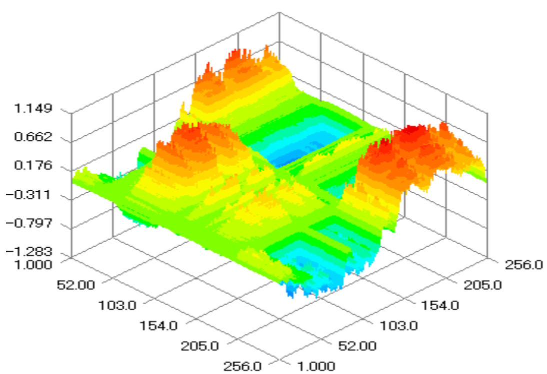
±1 encoding

HM, Phys. Rev. E85, 031101 (2012)

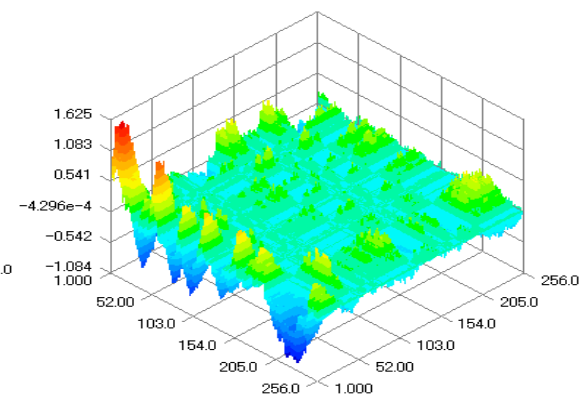
$$\psi(x,y)=\sum_{l=1}^{L}\psi^{(l)}(x,y) \qquad \psi^{(l)}(x,y)=U_l(x)\sqrt{\Lambda_l}V_l(y)$$



