Entanglement in CFTs at Finite Chemical Potential

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Outline

Statement of problem.

Spin-1 currents in 1+1 dimensions.

Higher spin currents in 1+1 dimensions.

Higher dimensions.

CFT with a global symmetry.

Conserved spin-1 Noether currents $\{J_a^\nu\}$.

Charges $Q_a = \int J_a^0 \, d^{d-1}x$ commute with H_{CFT}.

Finite chemical potential

$$\rho(\beta, \{\mu_a\}) \propto e^{-\beta H_{CFT} + \beta \sum_a \mu_a Q_a}$$

[$\beta \to \infty$ gives ground state]

Block entanglement: divide space into regions A and B and similarly the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\rho_A(\beta, \{\mu_a\}) = \mathrm{Tr}_{\mathcal{H}_B}\, \rho(\beta, \{\mu_a\})$$

Rényi entropies

$$S_A^{(n)}(\beta, \{\mu_a\}) = (1-n)^{-1} \log \mathrm{Tr}_{\mathcal{H}_A}\, \rho_A(\beta, \{\mu_a\})^n$$

Entanglement entropy $S_A(\beta, \{\mu_a\}) = \lim_{n\to 1} S_A^{(n)}(\beta, \{\mu_a\})$.

How does S_A behave for $\mu_a \neq 0$?

Other (un)related work

- *Holographic Charged Renyi Entropies*, Belin, Hung, Maloney, Matsuura, Myers, Sierens, arXiv:1310.4180: considered chemical potential coupling to charge in region A, gives Wilson line around entangling surface.
- *Charged Renyi entropies and holographic superconductors*, Belin, Hung, Maloney, Matsuura, arXiv:1407.5630
- *Higher Spin Entanglement and W_N Conformal Blocks*, de Boer, Castro, Hijano, Jottar, Kraus, arXiv:1412.7520
- *A new spin on entanglement entropy*, Hijano, Kraus, arXiv:1406.1804

Expectations

(1) μ has dimensions of mass so is a relevant perturbation.
Based on entropic c-theorems, S_A should decrease.

But adding $\mu \int J^0 d^{d-1}x$ to the action breaks Lorentz invariance.

(2) When $\beta^{d-1} \ll \mathrm{Vol}(A) \ll \mathrm{Vol}(B)$, $S_A \sim S_{\mathrm{Gibbs}}$.

$$\text{Grand partition function} \quad \Xi = \mathrm{Tr}\, e^{-\beta H + \beta\mu Q}$$

To $O(\mu^2)$

$$\delta S_{\mathrm{Gibbs}} = \mu^2(\beta\partial_\beta - 1)\left(\beta^2\langle Q^2\rangle\right)$$

For a CFT,

$$\langle Q^2\rangle \propto \frac{L^{d-1}}{\beta^{d-1}}$$

$$\delta S_{\mathrm{Gibbs}} \propto -(d-2)\mu^2\beta^2(L/\beta)^{d-1} < 0,$$

except in $d = 2$, where it vanishes.

Main result

For a 1+1-dimensional CFT

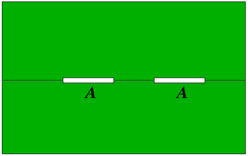
$S_A^{(n)}(\beta, \{\mu_a\})$ is independent of the $\{\mu_a\}$

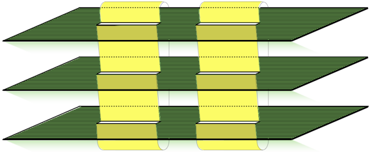
(and therefore of $\langle Q_a \rangle$)

Argument is in two parts:

- show this holds for a free scalar field
- argue that it is also true more generally

Replicas





$\mathrm{Tr}_{\mathcal{H}_A}\,\rho_A^n$ can be computed by taking n replicas of euclidean space-time and sewing them together cyclically along $A \cap \{\tau = 0\}$

$$\mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z_1^n}$$

Note that we should regularize the behavior near the conical singularities, e.g. by removing a small disc around them: this necessitates choosing a boundary condition, which we take to be conformal.

