BPS states, permutations and information

Sanjaye Ramgoolam

Queen Mary, University of London

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“Permutation centralizer algebras,” Mattioli and Ramgoolam arxiv:1601.06086, Phys. Rev. D.
“Enhanced symmetries in gauge theory and resolving the spectrum of gauge invariant operators,” Kimura and Ramgoolam, Phys. Rev. D 2010
More complete references are in these papers.

AdS/CFT

$$\mathcal{N} = 4 \text{ SYM} : g^2_{YM} \text{ and } U(N)$$
$$\updownarrow$$
$$\text{Strings in } AdS_5 \times S^5 \text{ with } g_s = g^2_{YM} \text{ and } R = (g_s N)^{1/4}$$

Classic example of emergent geometry. Graviton, strings, branes, back-reacted geometries in the bulk side. Quantum states expected to match. How does it work ?

Quantum states in CFT4 $\leftrightarrow$ local gauge invariant operators.

Half-BPS sector : Representations of the superconformal algebra containing states annihilated by half the Poincare super-charges Q. Ultra-short representations.

$Z = X_1 + iX_2$.

$\mathrm{tr}\, Z^k$ is in such a rep. Obeys $\Delta = J = k$. Generates a rep by acting with the Q's.

Likewise products $\mathrm{tr}\, Z^{k_1} \mathrm{tr}\, Z^{k_2}$.

Dimensions protected under change from $g^2_{YM} = 0$ to strong coupling. 3-point functions and extremal correlators non-renormalized.

Detailed map between CFT4 states and gravitons, branes, geometries (LLM) is known in the half-BPS sector.

Quarter-BPS sector: At $g^2_{YM} = 0$, the quarter-BPS states are holomorphic gauge invariant polynomials in two matrices Z, Y.

e.g. $\mathrm{tr}\, Z^2 Y^2, \mathrm{tr}\, ZYZY$.

$Z = X_1 + iX_2 \quad , \quad Y = X_3 + iX_4$

A subspace remains quarter-BPS beyond zero coupling. Those annihilated by the 1-loop dilatation operator.

Open problem: Complete finite N characterization of the ground states and matching with dual space-time.

More on this at the end of talk.

Bulk of the talk: Understanding the Hilbert space of quarter BPS states at $g_{YM}^2 = 0$.

A finite N labelling of the states: Young diagrams and Littlewood-Richardson numbers.

Measuring these labels using charges in the free theory.

Charges are constructed from enhanced symmetries in the free theory.

A quantitative measure of the complexity of this Hilbert space.

These complexity measures are given in terms of permutation groups.

Gauge invariant operators are constructed from adjoint fields by contracting upper and lower indices. The order can vary and is parametrized by permutations.

$$
\begin{aligned}
Z^{i_1}_{i_1} Z^{i_2}_{i_2} &= \mathrm{tr}\, Z \mathrm{tr}\, Z \\
Z^{i_1}_{i_2} Z^{i_2}_{i_1} &= \mathrm{tr}\, Z^2
\end{aligned}
$$

In general $\sigma \in S_n$ if we have n fields.

Distinct permutations do not always give distinct gauge invariant operators. Hence we have to look at equivalence classes of permutations.

In half-BPS sector, operators with n Z's, operators are built with permutations $\sigma \in S_n$

$$\sigma \sim \gamma \sigma \gamma^{-1}$$

for all $\gamma \in S_n$. Hence we look at conjugacy classes.

In the quarter BPS sector, looking at operators with n copies of Z and m copies of Y, we have $\sigma \in S_{n+m}$ and equivalence classes

$$\sigma \sim \gamma \sigma \gamma^{-1}$$

generated by $\gamma \in S_m \times S_n$. Hence "subgroup conjugacy classes".

Outline of the talk

1. Half-BPS sector: 1-matrix invariants, Conjugacy classes of permutations , associated commutative algebra, Young diagrams, Enhanced symmetries in the gauge theory, Charges for Young diagrams.