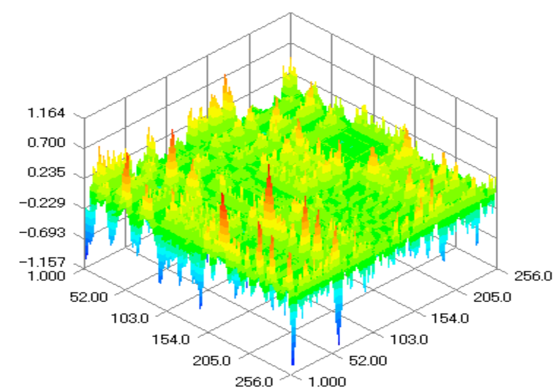




$\psi^{(2)}(x,y)$



$\psi^{(4)}(x,y)$

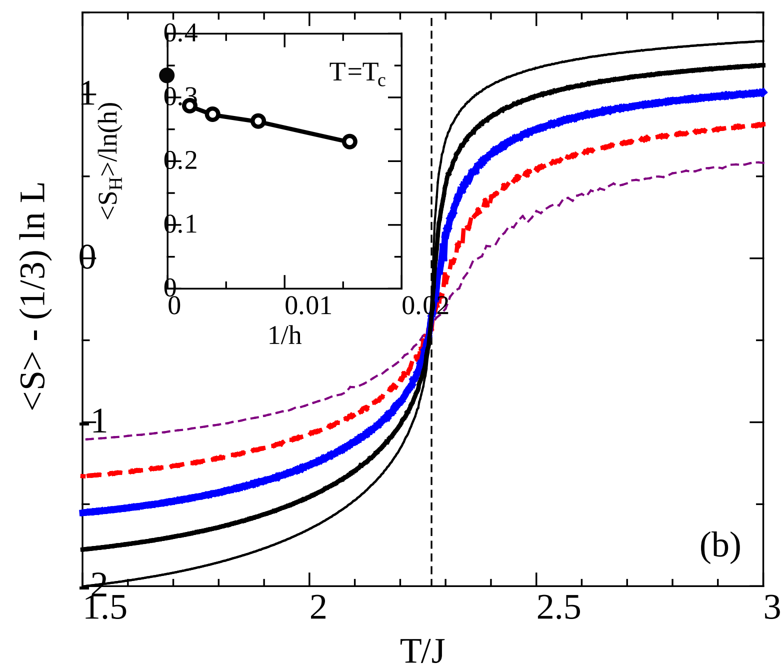


$\psi^{(8)}(x,y)$

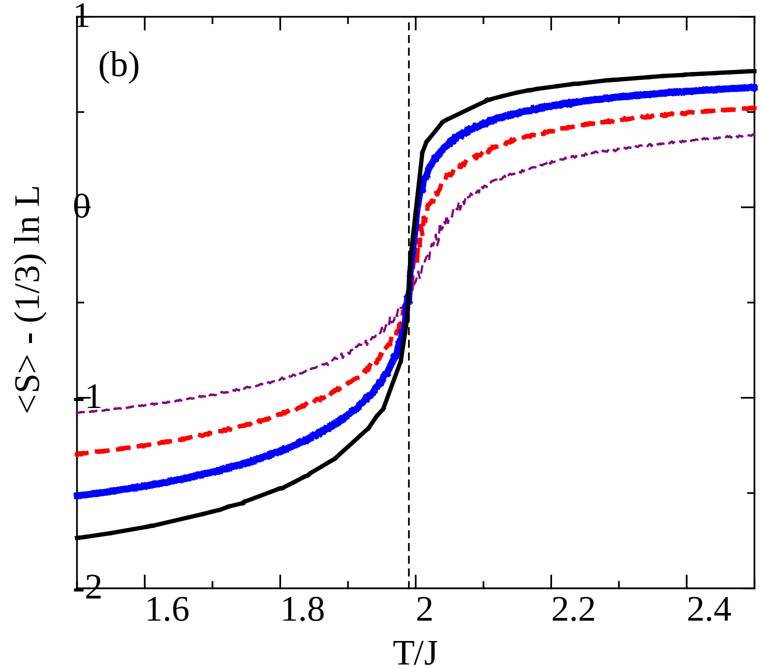
Scaling relation for the snapshot entropy

(0,1) encoding

HM and D.Ozaki, Phys. Rev. E92, 042167 (2015)



2D Ising model(c=1/2)



2D 3-states Potts model(c=4/5)

Scaling formula:

$$\langle S \rangle \approx \frac{1}{3} \log L - \frac{1}{2}$$

Similar to CFT result

Origin of the scaling

⇒layer number of scale decomposition

→ Consistent with RT formula

$$S_{EE} \sim c \langle S \rangle$$

Suzuki–Trotter decomposition

Trotter formula for non-commutative operators A and B

$$e^{A+B} = \lim_{M\to\infty}\left(e^{\frac{A}{M}}e^{\frac{B}{M}}\right)^M$$

1D transverse-field Ising model

$$H = J\sum_i \sigma_i^z \sigma_{i+1}^z + \lambda \sum_i \sigma_i^x$$

$$Z = \sum_{\{\sigma_1\}} \langle \{\sigma_1\} | e^{\beta H} | \{\sigma_1\} \rangle$$

$$|\{\sigma_1\}\rangle = |\{\sigma_{M+1}\}\rangle$$

$$= \sum_{\{\sigma_1\}} \langle \{\sigma_1\} | \left[e^{\frac{\beta J}{M}\sum_i \sigma_i^z \sigma_{i+1}^z} e^{\frac{\beta\lambda}{M}\sum_i \sigma_i^x} \right]^M | \{\sigma_{M+1}\} \rangle$$

$$= \sum_{\{\sigma_1\},\dots,\{\sigma_M\}} \langle \{\sigma_1\} | e^{\frac{\beta J}{M}\sum_i \sigma_i^z \sigma_{i+1}^z} e^{\frac{\beta\lambda}{M}\sum_i \sigma_i^x} | \{\sigma_2\} \rangle \cdots \langle \{\sigma_M\} | e^{\frac{\beta J}{M}\sum_i \sigma_i^z \sigma_{i+1}^z} e^{\frac{\beta\lambda}{M}\sum_i \sigma_i^x} | \{\sigma_1\} \rangle$$

