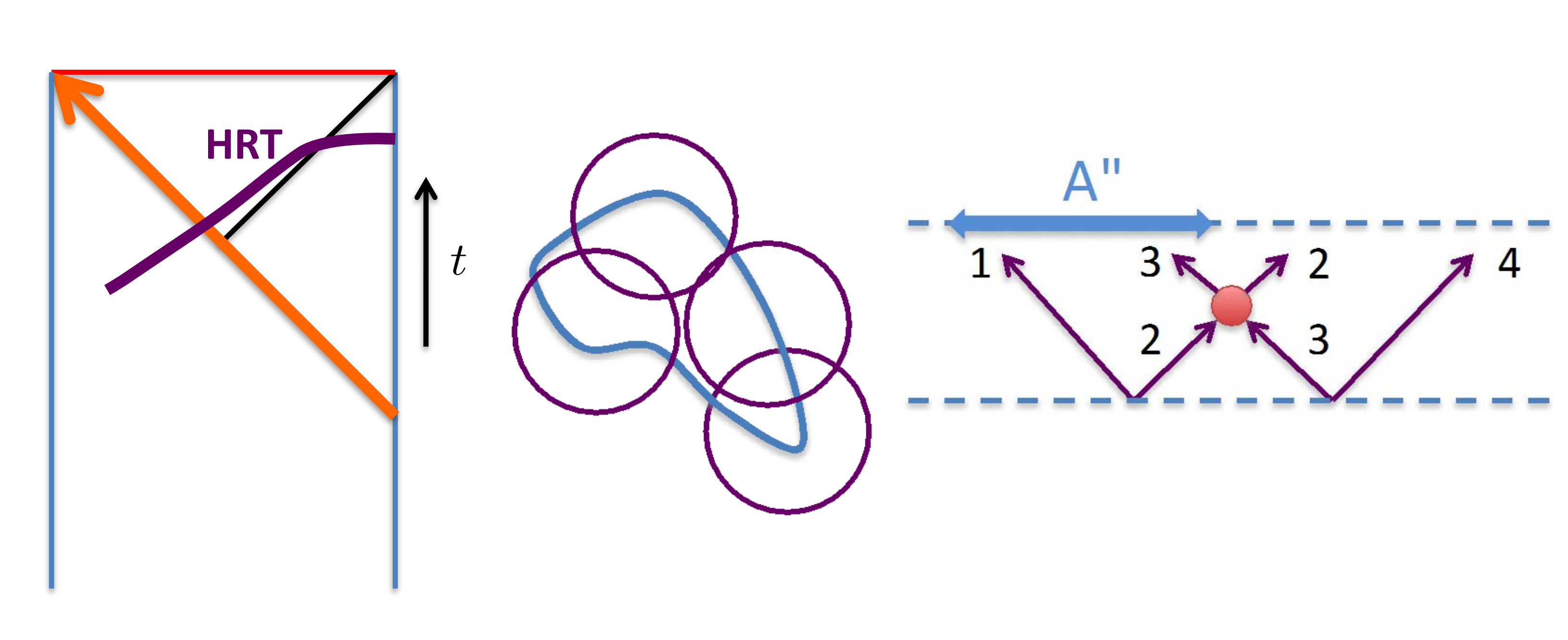
Spread of entanglement and chaos



Márk Mezei (Princeton)

MM, Stanford [to appear]; Casini, Liu, MM [1509.05044]

Quantum Information in String Theory and Many-body Systems YITP, 6/8/2016

Outline

Entanglement generation and chaos

- The three velocities

Relations between the velocities

- General considerations
- Holographic results
- Spin chain results

Benchmarking and interpretation

- Free streaming
- Free scalar theory
- Operator and tensor network models

Summary and open questions

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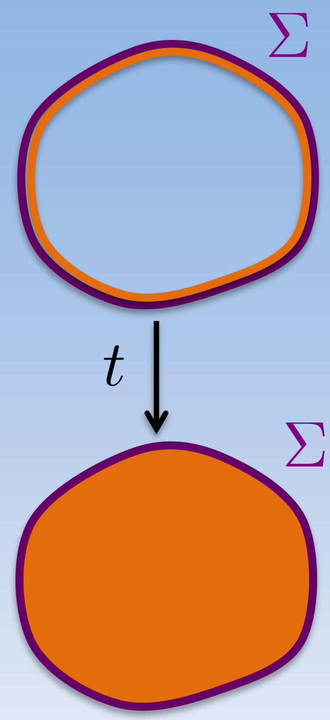
Entanglement generation in global quenches

Global quench:

- Thermalization in a pure state $|\psi(t)\rangle$
- Start with QFT in a short-range entangled state at t=0. (E.g. inject uniform energy density or change the Hamiltonian)
- One-point functions reach thermal value $t_{\text{loc}} \sim 1/T$
- EE (similarly to $\langle \phi(R)\, \phi(0)\rangle$) take $t_s \sim R$ to saturate to thermal value
- Good diagnostic of thermalization is how close $\rho_r\,(|\psi(t)\rangle)$ is to $\mathrm{Tr}_{\bar{V}} e^{-\beta(E)\, H}$



$$S_0 = \frac{A_\Sigma}{\delta^{d-2}} + \ldots$$

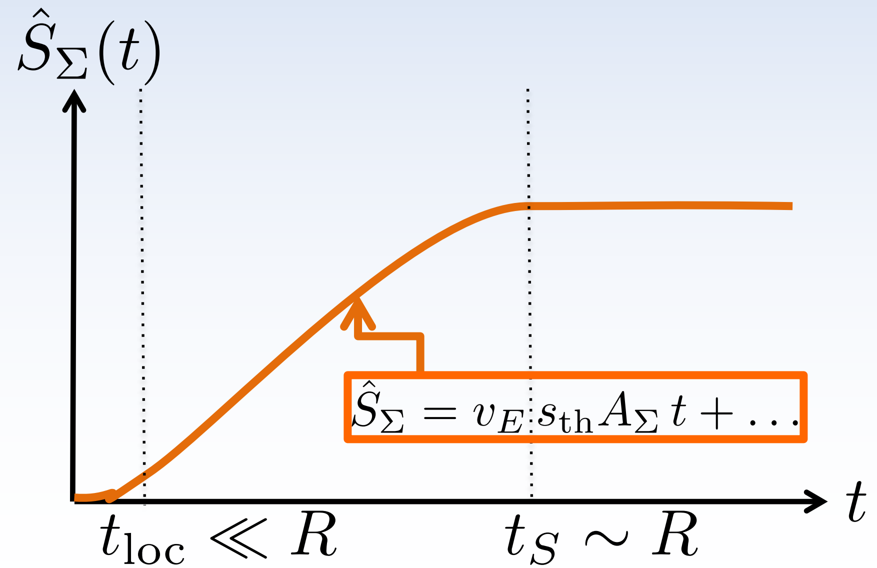
Typical point inside is unentangled with outside

