Surface transport properties in stationary relativistic fluids

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based on: 1512.08514, with Jay Armas and Nilay Kundu.
and work in progress with Felix Haehl, Jay Armas and Nilay Kundu.

Motivation

- Hydrodynamics is the effective long-wavelength description of an underlying finite temperature quantum field theory.
- The effective description is provided in terms of a few universal degrees of freedom such as $\{u_\mu, T, \mu\}$.
- Information regarding the microscopics is present in the parameters of the effective theory called *transport coefficients*.
- There has been a lot of recent development on the structural aspects of equations governing the effective theory.

- Most of the focus so far, has been to construct the effective theory of the states which describes space-filling configurations on non-compact manifolds.
- We will discuss the necessary modifications of the effective theory, when we also incorporate the states describing finite lumps of matter, in the fluid description: fluids with a *surface*.
- The situation is similar to describing the fluids on a manifold with a boundary, which itself is dynamical.
- The plasma-balls of $\mathcal{N} = 4$ SYM are a concrete example for which our effective description would be applicable.
  [Aharony, Minwalla, Wisemen]

Outline

Introduction

- The dynamics of the fluid are determined by symmetry principles, besides thermodynamics

$$\nabla_\mu T^{\mu\nu} = 0, \ \nabla_\mu J^\mu = 0.$$

- These are insufficient conditions to determine all the independent component of the currents.
- So we need to express the currents in terms of the fluid variables $\{u_\mu, T, \mu\}$, through the *constitutive relations*
- This is performed in a derivative expansion

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + \ldots, \ J_\mu = J^{(0)}_\mu + J^{(1)}_\mu + \ldots$$

- For example

$$T_{\mu\nu} = \mathcal{E}\ u_\mu u_\nu + \mathcal{P}\ P_{\mu\nu} + \eta\ \sigma_{\mu\nu} + \zeta\ \Theta\ P_{\mu\nu} + \ldots$$

- Fluid variables have an ambiguity of definition $\rightarrow$ fixed by the *choice of frame*

$$\text{Landau frame}: u_\mu T^{\mu\nu} = -\mathcal{E} u_\nu$$

Equilibrium partition function

[Banerjee,JB,Bhattacharyya,Jain,Minwalla,Sharma]

- All fluid equations must admit a stationary solutions when studied on slowly varying stationary background.
- One should be able to generate the stress tensor (evaluated on this stationary solution) from a partition function written purely in terms of sources.
- Let us consider system in thermal equilibrium on the most general stationary background geometry

$$ds^2 = G_{\mu\nu} dx^{mu} dx^{\nu} = -e^{2\sigma(\vec{x})} \left(dt + a_i(\vec{x}) dx^i\right)^2 + g_{ij}(\vec{x}) dx^i dx^j .$$

- Here $\{\sigma, a_i, g_{ij}, T_0, A_0, A_i\}$ constitutes the set of background data.
- Fluid limit $\Rightarrow$ Background fields are slowly varying compared to length scale set by T_0.

Procedure

- The partition function written in terms of the sources

$$\mathcal{W} = \ln \mathcal{Z} = \int dx^3 S(\sigma, a^i, g_{ij}, T_0)$$

- S is again expanded in a derivative expansion

$$S = S_0 + S_1 + S_2 + \dots$$

- We need the most general S_k so that it is invariant under all transformations that keeps the metric time independent.
  - S_k must be invariant under $\vec{x}$ diffeomorphism.
  - S_k must be invariant under KK gauge transformation

$$t \rightarrow t + \Lambda(\vec{x}) \Rightarrow a_i \rightarrow a_i + \partial_i \Lambda$$

    which means the dependence on a_i is only through the combination $f_{ij} = \partial_i a_j - \partial_j a_i$.
- Let p_k be the total number of terms that can be written down at any order k.

Procedure

- Let $T^F_{\mu\nu}$ be the most general symmetry-allowed fluid stress tensor, evaluated on general stationary fluid solutions, in a given *frame*.
- This has t_k number of transport coefficients that survives time independent limit.
- We demand this stress tensor is same as that obtained from the partition function $T^{\mathcal{W}}_{\mu\nu}$

$$T^{\mathcal{W}}_{\mu\nu} = T^F_{\mu\nu}$$

- Using this we can do two things
  - Determine t_k transport coefficients in terms of p_k arbitrary functions of the partition function reducing the number of independent transport coefficients to p_k.
  - Determine the fluid variables (the equilibrium solution),in the chosen *frame* , order by order in terms of the background sources.