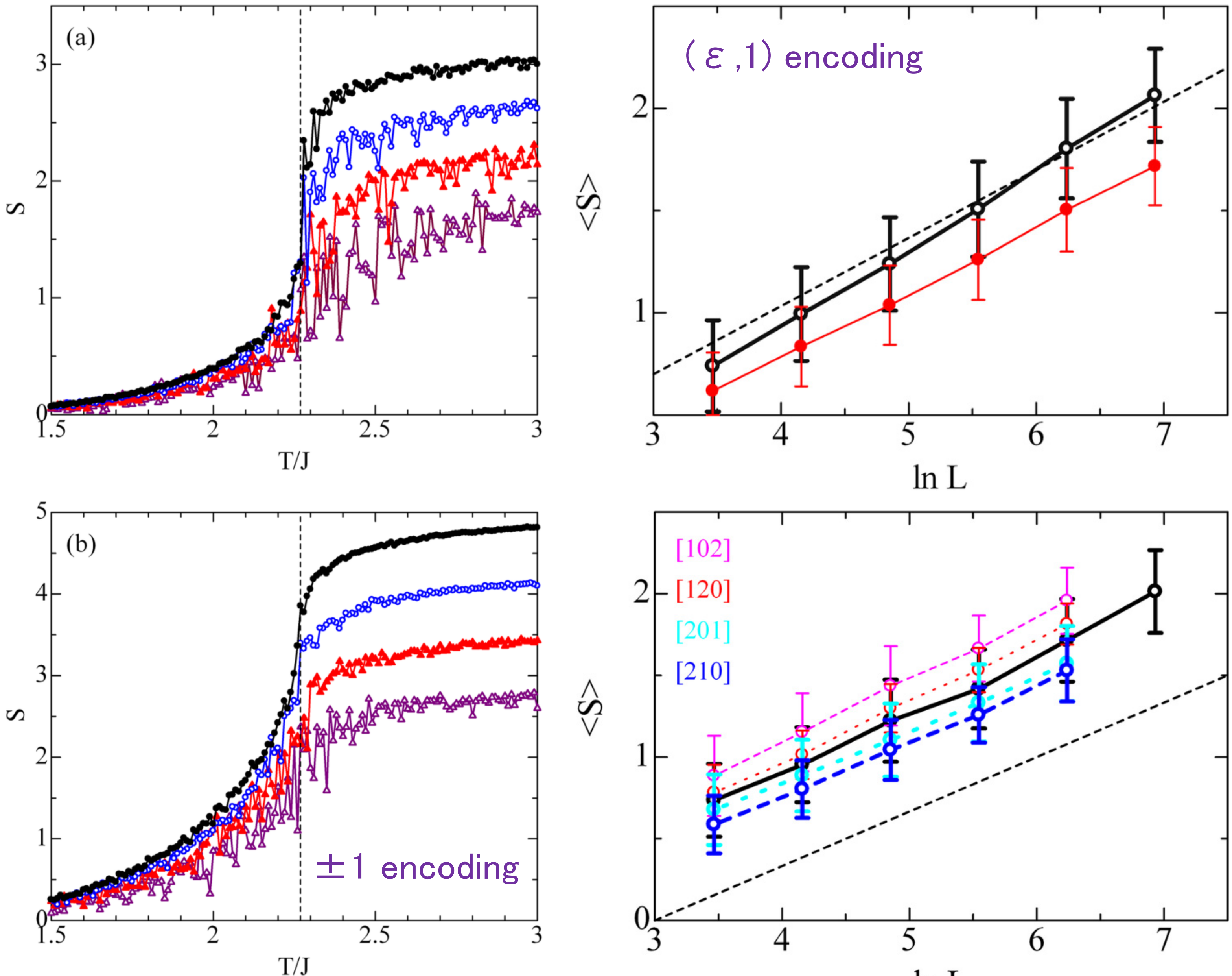
$$= \sum_{\{\sigma_1\},\dots,\{\sigma_M\}} \prod_{k=1}^{M} e^{\frac{\beta J}{M}\sum_i \sigma_i^k \sigma_{i+1}^k} \langle \{\sigma_k\} | e^{\frac{\beta\lambda}{M}\sum_i \sigma_i^x} | \{\sigma_{k+1}\} \rangle$$

$$\langle \sigma | e^{\frac{\beta\lambda}{M}\sigma^x} | \sigma' \rangle = A\exp\left(-\frac{1}{2}\ln\left(\tanh\frac{\beta\lambda}{M}\right)\sigma\sigma'\right)$$