2. Quarter BPS: sector at zero coupling: 2-matrix invariants. subgroup conjugacy classes, non-commutative algebra, Young diagram labels, Charges, enhanced symmetries.

3. Open problems.

Half-BPS sector : Gauge invariants and permutations

Gauge invariants built from Z.

$$\mathcal{O}_\sigma(Z) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}$$

e.g. $n = 2$, $\sigma = (1)(2)$, then

$$\mathcal{O}_\sigma(Z) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} = Z^{i_1}_{i_1} Z^{i_2}_{i_2} = (\mathrm{trZ})^2$$

$\sigma = (12)$

$$\mathcal{O}_\sigma(Z) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} = Z^{i_1}_{i_2} Z^{i_2}_{i_1} = \mathrm{trZ}^2$$

Half-BPS sector : Permutation equivalences

One proves that

$$\mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z) = \mathcal{O}_\sigma(Z)$$

Index-free way to think about it :

$$Z : V_N \to V_N$$
$$Z^{\otimes n} : V_N^{\otimes n} \to V_N^{\otimes n}$$

$$\sigma|e_{i_1} \otimes e_{i_2} \cdots e_{i_n} >= |e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots e_{i_{\sigma(n)}} >$$

$$\mathcal{O}_\sigma(Z) = \mathrm{tr}_{V_N^{\otimes n}}(Z^{\otimes n}\sigma)$$

Then

$$\gamma Z^{\otimes n}\gamma^{-1} = Z^{\otimes n}$$

Summing over equivalence classes

Given this equivalence, we can form formal sums of all elements in the class

$$[\sigma] = \sum_{\gamma \in S_n} \gamma \sigma \gamma^{-1}$$

These sums do not live in S_n but in the group algebra $\mathbb{C}(S_n)$.
In fact they live in the subspace of the algebra which commutes with everything in $\mathbb{C}(S_n)$.

$$[\sigma]\tau = \tau[\sigma]$$

The $[\sigma]$ span the centre $\mathcal{Z}[\mathbb{C}(S_n)]$.

$\mathcal{Z}(S_n)$: A physically interesting commutative algebra

Central elements form a closed commutative sub-algebra.

$$T_i T_j = \sum_k C_{ij}^k T_k$$

Example : $n = 3$

$$\begin{aligned} T_1 &= (1)(2)(3) \\ T_2 &= (1,2)(3) + (1,3)(2) + (1)(2,3) \\ T_3 &= (1,2,3) + (1,3,2) \end{aligned}$$

Sums of permutations with fixed cycle structure.

The multiplication is the combinatorics of splitting and joining of cycles e.g.

$$\begin{aligned} T_2 T_2 &= \sum_{i \neq j} (i,j) \sum_{k \neq l} (k,l) \\ &= n(n-1)/2 + \# \sum_{i \neq j \neq k} (ijk) + \sum_{i \neq j \neq k \neq l} (ij)(kl) \end{aligned}$$

These structure constants control the $1/N$ expansion of correlators in the half-BPS sector.

Also in orbifold theoiries and topological strings (e.g. 2d YM).

$\mathcal{Z}(S_n)$: Projector basis - Young diagrams

The centre $\mathcal{Z}(S_n)$ (in fact for any finite group) has a projector basis. One projector for each irreducible representation R

$$P_R = \frac{d_R}{n!} \sum_{\sigma} \chi_R(\sigma)\sigma$$

The R's are labelled by Young diagrams with n boxes. d_R is the dimension of the irrep. The character $\chi_R(\sigma)$ is the trace of the matrix representing σ in the representation R.

$$P^R P^S = \delta_{RS} P^S$$

The corresponding gauge invariant half-BPS operators are

$$\mathcal{O}_R(Z) = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\mathcal{O}_\sigma(Z)$$

Young diagram basis gives orthogonal 2-point functions

It can be shown that

$$\langle \mathcal{O}_R(Z(x_1))\mathcal{O}_S(Z^\dagger(x_2))\rangle = \frac{\delta_{RS} f_R}{(x_1 - x_2)^{2n}}$$

Corley, Jevicki, Ramgoolam 2001

This orthogonality allows a systematic map - for dimensions of order N - between gauge invariant operators and giant graviton states - half-BPS brane configurations in the dual $AdS_5 \times S^5$.