Ideal fluids

- The ideal fluid stress tensor is

$$T^{F\,\mu\nu} = \epsilon(T)u^\mu u^\nu + p(T)P^{\mu\nu}.$$

- The partition function at this order following our prescription

$$\mathcal{W} = \ln \mathcal{Z} = \int d^d x \sqrt{g} \frac{e^\sigma}{T_0} \mathcal{P}(T_0 e^{-\sigma}) + \ldots$$

- The stress tensor from the partition function is

$$[T^{\mathcal{W}}]^i_0 = 0, \quad [T^{\mathcal{W}}]_{00} = T_0 e^\sigma \mathcal{P}' - e^{2\sigma}\mathcal{P}, \quad [T^{\mathcal{W}}]^{ij} = \mathcal{P} g^{ij}$$

- Comparing we find
  - $u^\mu = e^\sigma\{1, 0, 0, \ldots\}$, and $T = T_0 e^{-\sigma}$
  - $\epsilon = T\ d\mathcal{P}/dT - \mathcal{P}, p = \mathcal{P} \Rightarrow \epsilon + p = T(dp/dT)$

Subleading stationary corrections

- For uncharged fluids there are no terms that one can write down in the partition function at first order.
- At second order we can write 3 terms

$$\mathcal{W} = \log \mathcal{Z} = \int_{\mathcal{M}_s} d^3x\sqrt{g}\left(\frac{e^{\sigma}}{T_0}\,\mathcal{P}\left(T_0 e^{-\sigma}\right) - \frac{1}{2}\left[P_1(\sigma)R + T_0^2 P_2(\sigma) f_{ij}f^{ij} + P_3(\sigma)(\partial\sigma)^2\right]\right).$$

- Purely on symmetry grounds, we can write 8 stationary terms in the stress tensor at second order in the Landau frame.

$$\begin{aligned} T_{\mu\nu} =& T\Big(\ \kappa_1 R_{\langle\mu\nu\rangle} + \kappa_2 \mathcal{R}_{\langle\mu\nu\rangle} + \lambda_3\ \omega^{\ \alpha}_{\langle\mu}\omega_{\alpha\nu\rangle} + \lambda_4\ \mathfrak{a}_{\langle\mu}\mathfrak{a}_{\nu\rangle} \\ & + P_{\mu\nu}(\zeta_2 R + \zeta_3 R_{\mu\nu}u^\mu u^\nu + \xi_3\omega^2 + \xi_4\mathfrak{a}^2)\ \Big) \quad . \end{aligned}$$

- Comparing we get 5 relations among transport coefficients in addition to the *second order corrections* to the fluid fields in the Landau frame.
- These relations are precisely coincide with those implied by the second law of thermodynamics, when we perform a complete entropy current analysis including dissipation.

Introducing the surface

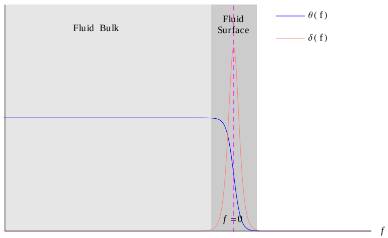
- The stress tensor and the currents take the form

$$T^{\mu\nu} = \theta(f)\,T^{\mu\nu}_{\text{blk}} + \tilde{\delta}(f)\,T^{\mu\nu}_{\text{sur}} + \ldots,\;\; J^{\mu} = \theta(f)\,J^{\mu}_{\text{blk}} + \tilde{\delta}(f)\,J^{\mu}_{\text{sur}} + \ldots$$

- The bulk currents are conserved as usual, while at the surface

$$\nabla_\mu T^{\mu\nu}_{\text{sur}} - n_\mu T^{\mu\nu}_{\text{sur}} = 0,\;\; \nabla_\mu J^{\mu}_{\text{sur}} - n_\mu J^{\mu}_{\text{sur}} = 0$$

- The location of the surface is given by $f = 0$.
- In general $\theta(f)$ and $\delta(f)$ also depends on the dimensionless ratio τ/T.