$$S_{\text{eq}} = s_{\text{th}}\, V_\Sigma + \ldots$$

Typical point inside is entangled with outside

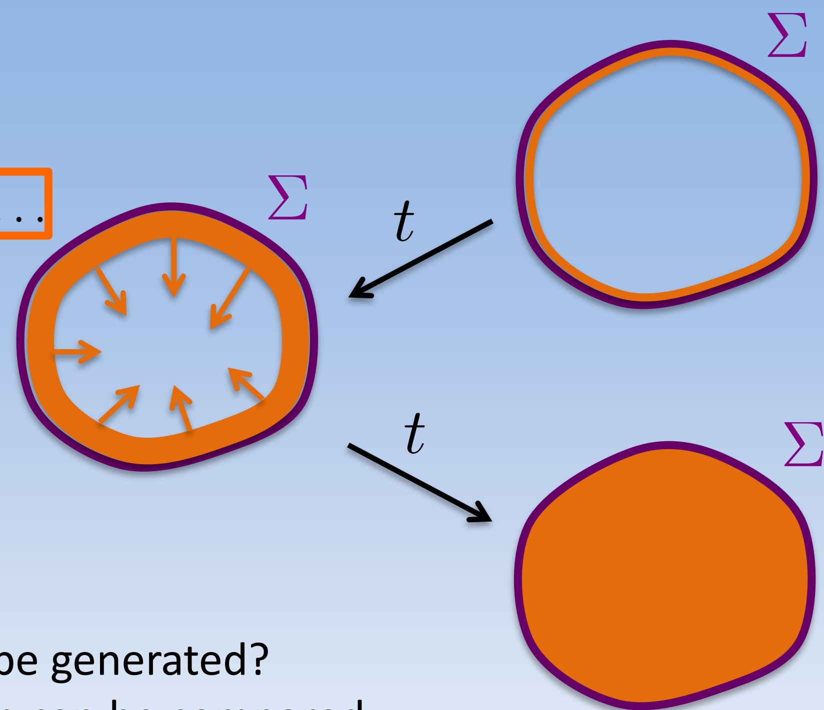
What is the time evolution of EE?

- 2d: numerics, CFT techniques [Huse, Kim; MM, Stanford; Calabrese, Cardy]
- d>2: holography, free field theory [Hartman, Maldacena; Liu, Suh; Cotler, Hertzberg, MM, Mueller]



Entanglement generation in global quenches

$$\boxed{\hat{S}_\Sigma = v_E\, s_{\rm th} A_\Sigma\, t + \ldots}$$



$$S_0 = \frac{A_\Sigma}{\delta^{d-2}} + \ldots$$

Typical point inside is unentangled with outside

$$S_{\rm eq} = s_{\rm th}\, V_\Sigma + \ldots$$

Typical point inside is entangled with outside

Suggests a entanglement wave moving in with v_E. Natural picture for a local Hamiltonian.

How fast can entanglement be generated?

- Normalized rate of growth can be compared across systems and regions:

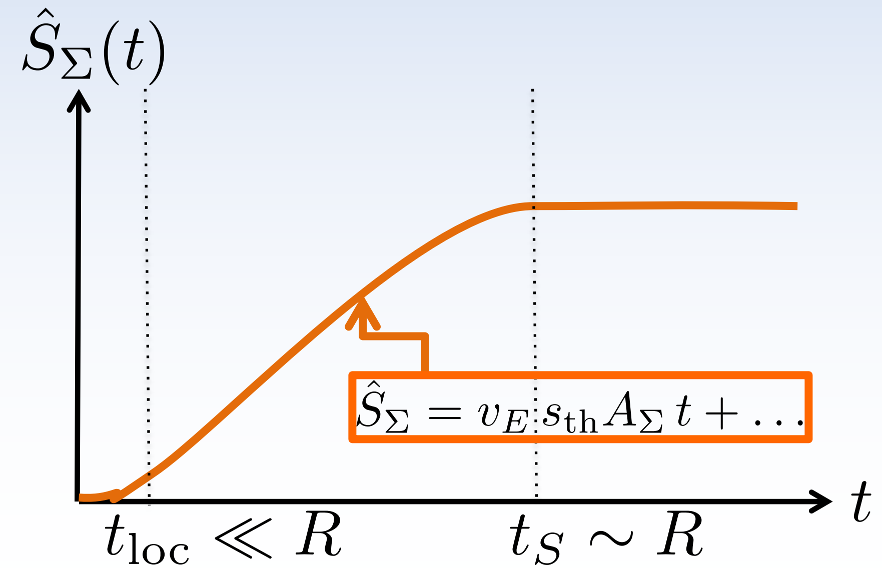
$$v_E = \frac{1}{s_{\rm th} A_\Sigma} \frac{dS_\Sigma}{dt} \qquad (t_{\rm loc} \ll t \ll R)$$

- Entanglement saturates with speed

$$c_E \equiv \frac{R}{t_S} \qquad (t_{\rm loc} \ll R, t_S)$$

  Should be constrained by causality.

- What is the relation to data characterizing chaos?



Chaos and the emergent light cone at finite temperature

- Lieb-Robinson bound: even in nonrelativistic systems quantum information propagates with a finite speed

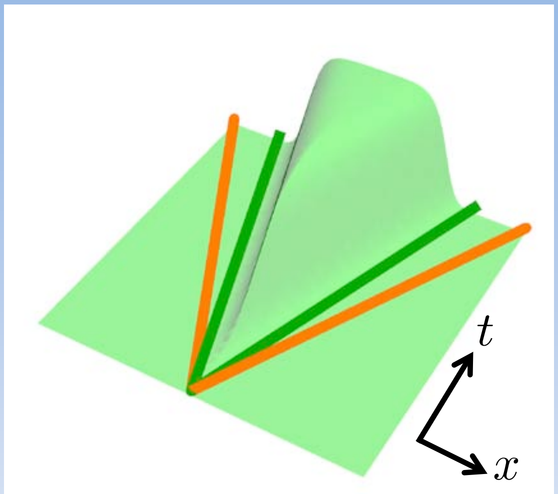
$$\frac{\|[W(x,t),V(0,0)]\|}{\|W\|\;\|V\|} \leq c_0 \exp\left[\lambda_L\left(t-\frac{x}{v_B}\right)\right]$$