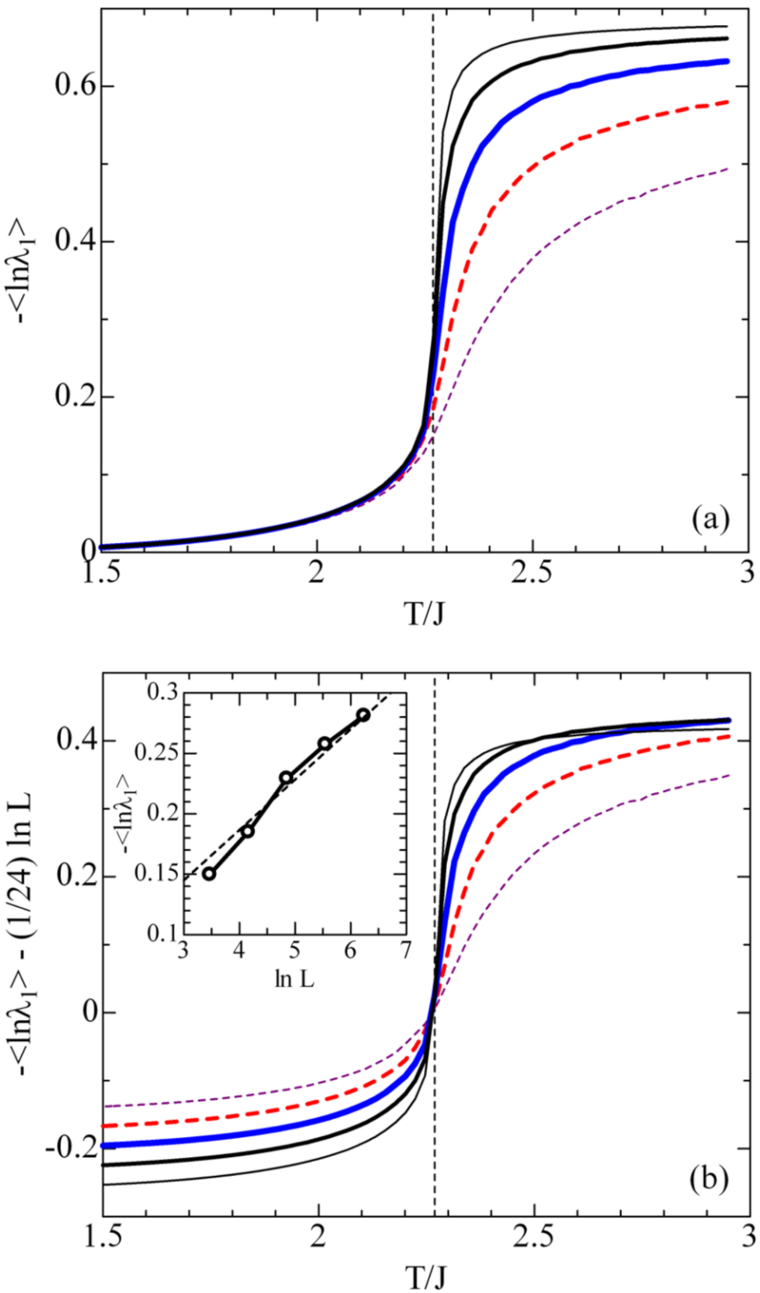
2D classical Ising model

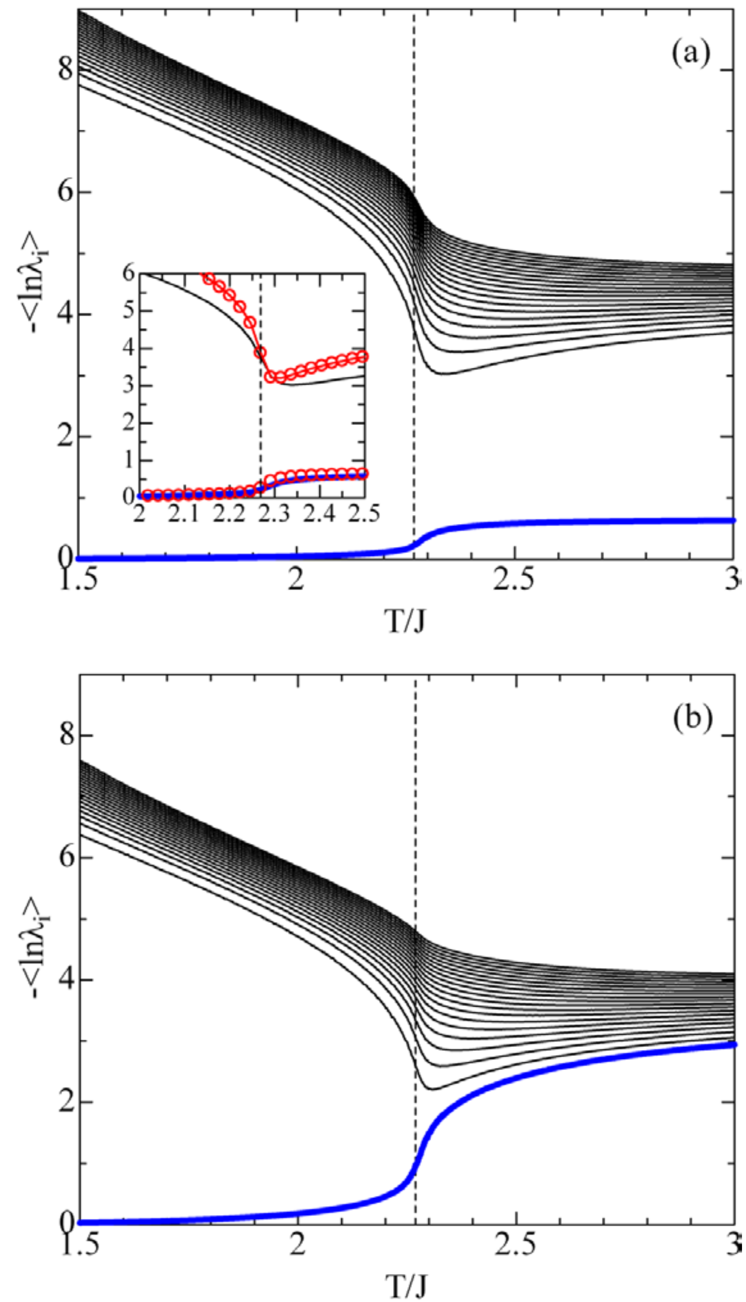
$$Z = A^M \exp\left(K_R \sum_{i,k} \sigma_i^k \sigma_{i+1}^k + K_I \sum_{i,k} \sigma_i^k \sigma_i^{k+1} \right)$$