Free field theory in 1+1 dimensions

Free scalar field, euclidean action

$$A_{CFT} = \frac{1}{2} \int\int_0^\beta \left((\partial_x \phi)^2 + (\partial_\tau \phi)^2\right) dx d\tau$$

[We can think of this as describing the phase fluctuations of a complex scalar $\Phi = |\Phi| e^{i\phi}$, or coming from bosonization.]

Symmetry $\phi \to \phi +$const., current $J^\mu = \partial^\mu \phi$

$$H = H_{CFT} - \mu \int \pi(x) dx$$

$$A = \frac{1}{2} \int\int_0^\beta \left((\partial_x \phi)^2 + (\partial_\tau \phi + i\mu)^2\right) dx d\tau = A_{CFT} + i\mu \int\int_0^\beta J_0 dx d\tau - \beta L \mu^2$$

$\phi(x, \beta) = \phi(x, 0)$ so the second term integrates to zero, apart from boundary terms.

$\delta\Omega = -L\mu^2$, so $\delta S_{\text{thermo}} = 0$.

Replicated action

$$A = \frac{1}{2} \int\int_0^\beta \sum_{j=0}^{n-1} \left((\partial_x \phi_j)^2 + (\partial_\tau \phi_j)^2 + i\mu \partial_\tau \phi_j\right) dx d\tau - n\beta L\mu^2$$

$\phi_j(x, 0-) = \phi_{j+1 (\mathrm{mod}\, n)}(x, 0+)$ for $x \in A$,

$\phi_j(x, 0-) = \phi_j(x, 0+)$ for $x \in B$.

Define linear combinations $\tilde{\phi}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{2\pi \mathrm{i} kj/n} \phi_j$.

These fields decouple in the quadratic part of the action.

They satisfy $\tilde{\phi}_k(x, 0-) = e^{2\pi \mathrm{i} k/n} \tilde{\phi}_k(x, 0+)$ across A and are continuous across B.

They obey the same (homogeneous) boundary conditions near the conical singularities.

The chemical potential μ couples only to $\sqrt{n}\tilde{\phi}_0$.

$$\frac{Z_n(\mu)}{Z_1(\mu)^n} = \frac{\widetilde{Z}_0(\sqrt{n}\mu)\prod_{k=1}^{n-1}\widetilde{Z}_k(\mu)}{Z_1(\mu)^n} = \frac{\widetilde{Z}_0(\sqrt{n}\mu)\prod_{k=1}^{n-1}\widetilde{Z}_k(0)}{Z_1(\mu)^n}$$

$$= \frac{\widetilde{Z}_0(\sqrt{n}\mu)\,Z_n(0)}{Z_1(\mu)^n\,\widetilde{Z}_0(0)} = \frac{Z_n(0)}{Z_1(0)^n}\,\frac{Z_1(\sqrt{n}\mu)/Z_1(0)}{\big(Z_1(\mu)/Z_1(0)\big)^n}$$

But $Z_1(\mu)/Z_1(0) = e^{-f(\beta)\mu^2}$ because the path integral is gaussian.

So the 2nd factor is $e^{-f(\beta)(\sqrt{n}\mu)^2}/\big(e^{-f(\beta)\mu^2}\big)^n = 1$. QED.

Extension to general 1+1-dimensional CFTs

Rotate in internal space so that $\sum_a \mu_a J_a^\nu = \mu J^\nu$.

$$\text{In 2d CFT} \quad J(z) = \tfrac{1}{2}(J^1 - iJ^0), \quad \overline{J}(\bar{z}) = \tfrac{1}{2}(J^1 + iJ^0)$$

are respectively holomorphic and antiholomorphic.

$$A = A_{CFT} + \mu \int (J + \overline{J}) d^2x + \mu^2 \int \mathcal{O} dx d\tau + \cdots$$

But $\mathcal{O}$ must have dimension zero and therefore be a constant in a unitary CFT.