And, for dimensions of order N^2, a map between gauge invariant operators and backreacted LLM geometries which are half-BPS and asymptotically $AdS \times S^5$

Lin, Lunin, Maldacena 2004

This system is also related to a system of N free fermions in harmonic oscillator potential.

Corley Jevicki Ramgoolam 2001, Berenstein 2004.

Enhanced symmetries and commuting charges

Large degeneracy of states of fixed dimension - eigenstates of the scaling operator. There should be a large number of commuting operators.

The free theory of complex scalar

$$\int d^4x \;\; \mathrm{tr}\,(\partial_\mu Z \partial_\mu Z^\dagger)$$

has separate left and right $U(N)$ global symmetry.

$$\begin{aligned} Z &\to UZV \\ Z^\dagger &\to V^\dagger Z^\dagger U^\dagger \end{aligned}$$

If $U = V^\dagger$ this is the global gauge symmetry. If $V = 1$ but U general, we have a left $U(N)$ symmetry. It has Lie algebra generators

$$E^i_j = Z^i_k \Pi^k_j$$

which can be realized as Noether charges .

Casimirs of left $U(N)$

Casimirs

$$C_2 = E^i_j E^j_i$$
$$C_3 = E^i_j E^j_k E^k_i$$
$$\vdots$$

The Casimirs commute with scaling operator. The Young diagram operators are eigenstates $C_k(R)$ are the eigenvalues.

The Casimirs are related to asymptotic multipole moments of the LLM geometries obtained from back-reaction of the giant graviton branes.

Balasubramanian, Cech,Larjo, Simon, 2006, “Integrability vs Information Loss: A simple example”

Not all the multipole moments are measurable to an observer who has access to less than Planck scale energies. Only order $N^{1/4}$ as opposed to N. Hence an information loss.

The Quarter BPS sector : 2-Matrix Invariants

Gauge invariants are again constructed from permutations

$$\mathcal{O}_\sigma(Z, Y) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}$$

Now $\sigma \in S_{n+m}$. But

$$\gamma Z^{\otimes n} \otimes Y^{\otimes m} \gamma^{-1} = Z^{\otimes n} \otimes Y^{\otimes m}$$

only for $\gamma \in S_n \times S_m$.

Hence subgroup conjugation equivalences

$$\mathcal{O}_\sigma = \mathcal{O}_{\gamma\sigma\gamma^{-1}}$$

for γ in the subgroup.

Algebra of equivalence classes

As before, we can construct sums over the equivalence classes

$$[\sigma] = \sum_{\gamma \in S_n \times S_m} \gamma \sigma \gamma^{-1}$$

These live in $\mathbb{C}(S_{n+m})$, commute with $S_n \times S_m$ but not with all of S_{n+m}. They form a closed associative algebra $\mathcal{A}(n, m)$. Now this algebra is non-commutative.

It has a sub-algebra which commutes with everything, which we call $\mathcal{Z}(n, m)$.

Projector basis for the centre $\mathcal{Z}(n, m)$

Just as $\mathcal{Z}(S_n)$ - centre of $\mathbb{C}(S_n)$)- has a projector basis, the centre $\mathcal{Z}(n, m)$ has a projector basis.

$$P^{R,R_1,R_2} = P^R P^{R_1} P^{R_2}$$

P^R is labelled by irreps R of S_{n+m} - Young diagrams with $n + m$ boxes. P^{R_1} is labelled by irreps R_1 of S_n - Young diagrams with n boxes.

Each P : a sum over permutations weighted by characters. The product in$\mathbb{C}(S_{n+m})$.