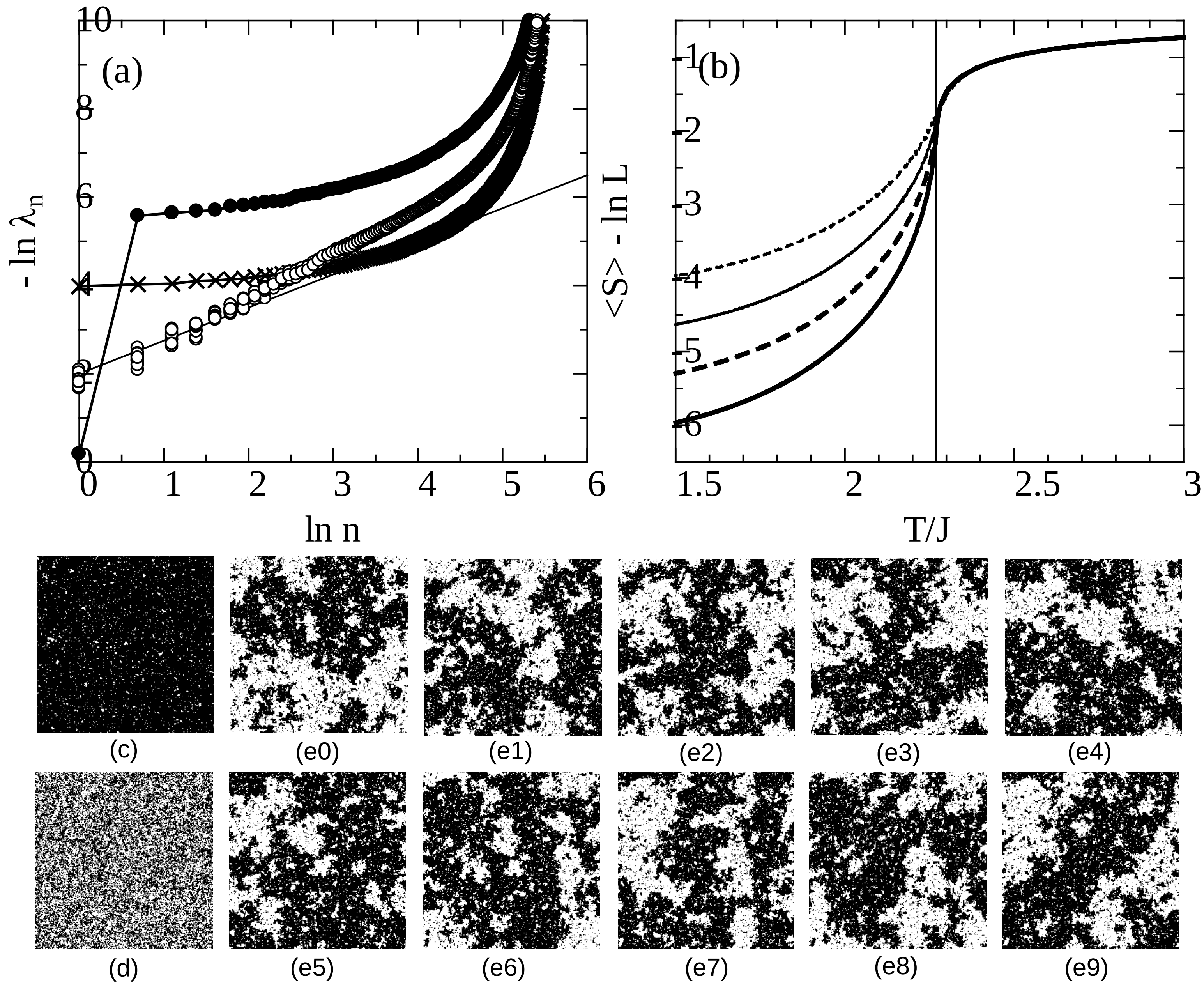
$$A = \frac{1}{2}\sinh\frac{2\beta\lambda}{M}$$



(a)
S
T/J
(ε,1) encoding
⟨S⟩
ln L
(b)
S
T/J
±1 encoding
[102]
[120]
[201]
[210]
⟨S⟩
ln L


-<lnλ1>
T/J
(a)
-<lnλ1> - (1/24) ln L
ln L
(b)


-<lnλi>
T/J
(a)
(b)