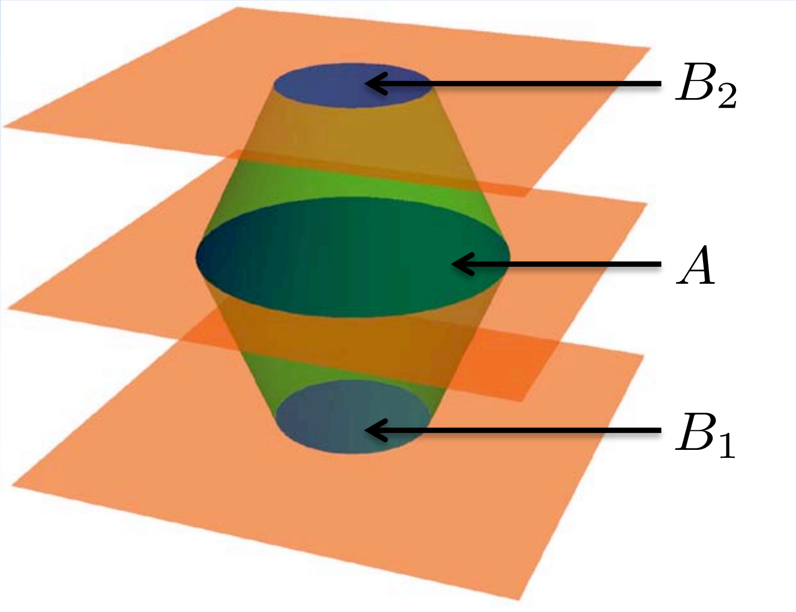
- Field theory generalization OTO thermal 4-point function [Shenker, Stanford; Roberts, Stanford, Susskind]

$$\|[W(x,t),V(0,0)]\|_2 \leq c_0 \exp\left[\lambda_L\left(t-\frac{x}{v_B}\right)\right]$$

$$\|\mathcal{O}\|_2 \equiv \mathrm{Tr}\left(\rho\,\mathcal{O}^\dagger\mathcal{O}\right)$$

- Emergent light cone (with $v_B<1$) at finite energy density
- $B_{1,2}$ are subsystems of A





We have defined the three velocities

v_E **c_E** **v_B**

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Relations between the three velocities

Using the fact that EE is the property of causal diamonds one can show that [Casini, Liu, MM; Afkhami-Jeddi, Hartman]

$\boxed{v_E,\, c_E \le 1}$

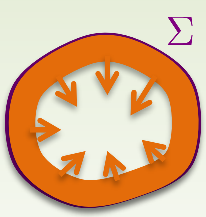
Using the emergent light cones with speed v_B we can strengthen this bound to

$\boxed{v_E,\, c_E \le v_B}$

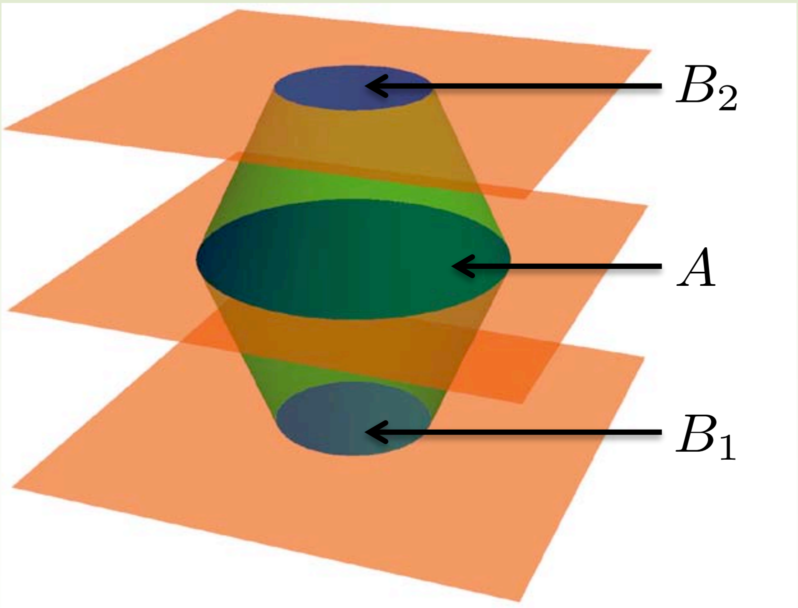
- Define thermal relative entropy $S_{\rm rel}(A) \equiv S\left(\rho_A \middle| \rho_A^{\rm th}\right) = s_{\rm th}(\beta)\, V_A - \hat{S}(\rho_A)$

- Use monotonicity of relative entropy for subregion B_1 of A to obtain

  $$\hat{S}_A(t) \le s_{\rm th}(\beta)\, V_{\rm tsunami}(t)$$



  The tsunami wave front propagates with v_B.
- Analogous lower bound exists.
- The inequality between the speeds follows.



Holographic results on entanglement

Holographic models of quenches

- Dual of Cardy-Calabrese boundary state is eternal BH with end of world brane [Hartman, Maldacena]

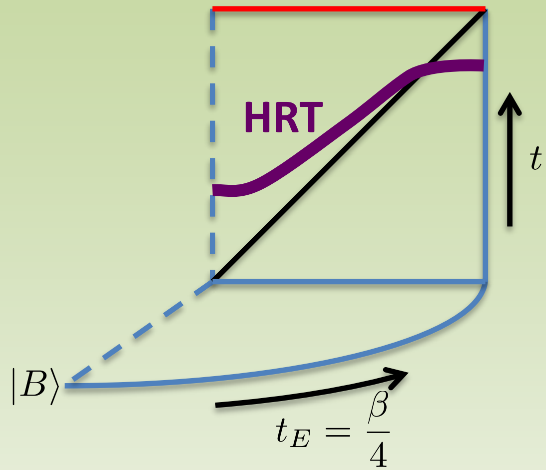
$$ds^2 = \frac{1}{z^2}\left[-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right]$$

$$f(z) = f_1(1-z) + f_2(1-z)^2 + \dots$$

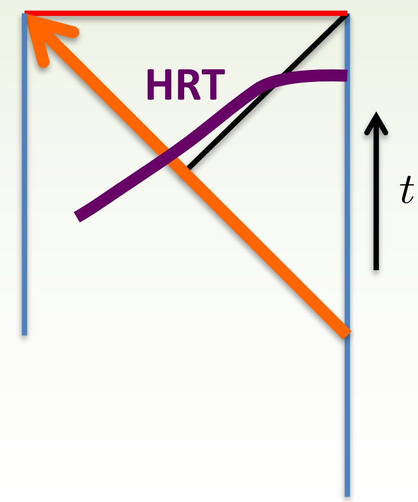
- Injecting energy density is dual to a collapsing shell. Saturation happens when the HRT surface touches the shell [Liu, Suh]
- The two setups are equivalent for large R
- v_E is determined by behind the horizon physics
- c_E is determined by near horizon physics
- Using the NEC, we can show that there are non-trivial constraints on these velocities:

$$\boxed{\begin{gathered} v_E \le v_E^{(\mathrm{SBH})}, \quad c_E \le c_E^{(\mathrm{SBH})}, \\ v_E \le c_E \end{gathered}}$$