$Z_n(\mu)/Z_1(\mu)^n$ can be expanded as a sum of integrals of correlators of J and $\overline{J}$ on the n-fold cover.

The OPEs

$$J(z)\cdot J(z') = \frac{\kappa}{(z-z')^2} + O(1)\,, \quad \overline{J}(\bar{z})\cdot\overline{J}(\bar{z}') = \frac{\kappa}{(\bar{z}-\bar{z}')^2} + O(1)$$

completely fix the form of the $2N$-point correlators in the plane

$$\langle J(z_1)J(z_2)J(z_3)J(z_4)\cdots\rangle$$

to be the same as for the $U(1)$ current of a free scalar field, up to the constant κ.

This extends to any conifold of genus zero, which includes those used to compute the Rényi entropies in simple cases.

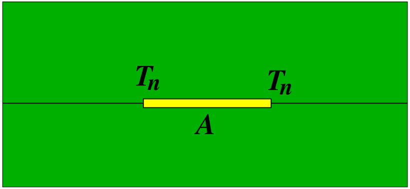
$Z_n(\mu)/Z_1(\mu)^n$ is therefore independent of μ for any CFT, as long as suitable boundary conditions are chosen at the conical singularities. QED.

Perturbative calculation

Consider the case where $A=(-\frac{\ell}{2},\frac{\ell}{2})$ with $\ell\ll\beta$.

In that case the dimensionless expansion parameter is $\mu\ell$.

The coupling between the replicas along A may be viewed as the insertion of *twist operators* at the ends of A.



These behave like local operators of dimension $(c/24)(n-1/n)$.

Short interval expansion [Headrick; Calabrese, JC, Tonni]

$$\overline{\mathcal{T}}_n(-\tfrac{\ell}{2})\cdot\mathcal{T}_n(\tfrac{\ell}{2})=\frac{1}{\ell^{\frac{c}{12}(n-1/n)}}\left(1+\sum_{\{k_j\}}C_{\{k_j\}}\prod_{j=0}^{n-1}\Phi_{k_j}(0)\right)$$

Φ_{k_j} is a local operator in the *j*th replica.

The $C_{\{k_j\}}$ may be evaluated by taking the correlation function of both sides with $\prod_j \Phi_{k_j}(\infty)$.

$C_{\{k_j\}}\propto \ell^{\sum_j \Delta_{k_j}}$ so this gives a series of increasing powers of ℓ.

We can use this to compute $\mathrm{Tr}\,\rho_A^n$ in *any* state

$$\mathrm{Tr}\,\rho_A^n=\langle\overline{\mathcal{T}}_n(-\tfrac{\ell}{2})\mathcal{T}_n(\tfrac{\ell}{2})\rangle=\frac{1}{\ell^{\frac{c}{12}(n-1/n)}}\Big(1+\sum_{\{k_j\}}C_{\{k_j\}}\prod_{j=0}^{n-1}\langle\Phi_{k_j}(0)\rangle\Big)$$

where $\langle\Phi_{k_j}(0)\rangle$ is evaluated in the *decoupled* *j*th replica.

In our case the interesting terms in the OPE are

$$1+\ell^2\sum_j c_j(T_j+\overline{T}_j)+\ell^2\sum_{j\neq k} b_{jk}(J_jJ_k+\overline{J}_j\overline{J}_k)+\cdots$$

We find

$$c_j=\frac{n^2-1}{12n}\,,\qquad b_{jk}\propto\frac{1}{[2-2\cos(\frac{2\pi(j-k)}{n})]^2}$$

$\langle T_j\rangle=\langle\overline{T}_j\rangle$ has two pieces:

– $\pi c/12\beta^2$ which gives the usual $O((\ell/\beta^2))$ term when $\mu=0$;

– a term due to the $O(\mu^2)$ change in the energy density

On the other hand $\langle J\rangle^2=O(\mu^2)$.