±1 encoding

HM, Ching-Hua Lee, and Y. Hashizume (2014)

SVD spectrum → algebraic decay near Tc

* exponent → anomalous dimension

$$f(\lambda) = \sum_n \delta(\lambda - \lambda_n) = A\lambda^{-\alpha} \qquad \alpha = \frac{2-\eta}{1-\eta}$$

Okunishi's work

$$\lambda_n = \frac{a}{n^{\Delta}} \qquad \Delta = 1-\eta$$

$$S_\chi = -\sum_{n=1}^{\chi} \lambda_n \ln \lambda_n$$

$$\approx \frac{\chi^{1-\Delta}-1}{N^{1-\Delta}-1}\left\{\ln\left(N^{1-\Delta}-1\right)-\gamma(\Delta)\right\}+\frac{\Delta}{N^{1-\Delta}-1}\chi^{1-\Delta}\ln\chi$$

$$S_L = \ln L - \gamma(\Delta) \qquad \gamma(\Delta) = \ln(1-\Delta)+\frac{\Delta}{1-\Delta}$$

High-T limit⇒RMT

$$S_L = \ln L - \frac{\pi}{4}$$

Tensor-product construction of Sierpinski carpet

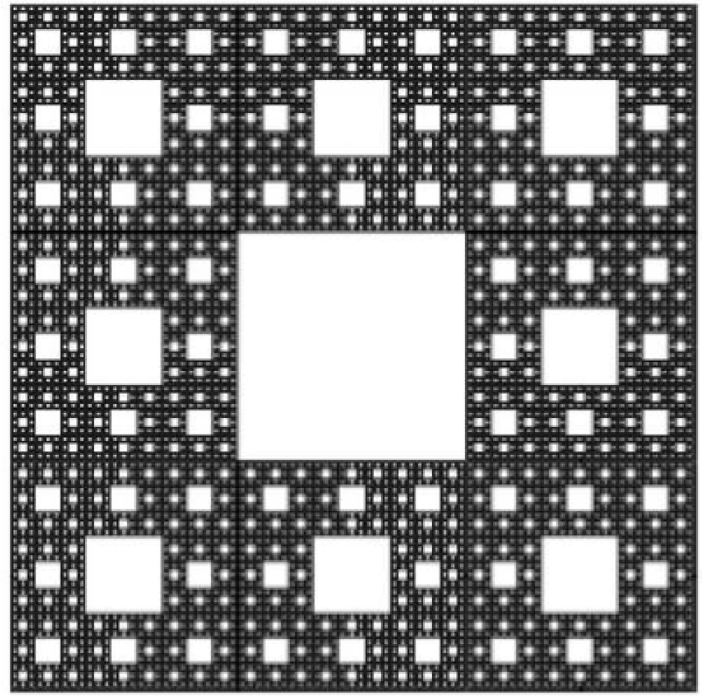
h × h(=3 × 3) unit cell

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Factorized form

$$M = H \otimes H \otimes \cdots \otimes H \otimes H$$

$$H \otimes H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



Fractal image

→ L × L matrix

→ N different scales

$$L = h^N$$

SVD spectrum of Sierpinski carpet

$$M = H \otimes H \otimes \cdots \otimes H \otimes H$$

$$\left(-\sum_{i=\pm} \gamma_i \ln \gamma_i \right) \frac{1}{\ln h} = \frac{c}{3}$$

$$M^2 = H^2 \otimes H^2 \otimes \cdots \otimes H^2 \otimes H^2$$

Two non-zero eigenvalues of H²: $\Gamma_\pm = 4 \pm 2\sqrt{3}$

Normalization of Γ: $\gamma_\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{4}$ $\quad \gamma_- = 1 - \gamma_+$

Eigenvalues of M²: $\lambda_j = \gamma_+^j \gamma_-^{N-j} = \gamma_+^j (1-\gamma_+)^{N-j}$ (Degeneracy : ${}_N C_j$)

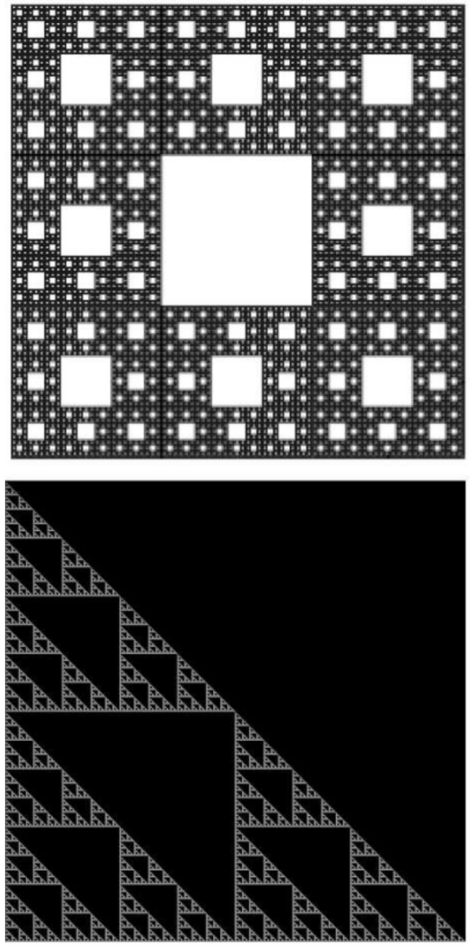
Snapshot entropy ⇔ entanglement entropy of 1D free fermions