- The partition function should take the general form

$$\log \mathcal{Z} = \int \theta(f)\; S_{\text{blk}} + \tilde{\delta}(f)\; S_{\text{sur}}$$

Fluid variables and frames choice

- We would like to have continuous fluid variables near the surface and we must also have $u^\mu n_\mu|_{f=0} = 0$.
- We should view the surface fluid variables as dynamical boundary conditions on the bulk fluid variables

$$(u^\mu_{\text{blk}})n_\mu|_{f=0} = 0, \text{ and } \{e^\mu_a u_\mu, T\}_{\text{blk}}|_{f=0} = \{u^{\mathfrak{s}}_a, T^{\mathfrak{s}}\}_{\text{sur}},$$

- Since the stress tensor and the current is discontinuous at the surface, choosing a Landau frame would also make the fluid variables discontinuous.
- Choose the same frame on the surface as in the bulk.
- Some natural choices are
  - The fluid velocity is identified with the time-like killing vector everywhere and with n_μ being orthogonal to this killing vector on the surface $u^\mu n_\mu|_{f=0}$ automatically vanishes.
  - A modified *Orthogonal Landau frame*.

Leading order at the surface: Surface tension

- At the leading order, we just have one term, same in the fluid bulk

$$\mathcal{W} = \log \mathcal{Z} = \int_{\mathcal{N}_s} d^3x\sqrt{g}\left(\theta(f)\,\frac{e^{\sigma}}{T_0}\,\mathcal{P}\left(T_0 e^{-\sigma}\right) + \tilde{\delta}(f)\,\frac{e^{\sigma}}{T_0}\,\mathcal{C}\left(T_0 e^{-\sigma}\right)\right)$$

- The surface stress tensor takes the form

$$T^{\mu\nu}_{\text{sur}} = \chi_E(T)\,u_\mu u_\nu - \chi(T)\,\mathcal{P}_{\mu\nu} + \ldots, \text{where } \chi = -\mathcal{C}, \chi_E = -\mathcal{C} + T\frac{\partial \mathcal{C}}{\partial T}.$$

- We immediately have a surface thermodynamics with

$$\chi_E = \chi + T\chi_S$$

Laplace-Young equation

- The normal component of the stress tensor conservation equation at the surface is non-trivial

$$P(T)|_{f=0} = \chi K + (\chi_E - \chi)\ \ n_\mu \mathfrak{a}^\mu|_{f=0},$$

- This equation is identical to the equation of motion of the function $f(\vec{x})$, if we were to consider it as a dynamical field.
- If $\chi_E = \chi$ or equivalently $\chi_s \equiv \partial\chi/\partial T = 0$ then this reduces to the familiar Laplace-Young equation.
- The new term in the modified Laplace-Young equation can be explained as a centripetal acceleration arising out of non-negligible surface degrees of freedom.

Subleading order at the surface

- At first order on the surface, we can write 3 new terms in the partition function

$$\mathcal{W} = \log \mathcal{Z} = \int_{\mathcal{M}_s} d^3x\sqrt{g}\left(\frac{e^{\sigma}}{T_0}\,\mathcal{P}\left(T_0 e^{-\sigma}\right) - \frac{1}{2}\left[P_1(\sigma)R + T_0^2 P_2(\sigma) f_{ij} f^{ij} + P_3(\sigma)(\partial\sigma)^2\right]\right)$$
$$+ \int_{\partial\mathcal{M}_s} d^2x\sqrt{\gamma}\frac{e^{\sigma}}{T_0}\left(\mathcal{C}\left(T_0 e^{-\sigma}\right) + \mathcal{B}_1\left(T_0 e^{-\sigma}\right) n^i\, \partial_i\sigma + \mathcal{B}_2\left(T_0 e^{-\sigma}\right) \epsilon^{ijk} n_i f_{jk} + \mathcal{B}_3\left(T_0 e^{-\sigma}\right)\mathcal{K}\right)\Big|_{f=0}$$

- Stress tensor is straightforwardly obtained by varying the partition function.
- There are 31 symmetry allowed terms that can be written down in the stress tensor $\Rightarrow$ 31 transport coefficients.
- Comparison with the stress tensor from the partition function gives us 28 relations among the transport coefficients.
- The bulk second order transport coefficients enter these relations non-trivially.

A close look at the parity odd term

- We have a parity odd surface term in the partition function

$$\mathcal{W} \supset \int_{\partial \mathcal{M}_s} \mathcal{B}_2 \left(T_0 e^{-\sigma} \right) \epsilon^{ijk} n_i f_{jk}$$

- This gives rise to two terms in the surface stress tensor

$$T^{\mu\nu}_{\text{sur}} = \cdots + \mathfrak{s} \; u^\mu u^\nu \; n_\nu \ell^\nu + \mathfrak{v} \; u^{(\mu} \left(\epsilon^{\nu)\sigma\rho\lambda} u_\sigma n_\rho \mathfrak{a}_\lambda \right) + \ldots$$

- The two transport coefficients are related $\mathfrak{s} + \mathfrak{v} = 0$, since both are determined by $\mathcal{B}_2$.