Matrix-Like Basis for $\mathcal{A}(n, m)$

There is a matrix-like basis for the $\mathcal{A}(n, m)$.

$$Q_{ij}^{R,R_1,R_2}$$

which can be constructed using group theory of S_{n+m} and reduction to $S_n \times S_m : R \to R_1 \otimes R_2$.

The projector P and the matrix-like Q are non-zero only when we can compose the R from R_1 and R_2 by the Littlewood-Richardson rule.

The LR rule gives the number $g(R_1, R_2, R)$ that the irrep $R_1 \otimes R_2$ appears in R when we reduce the irrep R of S_{n+m} to the subgroup.

$$|R, M> = \sum_{R_1,R_2,M_1,M_2,i} |R_1, M_1, R_2, M_2, i >< R_1, M_1, R_2, M_2, i|R, M>$$

Matrix Like Basis for $\mathcal{A}(n, m)$

Construction of Q uses these branching coefficients and the indices i, j range over

$$1 \leq i, j \leq g(R_1, R_2, R)$$

The R's label matrix-blocks and the i, j matrix-entries :

$$Q^{\vec{R}}_{ij} Q^{\vec{S}}_{kl} = \delta_{jk} \delta^{\vec{R}, \vec{S}} Q^{\vec{R}}_{il}$$

Wedderburn-Artin theorem in general. Here group theory and orthogonality relations of group theory.

Matrix-like Q's and corresponding 2-matrix operators

These Q's are some linear combinations of permutations in S_{n+m}, which are invariant under conjugation by $S_n \times S_m$.

$$Q_{ij}^{\vec{R}} = \sum_{\sigma \in S_{n+m}} \chi_{ij}^{\vec{R}}(\sigma) \ \sigma$$

There are corresponding operators

$$\mathcal{O}_{ij}^{\vec{R}}(Z, Y) = \sum_{\sigma \in S_{n+m}} \chi_{ij}^{\vec{R}}(\sigma) \ Q_{\sigma}^{\vec{R}}(Z, Y)$$

Orthogonality of operators corresponding to Q's

It is found that

$$\langle \mathcal{O}^{\vec{R}}_{ij}(Z,Y)(x_1)(\mathcal{O}^{\vec{S}}_{kl}(Z,Y))^{\dagger}(x_2)\rangle = \frac{f_R \delta^{\vec{R},\vec{S}} \delta_{ik}\delta_{jl}}{((x_1 - x_2)^2)^{m+n}}$$

The f_R vanish for $l(R) > N$.

This is one of a class of closely related orthogonal bases for multi-matrix systems in free field theories found in 2007-2008.

Yusuke Kimura, S. Ramgoolam, 2007
Brown, Heslop, Ramgoolam, 2007
Bhattaccharrya, Collins and de Mello Koch, 2008

Enhanced symmetries and Charges

Left action on Z :

$$(E_z)^i_j \rightarrow \text{ Casimirs } \rightarrow C_k(R_1)$$

Left action on Y:

$$(E_y)^i_j \rightarrow \text{Casimirs } \rightarrow C_k(R_2)$$

Left action on Z, Y:

$$((E_z)^i_j + (E_y)^i_j) \rightarrow \text{ Casimirs } \rightarrow C_k(R)$$

More general combinations of these e.g.

$$(E_x)^i_j(E_y)^j_k(E_y)^k_i$$

can be used to distinguish the more subtle LR-multiplicity indices on the

$$Q^{\vec{R}}_{ij}$$

Commutant algebras and Schur-Weyl duality

Unitary group acts on $V_N^{\otimes n}$ as

$$U \otimes U \otimes \cdots \otimes U$$

This commutes with permutations γ

$$\gamma \, (U \otimes U \otimes \cdots \otimes U) \, \gamma^{-1} = (U \otimes U \otimes \cdots \otimes U)$$

In fact anything that commutes with $U(N)$ in $V_N^{\otimes n}$ comes from the action of the $\mathbb{C}(S_n)$ group algebra.

One consequence is

$$V_N^{\otimes n} = \bigoplus_{R \vdash n} V_R^{U(N)} \otimes V_R^{S_n}$$

which is why Young diagrams have a dual role : labelling irreps of $U(N)$ (symmetry types of tensors) and irreps of S_n.