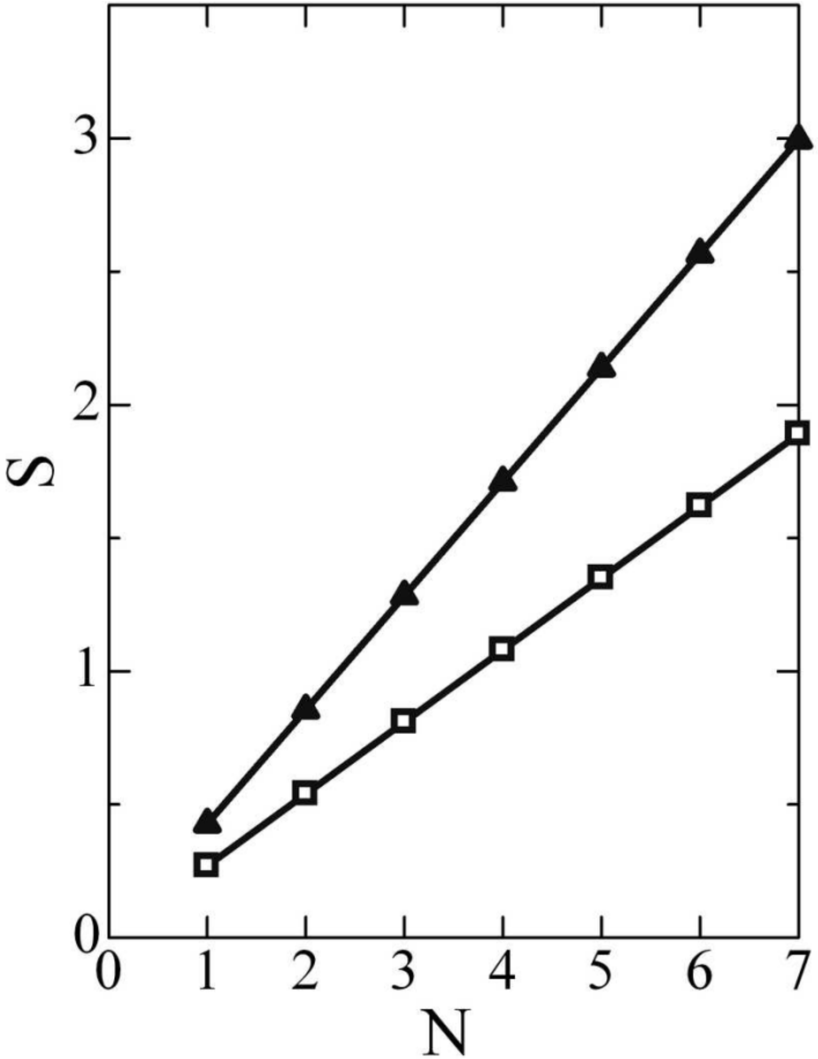
$$S = -\sum_{j=0}^{N} {}_N C_j (\lambda_j \ln \lambda_j) = \left(-\sum_{i=\pm} \gamma_i \ln \gamma_i \right) N = \left(-\sum_{i=\pm} \gamma_i \ln \gamma_i \right) \frac{\ln L}{\ln h}$$

C.H.Lee, Y.Yamada, K.Kumamoto, HM, JPSJ84, 013001 (2015)