$\sum_{j\neq k} b_{jk}\propto(n^2-1)$, and the last two pieces cancel exactly for all n !!

In retrospect this cancellation had to happen, otherwise we would have found a super-volume term

$$\delta S_A \sim \mu^2 \ell^2$$

at $\beta = \infty$.

This argument also implies the cancellation holds for cases when A is several intervals of lengths $\propto \ell$.

One way this might fail is if there is a phase transition at $\mu = 0$ (but there are no IR divergences in the perturbation expansion).

Higher spin currents in 2d

Many 2d CFTs possess holomorphic and antiholomorphic currents $(W, \overline{W})$ of integer spins $\pm s$ with $s > 2$, and corresponding conserved charges $Q = \int (W + \overline{W}) dx$.

If we add a coupling μQ to the hamiltonian how does the entanglement change?

In this case the important terms in the OPE are

$$1+\ell^2\sum_j c_j(T_j+\overline{T}_j)+\ell^{2s}\sum_{j\neq k} b_{jk}(W_jW_k+\overline{W}_j\overline{W}_k)+\cdots$$

where now

$$c_j=\frac{n^2-1}{12n}\,,\qquad b_{jk}\propto\frac{1}{[2-2\cos(\frac{2\pi(j-k)}{n})]^{2s}}$$

The change in $\langle T_j\rangle$ and $\langle\overline{T}_j\rangle$ is now $O(\mu^2\beta^{2-2s})$.

On the other hand $\langle W\rangle^2=O(\mu^2\beta^{4-4s})$.

So the 2 terms do not cancel, and $\delta S_A\propto\mu^2\ell^2/\beta^{2s-2}$.

[Also $\sum_{jk} b_{jk}$ is a polynomial in n of degree $2s\neq 2$.]

Higher dimensions

Free massless bosons Bose condense when $\mu \neq 0$.

Free massless fermions form a Fermi surface and there is an extra $\log R$ term in the area law.

If we add repulsive interaction $\lambda|\Phi|^4$ to the bosons and go into the condensed phase where $\Phi \sim |\Phi| e^{i\phi}$, we get a similar result as in 1+1 dimensions, but this is no longer typical.

If A is a region of size $\ell \ll \beta$ there is an analog of the small interval expansion:

If $\Sigma_A^{(n)}$ denotes the operator which sews together the replicas on the entangling surface,

$$\Sigma_A^{(n)} = e^{-\mathrm{Area}(\partial A)} \left(1 + \ell^d \sum_j c_j T_j^{\tau\tau} + \ell^{2(d-1)} \sum_{jk} b_{jk} J_j^0 J_k^0 + \cdots \right)$$

The coefficients are in principle computable for a sphere.

$\langle T^{\tau\tau} \rangle$ contains a thermal term σ/β^d, and a correction due to the chemical potential $\sim \mu^2/\beta^{d-2}$. On the other hand $\langle J \rangle \sim \mu/\beta^{d-2}$ so the second term is higher order in ℓ/β.
So the leading correction for $\ell/\beta \ll 1$ is entirely given by the equation of state (also for $\ell \gg \beta$). In this case one can show that $\delta S_A < 0$.

Direct perturbation theory theory also suggests a universal term $\sim \mu^2 \ell^2 \log \ell$ for $d = 4$.

Other approaches

– Entanglement hamiltonian (work in progress with M. Smolkin): in principle this allows computation of S_A in any state, but there is a subtlety since the perturbation breaks Lorentz invariance.

Summary

Adding a chemical potential to a 1+1-dimensional CFT has no effect on the entanglement.

For $d > 2$ there are no universal perturbative corrections at zero temperature (besides a possible log correction to area law in 4d), and the leading corrections in for $\ell \ll \beta$ are determined by the equation of state (as they are for $\ell \gg \beta$).

Exceptions to this can occur if turning on μ totally changes the physics.

Similiar ideas apply to entanglement in non-equilibrium steady states where $J_x \neq 0$.