End of the world brane quench



Vaidya quench

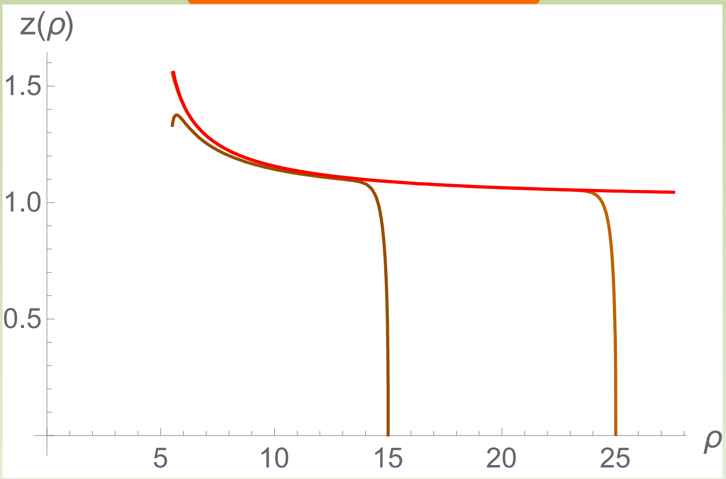


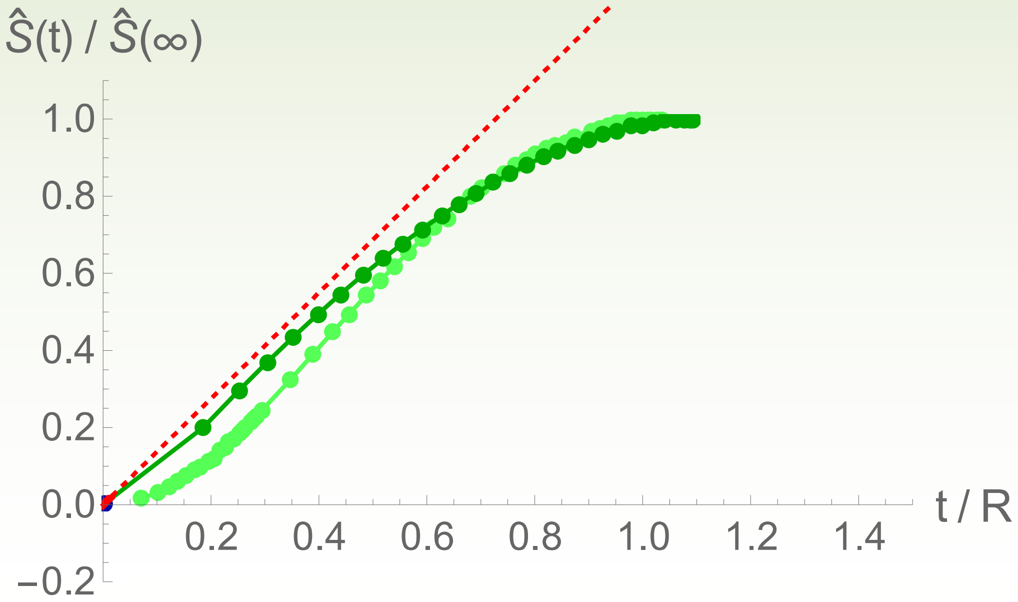
Holographic results on entanglement

Detailed understanding of how HRT surfaces are behaving

- For large R, we can understand the entropy analytically
- In both setups the minimal surfaces are close to a critical surface determined by an **algebraic equation**.
- They shoot out to the boundary exponentially fast.
- Entropy and time are given by the critical surface

Critical surface

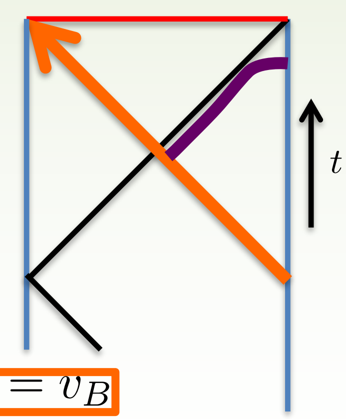




Holographic results on chaos

In holography we find that $\boxed{c_E = v_B\,, \qquad v_E \leq c_E}$

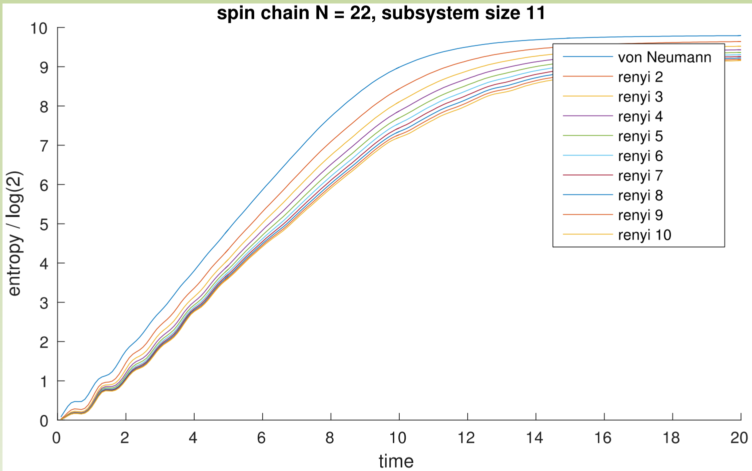
- That $c_E = v_B$ in Einstein gravity can be shown using explicit computation. Instead more insightful derivation based on entanglement wedge reconstruction.
- The size of an operator can be measured by studying its commutator with other operators: $\|[W(x,t), V(0,0)]\|_2$
- It can also be defined by the smallest ball from which it can be reconstructed.
- Holographic setup:
  - ➢ Thermal state is represented by BH
  - ➢ Acting with V(0,0) creates a particle near the boundary that falls into the BH
  - ➢ Want to find bulk subregion that contains the particle
  - ➢ By subregion-subregion duality this corresponds to a boundary subregion
  - ➢ **Summary:** find RT surface anchored on a boundary ball that contains the infalling particle
- Infalling particle's trajectory: $z(t) \approx 1 - \boxed{f_1/f_2\, e^{-f_1 t}}$
- Near horizon RT surface [Liu, MM]: $z(x) \approx 1 - \boxed{\epsilon}\, I_0\left(\sqrt{f_1}\, x\right)^2 \quad (d=3)$
  Reaches boundary at: $R \approx -\dfrac{\log \epsilon}{2\sqrt{f_1}}$
- Size of operator: $\boxed{R = v_B\, t = \dfrac{\sqrt{f_1}}{2}\, t}$
- This is the same RT surface that we needed to determine c_E ➡ $\boxed{c_E = v_B}$
- Survives Gauss-Bonnet correction

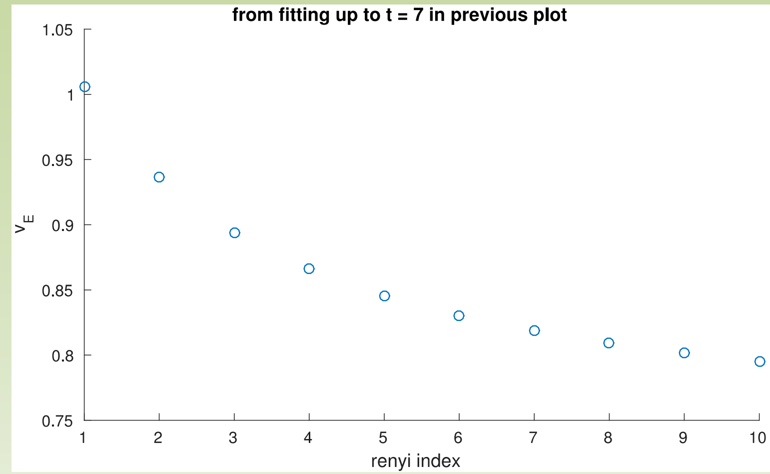


Spin chain results on entanglement and chaos

Chaotic spin chain Hamiltonian: $H = -\sum_i \left(Z_i Z_{i+1} - 1.05 X_i + 0.5 Z_i\right)$