I. Peschel, J. Phys. A: Math. Gen. 36, L205 (2003)

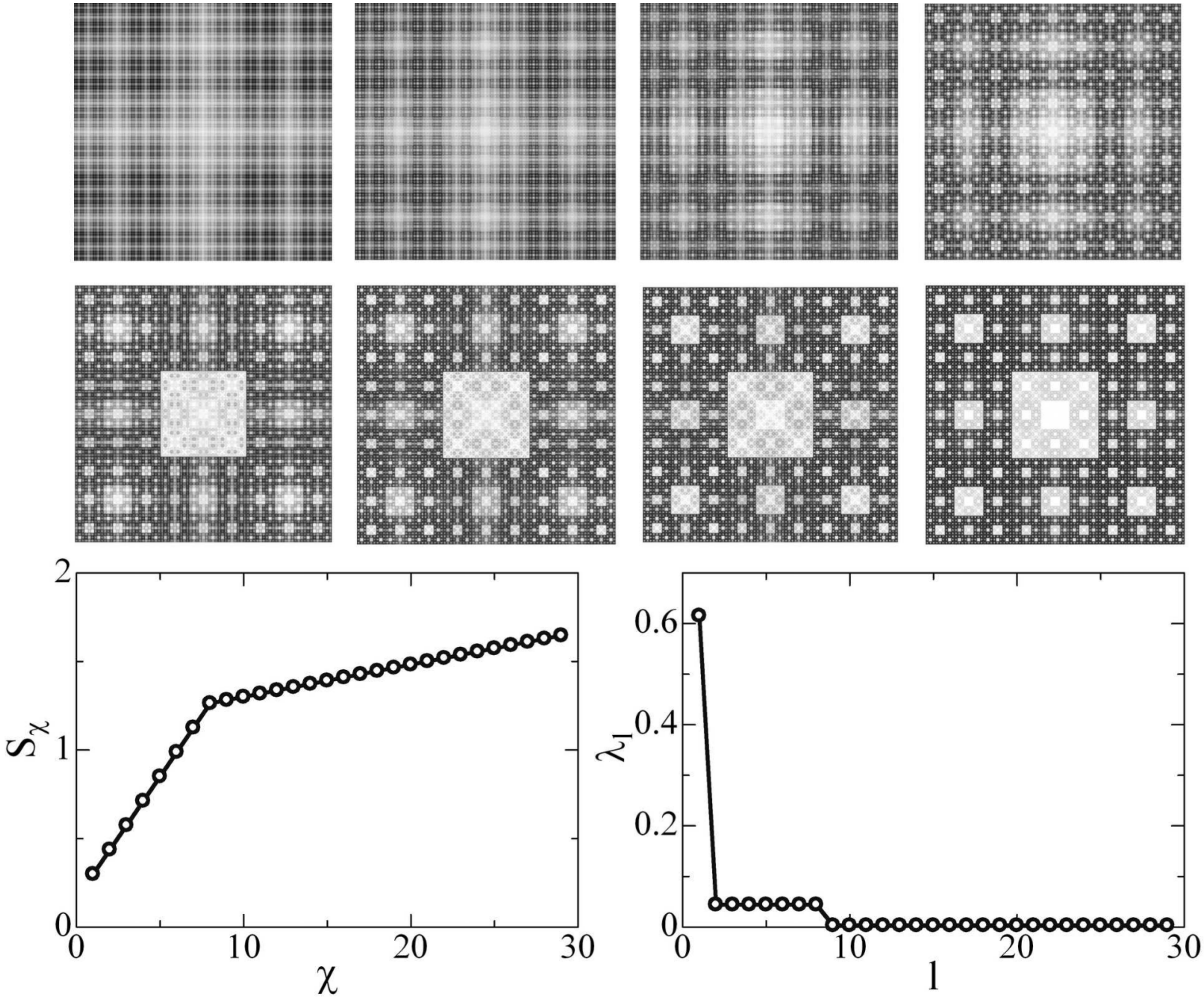
Snapshot entropy as a function of layer number N (Numerical calculation)





C.H.Lee, Y.Yamada, K.Kumamoto, HM, JPSJ84, 013001 (2015)

Coarse-grained snapshot entropy



C.H.Lee, Y.Yamada, K.Kumamoto, HM, JPSJ84, 013001 (2015)

Finite-χ scaling

When we look at the overall structure, the scaling relation seems to be logarithmic.

Fractal → degenerate eigenvalues

We focus on the first (N+1)-th eigenvalues

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{N+1}$$

$$S_\chi = -\lambda_1 \log \lambda_1 - (\chi - 1)\lambda_2 \log \lambda_2$$

$$\lim_{\chi \to 0} S_\chi = 0 \Rightarrow -\lambda_1 \log \lambda_1 = -\lambda_2 \log \lambda_2 \Rightarrow S_\chi = S_1 \chi$$

cf. finite-entanglement scaling near 1D quantum criticality

$$S_\chi = \frac{c\kappa}{6} \log \chi = \frac{1}{\sqrt{12/c} + 1} \log \chi$$

Their difference may come from violation on full conformal symmetry on the fractal image that has just scale invariance.

Summary

The SVD spectra for the snapshots of the 2D Ising & q=3 Potts models represent the information of the two-point correlator of spins.
Thus the SVD data are good benchmarks for the phase transition.

Near Tc, the snapshot entropy obeys the logarithmic scaling, which is consistent with the CFT formula.

The SVD data contain the information similar to the holographic entropy formula.