Superfluids

- In case of superfluids the partition function can also depend on the superfluid velocity ξ_μ [Jensen, Kaminski, Kovtun, Meyer, Ritz, Yarom] [Bhattacharyya,Jain,Minwalla,Sharma]

$$\mathcal{W} = \log \mathcal{Z} = \int_{\mathcal{M}} d^3x \sqrt{g} \frac{e^{\sigma}}{T_0} \mathcal{P} \left(T_0 e^{-\sigma}, A_0 e^{-\sigma}, \xi \right) + \int_{\partial \mathcal{M}} d^2x \sqrt{\gamma} \frac{e^{\sigma}}{T_0} \mathcal{C} \left(T_0 e^{-\sigma}, A_0 e^{-\sigma}, \xi \right) \Big|_{f=0} ,$$

- The bulk and surface currents both take the form

$$T^{\mu\nu} = \epsilon \; u^{\mu} u^{\nu} + P \; \mathcal{P}_{\mu\nu} + \lambda \; \xi^{\mu} \xi^{\nu}, \quad J^{\mu} = q \; u^{\mu} - \lambda \; \xi^{\mu}.$$

- The Laplace-Young equation now has another new contribution

$$P(T)|_{f=0} = -\chi K + (\chi_E + \chi) \; n_{\mu} \mathfrak{a}^{\mu}|_{f=0} + \lambda \; n^{\mu} \; \xi^{\nu} \; \nabla_{\nu} \xi_{\mu}|_{f=0} \quad .$$

Zeroth order parity odd effects for superfluids

- At the first order, in the bulk of a superfluid we can write two parity odd terms in the partition function

$$\mathcal{W} \supset S^{odd} = \int \sqrt{g} d^3x \left(g_1 \epsilon^{ijk} \zeta_i \partial_j A_k + T_0 g_2 \epsilon^{ijk} \zeta_i \partial_j a_k \right) + \dots$$

- This gives rise to zeroth order parity odd terms in the surface currents

$$T^{\mu\nu}_{\text{sur}} = \epsilon u^\mu u^\nu + P\mathcal{P}_{\mu\nu} + \lambda \xi^\mu \xi^\nu + \gamma_1 \ u^{(\mu} \epsilon^{\nu)\sigma\lambda\rho} u_\sigma n_\lambda \xi_\rho,$$
$$J^\mu_{\text{sur}} = q \ u^\mu - \lambda \ \xi^\mu + \gamma_2 \ \epsilon^{\mu\nu\lambda\rho} u_\nu n_\lambda \xi_\rho.$$

- Transformation of the measure of the path integral $\Rightarrow$ anomaly in the current conservation equation.

$$\nabla_\mu J^\mu = c\ (\star\ \ F \wedge F)\,.$$

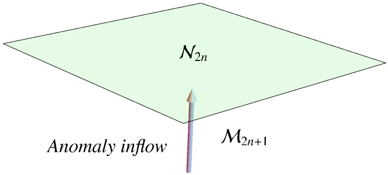
- It can also be understood by considering the system

$$\mathcal{W} = \int_{\mathcal{M}_5} A \wedge F \wedge F + \int_{\mathcal{N}_4} S$$

- The conservation equation for $\mathcal{W}$

$$\nabla_\mu J^\mu_{\text{cov}} = J^\perp_{\mathcal{H}}$$

- The Bardeen-Zumino shift is automatic when we vary $\mathcal{W}$ to obtain the current.



- For anomalous fluids the currents also contain a contribution from the anomaly.
- This effect may be captured by writing down a partition function

$$\mathcal{W} = \int_{\mathcal{M}_5} \frac{\mathfrak{u}}{2\boldsymbol{\omega}} \wedge \left(\mathcal{P} - \hat{\mathcal{P}}\right) = \int_{\mathcal{M}_5} \mathbf{I} - \hat{\mathbf{I}} + \int_{\mathcal{N}_4} \frac{\mathfrak{u}}{2\boldsymbol{\omega}} \wedge \left(\mathbf{I} - \hat{\mathbf{I}}\right)$$

$$\frac{\mathfrak{u}}{2\boldsymbol{\omega}} \wedge \left(\mathcal{P} - \hat{\mathcal{P}}\right) = -\mu\, \mathbf{u} \wedge \left(3\mathbf{B}^3 + 6\mu\boldsymbol{\omega}\mathbf{B} + 4\mu^2\boldsymbol{\omega}^2\right),\ \frac{\mathfrak{u}}{2\boldsymbol{\omega}} \wedge \left(\mathbf{I} - \hat{\mathbf{I}}\right) = \mathfrak{u} \wedge \left(-2\mu A \wedge \mathbf{B} - 2\mu^2 A \wedge \boldsymbol{\omega}\right)$$