- Entropy growth and v_E:





- Operator growth [Roberts, Susskind, Stanford]

$$Z_1(t_w) = Z_1 - i t_w\,[H, Z_1] - \frac{t_w^2}{2!}\,[H,[H,Z_1]] + \ldots$$

$$Z_1$$

$$Y_1$$

$$X_1 \quad Z_1 \quad X_1 Z_2$$

$$Y_1 \quad X_1 Y_2 \quad Y_1 Z_2$$

$$Y_1 \quad X_1 Y_2 \quad Y_1 X_2 \quad Y_1 Z_2 \quad Z_1 Y_2 \quad X_1 X_2 Y_3 \quad X_1 Y_2 Z_3 \quad Y_1 X_2 Z_3$$

$$\vdots$$

$$\boxed{v_B = 2.0 > v_E, c_E}$$

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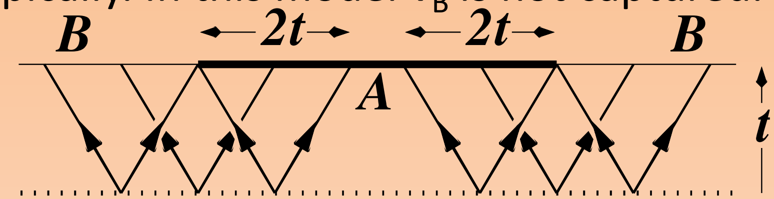
Free streaming model for entanglement spread

Calabrese-Cardy model: energy injection from quench creates a finite density of EPR pairs, subsequently travel freely at the speed of light isotropically. In this model v_B is not captured.

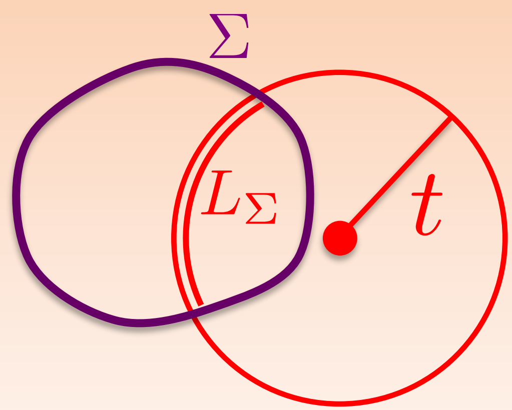
- Leads to linear growth with $v_E = 1$ in 2d



$2t < l$

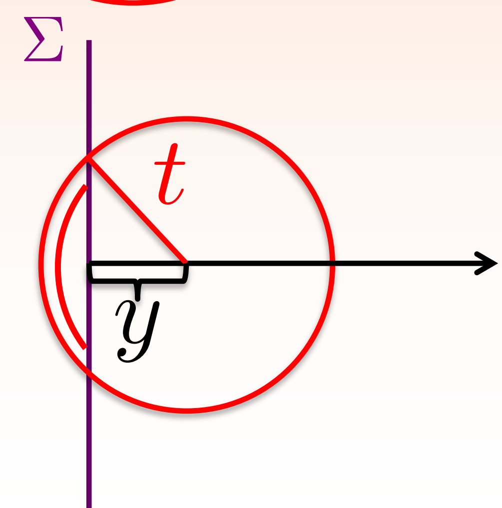
- Higher dimensions: entanglement spreading depends on entanglement pattern on the light cone $\mu[L_\Sigma]$
  Contribution from each light cone has to be added:

$$\hat{S}_\Sigma(t) = \int d^{d-1}x \, \mu[L_\Sigma(\vec{x}; t)]$$



- Properties of the measure:
  - Purity: $\mu[A] = \mu[\bar{A}]$
  - SSA: $\mu[A] + \mu[B] \geq \mu[A \cap B] + \mu[A \cup B]$
  - Thermalization: $\lim_{\Delta\theta \to 0} \mu[A]/\xi_A = s_{\text{th}}$
  - Upper bound: $\mu[A] \leq s_{\text{th}} \min(\xi_A, \xi_{\bar{A}})$
- Linear growth for $t_{\text{loc}} \ll t \ll R$

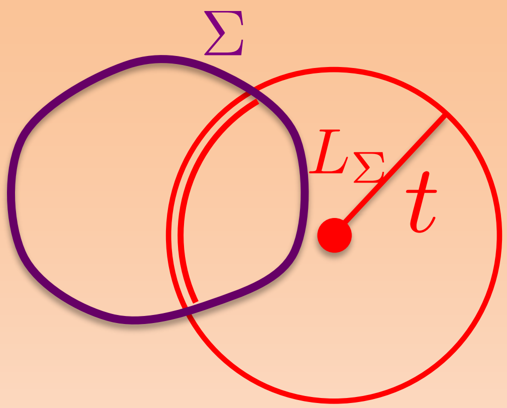
$$\boxed{S_\Sigma(t) = 2A_\Sigma \int_0^t dy \, \mu_{\text{cap}}(\xi(y/t)) = \underbrace{\frac{2}{s_{\text{th}}} \int_0^1 dx \, \mu_{\text{cap}}(\xi(x))}_{v_E} \, s_{\text{th}} A_\Sigma t}$$



Free streaming model for entanglement spread

Examples for measures

- EPR pairs (for small A): $\mu_{\rm EPR}[A] = s\,\xi_A$
- 2m particle GHZ block:
  $\mu_{\rm GHZ}[A;m] = \frac{s_{\rm th}}{2m}\left[1-(1-2\xi_A)^m\right]$
- Random Pure State (RPS) measure inspired by [Page]
  $\mu_{\rm RPS}[A] = s_{\rm th}\min(\xi_A,\xi_{\bar A})$
  Saturates the bound $\mu[A] \leq s_{\rm th}\min(\xi_A,\xi_{\bar A})$



Bound on the entanglement velocity

$$v_E = \frac{2}{s_{\rm th}}\int_0^1 dx\,\mu_{\rm cap}(\xi(x)) \leq 2\int_0^1 dx\,\xi(x) \equiv v_E^{\rm free}$$

- Slower than holography: $v_E^{\rm free} = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)} < v_E^{\rm (SBH)}$
- In strongly coupled systems, entanglement propagates faster than what's possible for free particles streaming at the speed of light!
- $c_E = 1 > c_E^{\rm (SBH)}$ is achievable, makes free streaming look even less effective
- Consider the effect of interactions: tensor network picture emerging from scattering particles is natural [Hartman, Maldacena; Casini, Liu, MM]

Entanglement spread in free scalar theory

In a free theory for Gaussian states we can use the correlation matrix to compute EE