SW duality : Casimirs of $U(N)$ and permutation algebra actions.

Casimir operators constructed from Lie algebra generators of $U(N)$ - by construction - commute with $U(N)$. When acting on $V_N^{\otimes n}$, they must be expressible in terms of permutations.

Example : The Casimir operator $\hat{C}_2$ can be expressed in terms of the sum of transpositions

$$T_2 = \sum_{i \neq j} (ij)$$

Another consequence The question "What is the minimal set of Casimirs that are needed to distinguish all Young diagram labels for n boxes ?" can be expressed in terms of how many of these central elements of $\mathcal{Z}(\mathbb{C}(S_n))$ are needed to generate (by adding and multiplying) - the whole centre.

Casimirs and structure of $\mathcal{A}(m, n)$

Action of the above Casimirs on $\mathcal{O}_{ij}^{\vec{R}}(Z, Y)$ can be mapped to actions of the algebra $\mathcal{A}(n, m)$ on itself by left multplication.

Recall that the algebra has been decomposed into matrix blocks.

$$Q_{ij}^{\vec{R}}$$

$$P^{\vec{R}} = \sum_i Q_{ii}^{R} = P^R P^{R_1} P^{R_2}$$

The projectors are 1 in the blocks and zero elsewhere. They span $\mathcal{Z}(m, n)$ - the centre of the algebra. The dimension of this algebra is the number of triples (R_1, R_2, R) such that $g(R_1, R_2, R) \neq 0$.

Measuring the multiplicity indices.

To completely measure all the multiplicity indices, we need the diagonal elements $Q^{\vec{R}}_{kk}$. These span a commutative algebra (a Cartan of $\mathcal{A}(m,n)$) which we will denote $\mathcal{M}(m,n)$).

$$Q^{\vec{R}}_{kk} Q^{\vec{R}}_{ij} = \delta_{ik} Q^{\vec{R}}_{kj}$$

The dimension of $\mathcal{M}(m,n)$ is equal to

$$\sum_{R_1 \vdash n} \sum_{R_2 \vdash m} \sum_{R \vdash m+n} g(R_1, R_2, R)$$

Minimal generating sets

The commutative algebra $\mathcal{M}(m,n)$ is an algebra over $\mathcal{Z}(m,n)$.

$$\sum_a z_a m_a \sum_b z_b m_b = \sum_{a,b,c} C^c_{a,b} z_c m_a m_b$$

The elements $\mathcal{Z}(m,n)$ can be treated as constants. In this treatment : Minimal size of a set of generators of $\mathcal{M}(m,n)$?

This gives a precise definition of how many charges are needed for the multiplicities.

Summary

We gave a characterization of the minimal number of charges needed to distinguish the holomorphic invariants constructed from two matrices Z, Y. These holomorphic invariants correspond to quarter-BPS states at zero coupling Yang Mills theory.

Enhanced symmetries and Casimir charges played a role.

The relation of these charges to permutation algebras and the structure of these algebras also played an important role.

Open problems.

Explicit computational results for the minimal mumbers of charges.

Beyond zero coupling, the quarter BPS states are those which are annihilated by the Hamiltonian of Minahan-Zarembo

$$H = tr[X, Y][\check{X}, \check{Y}]$$

Open problems

This is understood - using permutation methods - for states where $\Delta \leq N$.

Brown, Heslop, Ramgoolam, 2008.
Brown, 2009.
Pasukonis, Ramgoolam, 2010.

For states with $\Delta > N$, partial results are available.

Yusuke Kimura “Quarter BPS from Brauer algebra” - 2010

A complete understanding of finite N should make contact with quantization of moduli spaces of giant gravitons - where it is argued that geometric quantization leads to N bosons in a 3D harmonic oscillator.

Biswas, Gaiotto, Lahiri, Minwalla - 2006.

Perhaps new ideas from entanglement, emergence, condensed matter physics are needed ...