[Haehl,Loganayagam,Rangamani; Jensen,Loganayagam,Yarom]

- This gives a fluid current

$$J^{\mu}_{\mathcal{N}_4} = \cdots + \xi_B\, B^{\mu} + \xi_{\ell}\, \ell^{\mu} + \ldots$$

- The conservation of this current is violated by the current due to the term in $\mathcal{M}_5$ flowing into $\mathcal{N}_4$, which precisely accounts for the anomaly.

Surface of Anomalous fluids

- When anomalous fluids form a surface there are two questions
  - What happens to the LHS of the conservation equation

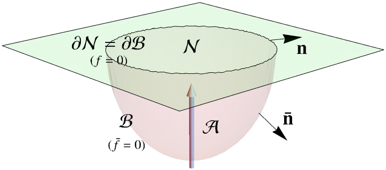
$$\nabla_\mu J^\mu = 0, \begin{cases} \nabla_\mu J^\mu_{\text{blk}} = 0 \\ \nabla_\mu J^\mu_{\text{sur}} - n_\mu J^\mu_{\text{blk}} = 0 \end{cases}, \quad \nabla_\mu J^\mu = c\,(\star \; F \wedge F)\,, \begin{cases} \nabla_\mu J^\mu_{\text{blk}} = ? \\ \nabla_\mu J^\mu_{\text{sur}} - n_\mu J^\mu_{\text{blk}} = ? \end{cases}$$

  - How are the anomalous terms within J^μ_{blk} balanced at the surface.
- No anomalies in odd dimensions so perhaps

$$\nabla_\mu J^\mu = c\,(\star \; F \wedge F)\,, \begin{cases} \nabla_\mu J^\mu_{\text{blk}} = c\,(\star \; F \wedge F) \\ \nabla_\mu J^\mu_{\text{sur}} - n_\mu J^\mu_{\text{blk}} = 0 \end{cases}$$

- We may extend the surface ∂N into $\mathcal{M}$ as a surface $\mathcal{B}$ and study the inflow of anomaly in this set up.

- Gauge invariance will require us to write down some additional terms on $\mathcal{B}$, in the partition function.



- We may now consider a system

$$\mathcal{W} = \int_{\mathcal{M}_5} \mathbf{I} - \hat{\mathbf{I}} + \int_{\mathcal{B}} \frac{\mathfrak{u}}{2\omega} \wedge \left(\mathbf{I} - \hat{\mathbf{I}}\right) + \int_{\mathcal{N}_4} \frac{\mathfrak{u}}{2\omega} \wedge \left(\mathbf{I} - \hat{\mathbf{I}}\right)$$

Conservation equations in presence of the surface

- The conservation equation in $\mathcal{N}_4$ splits up into bulk and surface pieces

$$\nabla_\mu J^\mu = J^\perp_{\mathcal{H}}, \quad \begin{cases} \nabla_\mu J^\mu_{\text{blk}} = J^{(b)\perp}_{\mathcal{H}} \\ \nabla_\mu J^\mu_{\text{sur}} - n_\mu J^\mu_{\text{blk}} = J^{(s)\perp}_{\mathcal{H}} \end{cases},$$

- The covariant surface current vanishes although the consistent surface current is non-trivial.
- Similarly for the stress tensor

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J^\mu + T^{\perp\nu}_{\mathcal{H}}, \quad \begin{cases} \nabla_\mu T^{\mu\nu}_{\text{blk}} = F^{\mu\nu} J^\mu_{(b)} \\ \nabla_\mu T^{\mu\nu}_{\text{sur}} - n_\mu T^{\mu\nu}_{\text{blk}} = T^{(s)\perp\nu}_{\mathcal{H}} \end{cases},$$

Discussion

- Our Partition function may pave way to understand how surfaces may effectively emerge from an underlying microscopics.
- Derive the surface transport coefficients from microscopics with Kubo-like formulae.
- Parity-odd transport coefficients may serve as simple ways to detect parity violation. Parity-odd effects may arise from some effective breaking of the parity-symmetry.
- It would be interesting to analyze the small fluctuations of the surface and try to understand dissipative effects in that context.
- A non-relativistic limit of our setup may provide interesting predictions for the surface behaviour of laboratory fluids.