- Time evolution of a Gaussian initial state is Gaussian (with time dependent complex kernel)
- Correlation matrix determines all correlation functions due to Wick's theorem

$$\chi_I = \begin{pmatrix} \phi_i \\ \pi_i \end{pmatrix} , \qquad [\chi_I, \chi_J] = i\, J_{IJ}$$

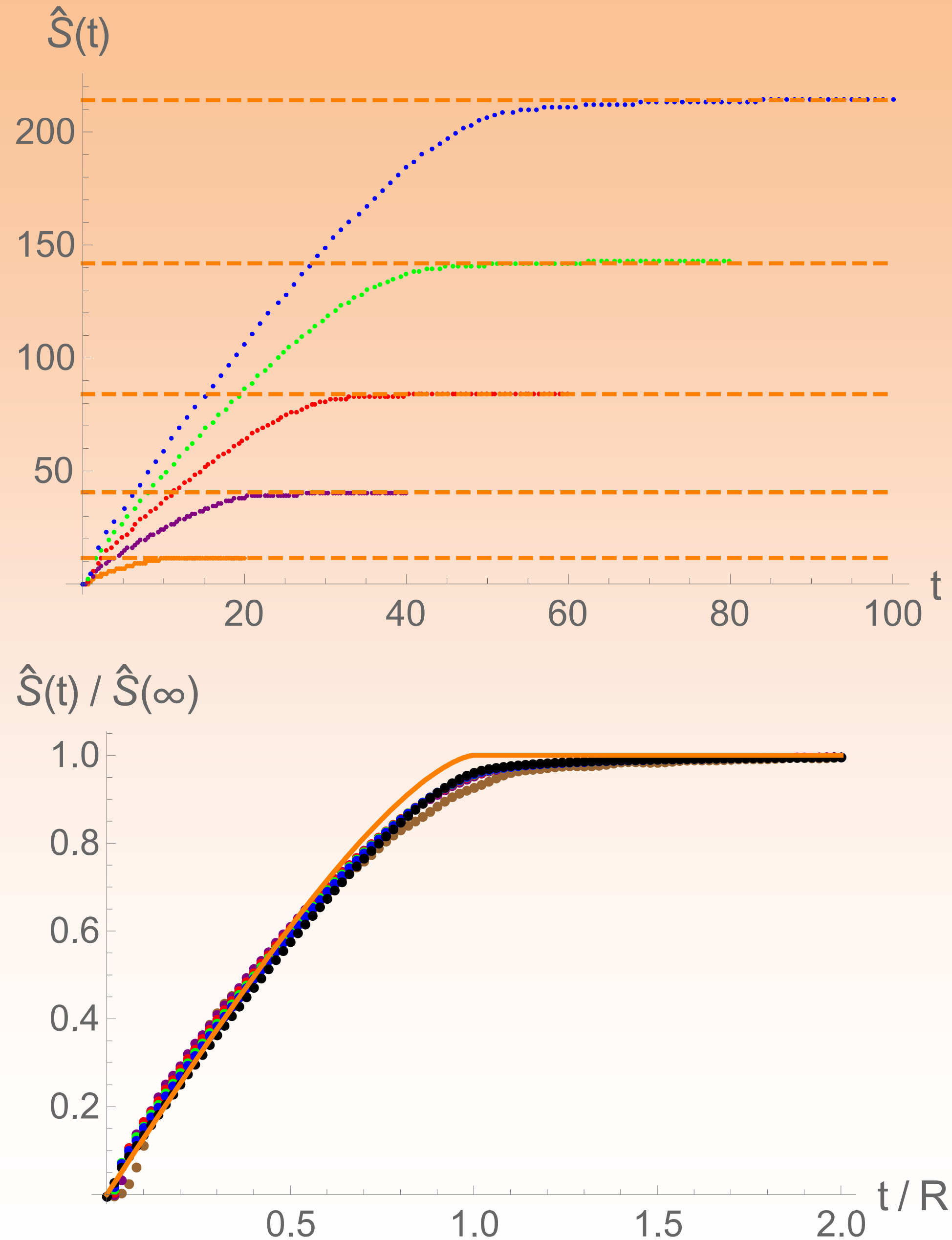
$$\Gamma_{IJ} = \frac{1}{2} \langle \psi | \{ \chi_I \,, \chi_J \} | \psi \rangle$$

- The symplectic eigenvalues of the correlation matrix give the eigenvalues of the reduced density matrix

$$\tilde{\chi} = S\chi \,, \qquad SJS^T = J \,,$$

$$\tilde{\Gamma} = S\Gamma S^T = \begin{pmatrix} \mathrm{diag}\,(\gamma_k) & 0 \\ 0 & \mathrm{diag}\,(\gamma_k) \end{pmatrix}$$

- Numerical results for 3d mass quench for scalar field [Cotler, Hertzberg, MM, Mueller]



Operator and tensor network models

Operator counting model [Abanin, Ho]

- Closer in spirit to spin chains, infinite temperature
- The reduced density matrix is an operator, so it also spreads

$$\rho(0) = |\uparrow\uparrow\ldots\uparrow\rangle\langle\uparrow\uparrow\ldots\uparrow| = \prod_i \frac{\mathbb{I}_i + Z_i}{2} = \frac{1}{2^{V/2}} \sum_{\mathcal{O}(0)} \mathcal{O}(0)$$

$$\longrightarrow \quad \rho_\Sigma(t) = \frac{1}{2^{V_\Sigma/2}} \sum_{\mathcal{O}(0)} \mathcal{O}(t)_\Sigma$$

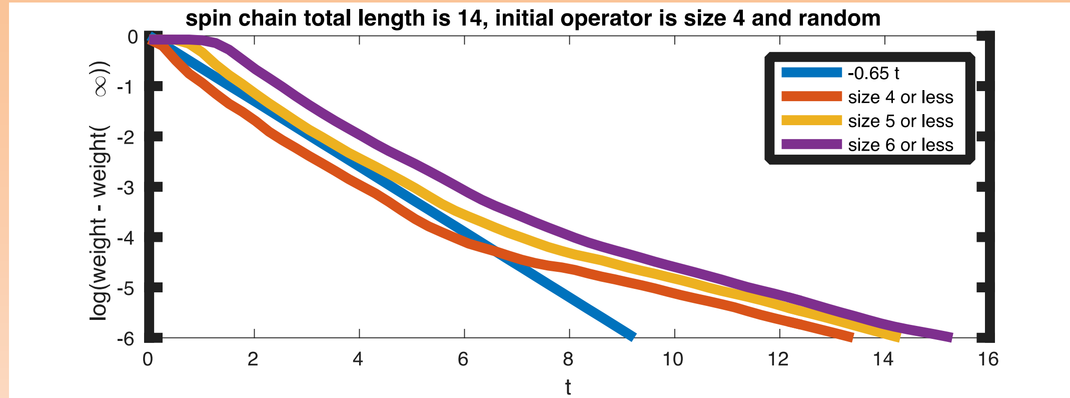
- Second Rényi entropy:

$$\begin{aligned}\mathrm{Tr}_\Sigma \rho_\Sigma(t)^2 &= \frac{1}{2^{V_\Sigma}} \sum_{\mathcal{O}(0),\mathcal{P}(0)} \mathrm{Tr}_\Sigma \left(\mathcal{O}(t)_\Sigma \mathcal{P}(t)_\Sigma\right) \\ &= \frac{1}{2^{V_\Sigma}} \sum_{\mathcal{O}(0)} \mathrm{Tr}_\Sigma \left(\mathcal{O}(t)_\Sigma^2\right)\end{aligned}$$

- Small operators contribution: 1
  Big operators: probability of staying inside $\mathrm{Tr}_\Sigma \left(\mathcal{O}(t)_\Sigma^2\right) = 2^{-\alpha\, A[\mathcal{O}(0)](t-t_{\mathrm{delay}})}$
- Have to sum over all operators

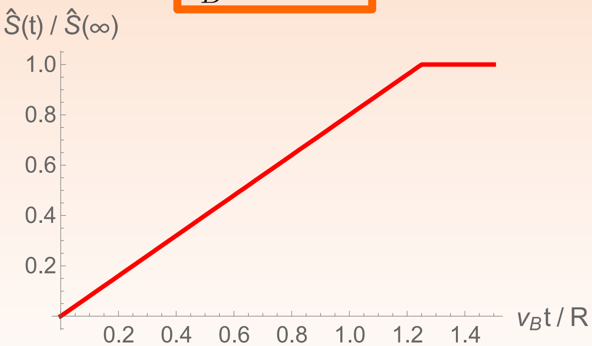
Operator and tensor network models

- Predictions:
  - d=2: linear growth until saturation with
    $v_E = \min(v_B, \alpha)$
    Can measure α independently
    $\alpha = \frac{0.65}{\log 2} = 0.94$
    Precise agreement with $S_2(t)$

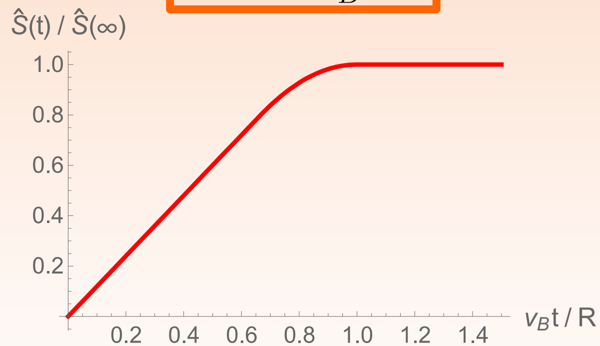


  - d>2: three regimes

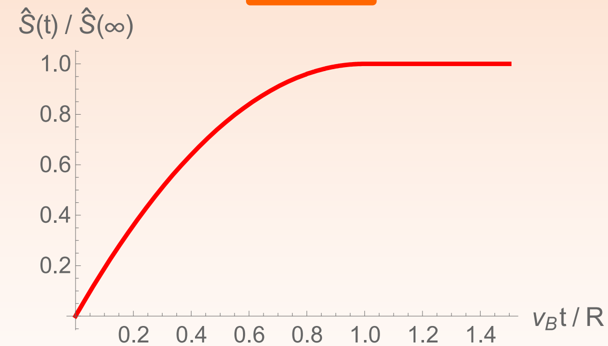
$\boxed{\frac{\alpha}{v_B} < \frac{1}{d-1}}$



$\boxed{\frac{1}{d-1} \leq \frac{\alpha}{v_B} \leq 1}$



$\boxed{1 < \frac{\alpha}{v_B}}$



Middle regime in good agreement with holographic theories

Tensor network model gives identical predictions

- Combines insight from thermal relative entropy and cuts in tensor networks
- New way of bounding the entropy introduced by a cut in a tensor network

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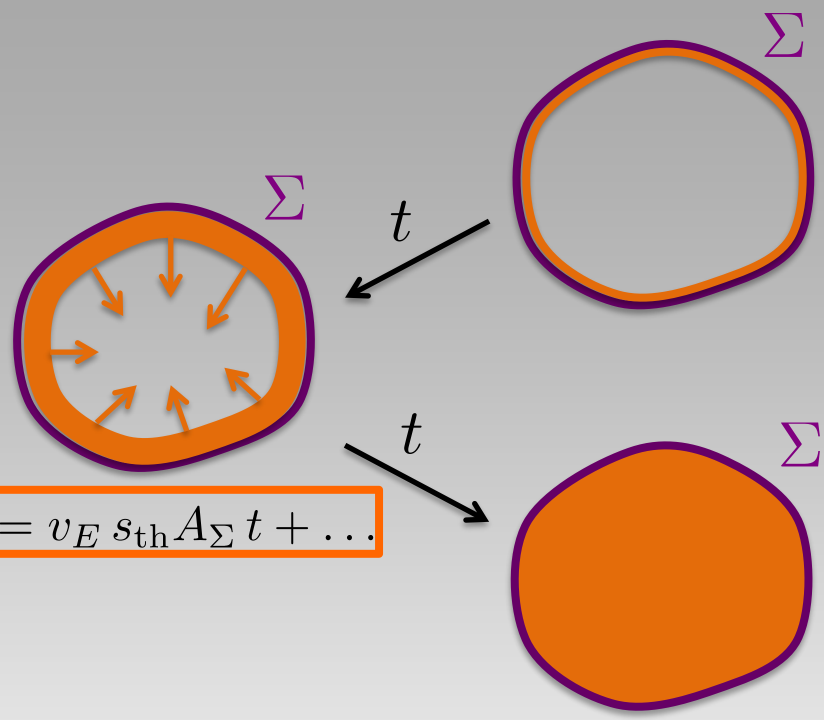
Summary

- Studied EE spread in a global quench
- Bound from chaos and thermal relative entropy: $\boxed{v_E,\, c_E \leq v_B}$
- In holography: $\boxed{c_E = v_B}$
- Can solve for the entire $\hat{S}(t)$ curve analytically
- In chaotic spin chain: $\boxed{v_E = c_E < v_B}$
- Free streaming is slower than holography
- Operator and tensor network models

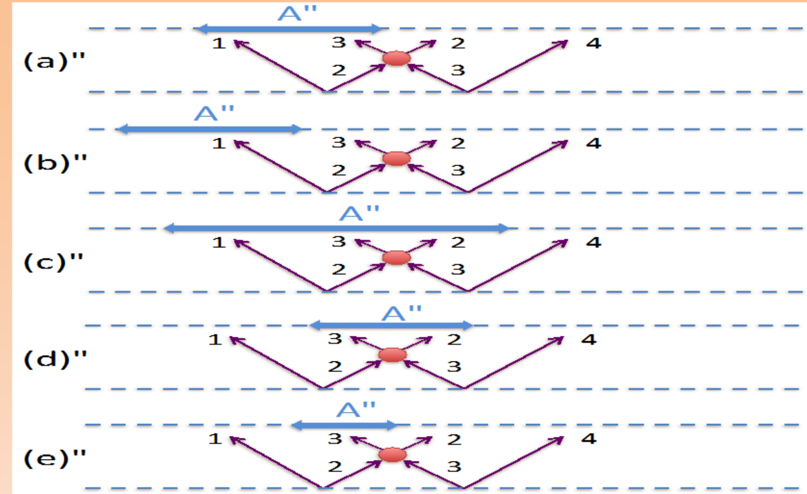


$$\boxed{\hat{S}_\Sigma = v_E\, s_{\text{th}} A_\Sigma\, t + \dots}$$

Open questions

- **What is the relation between v_E and v_B?**
- Can the bound from relative entropy be realized in a CFT? Are the holographic bounds $v_E \leq v_E^{(\text{SBH})}, \quad c_E \leq c_E^{(\text{SBH})}$ universal?
- The three velocities are new observables in a QFT. Are they calculable?
  - What are they in weakly coupled theories? [v_B: Stanford]
  - What are they for perturbed 2d CFTs? [v_E: Cardy]
  - What is v_E and c_E for free theories? [Cotler, Hertzberg, MM, Mueller]

Effect of interactions



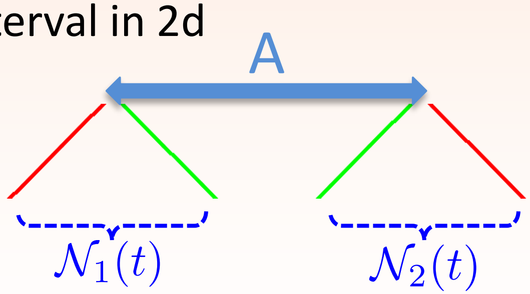
How does EE of A change in a scattering event between particles 2,3?

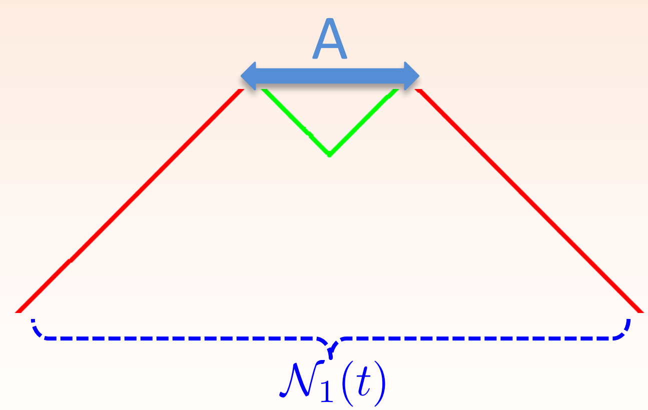
a) (1,2) and (3,4) are entangled initially. Scattering is taken into account as a unitary transformation on $\mathcal{H}_{23}$ EE can change. YES

b) 1 is not affected, scattering is a change of basis in $\mathcal{H}_{\bar{A}} = \mathcal{H}_{234}$. NO

Assume that the mean free path $\ell \ll R, t$

- Scatterings that happen in $\mathcal{D}_{-}(\bar{A})$ or $\mathcal{D}_{-}(A)$ don't matter for EE
- Scatterings like (a) are effective scatterings, only particles that originate in $\mathcal{N}(t) \equiv \mathcal{M} - (\mathcal{D}_{-}(A) \cap \mathcal{M}) - (\mathcal{D}_{-}(\bar{A}) \cap \mathcal{M})$ participate in them.
- One interval in 2d





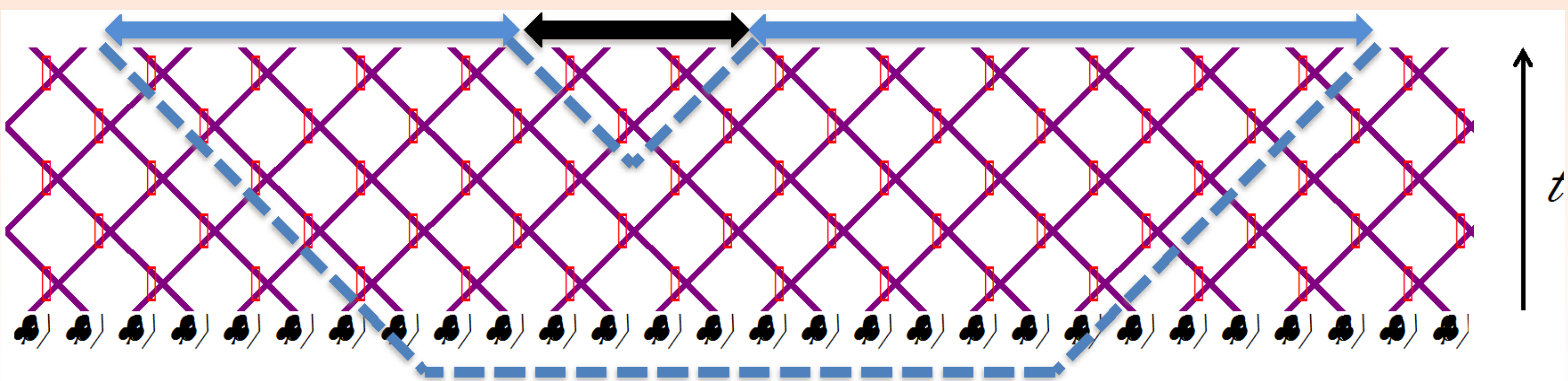
- Infinitely many scattering leads to loss of memory of the initial state, postulate RPS
- Leads to tsunami, caveat for concave regions. [Casini, Liu, MM; Leichenauer, Moosa]

Reinterpreting the infinite scattering model

The scattering picture can be reinterpreted as a tensor network:

$$|\psi(t)\rangle = \sum_{\{i_j=1\}}^{\chi} T_{i_1 i_2 i_3 \ldots i_N}(t)\,|i_1 i_2 i_3 \ldots i_N\rangle$$ [also: Hartman, Maldacena]

- Bound on entropy from minimal cut: $S_A \leq \ell_{\text{cut}} \log \chi$



- Reproduces holography for 2d for multiple intervals. Higher dimensions?

Geometric model inspired by tensor networks

Want an entropy function that satisfies all known criteria for a (relativistic) CFT:

- Rotational invariance and scaling property
- SSA even for boosted regions
- Linear regime and saturation

The model is defined by

- Spacetime ends at t=0
- Entropy is given by a minimal surface area:
  - Area is measured using the metric $ds^2 = d\vec{x}^2$
    Many degenerate surfaces
  - Slope is bounded $n = (n^0, \vec{n})$
    $$\frac{|\vec{n}|}{|n^0|} \leq \frac{1}{\alpha}, \quad \cancel{0 < \alpha < \infty} \rightarrow \boxed{0 < \alpha \leq 1}$$
  - Proof of SSA is similar to [Headrick, Takayanagi]
- Tsunami velocity $v_E = \alpha$

Choosing $\alpha = v_B$ saturates the bound obtained from thermal relative entropy $\hat{S}_A(t) = s_{\mathrm{th}}(\beta)\, V_{\mathrm{tsunami}}(t)$

Reinterpretation of this model is the next task.