Quantum quenches and boundary entropy: some new applications of relative entropy

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Relative entropy

$$S(\rho_V|\rho_V^0) = \mathrm{tr}\left(\rho_V \log \rho_V - \rho_V \log \rho_V^0\right)$$

Two states in the same Hilbert space or algebra

$$= \mathrm{tr}(\rho_V \log(\rho_V) - \rho_V^0 \log(\rho_V^0)) + \mathrm{tr}(\rho_V^0 \log(\rho_V^0) - \rho_V \log(\rho_V^0))$$

$$= \Delta\langle K\rangle - \Delta S$$

$$\rho_V^0 = c\, e^{-K}$$

K is called the modular Hamiltonian or entanglement Hamiltonian

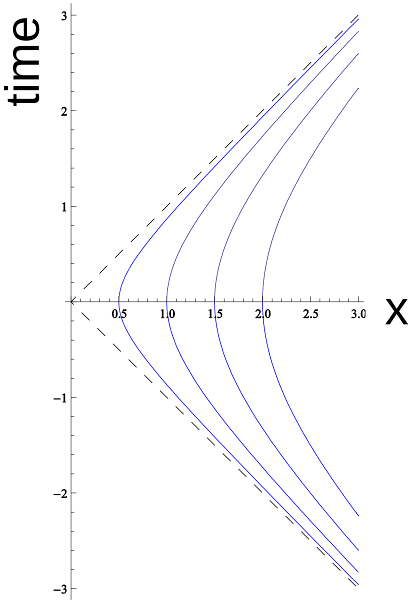
Relative entropy measures distinguishability of two states (roughly inverse of number of experiments needed to distinguish them)

It is positive and increasing with the size of the region: States become more distinguishable for larger regions.

$$\boxed{\Delta S \leq \Delta\langle K\rangle}$$

Preestablished relation between energy and entropy: Vacuum state in half space determined by the energy density operator for all relativistic quantum field theories.

$$\rho_V^0 \sim e^{-2\pi \int_V x^1 T_{00}(x)}$$



Bisognano Wichmann (1975). Unruh (1976):

$$T = \frac{a}{2\pi} = \frac{1}{2\pi x}$$

$$K = 2\pi \int_V x^1 T_{00}$$

$$S(\rho_V|\rho_V^0) = \Delta K - \Delta S \;= 2\pi \int_V x \, \langle \rho_E \rangle - (S(V) - S_0(V)) \geq 0$$

This is Bekenstein's bound π E R> S .
In general K is expected to be non local (but localized inside V), and model dependent.

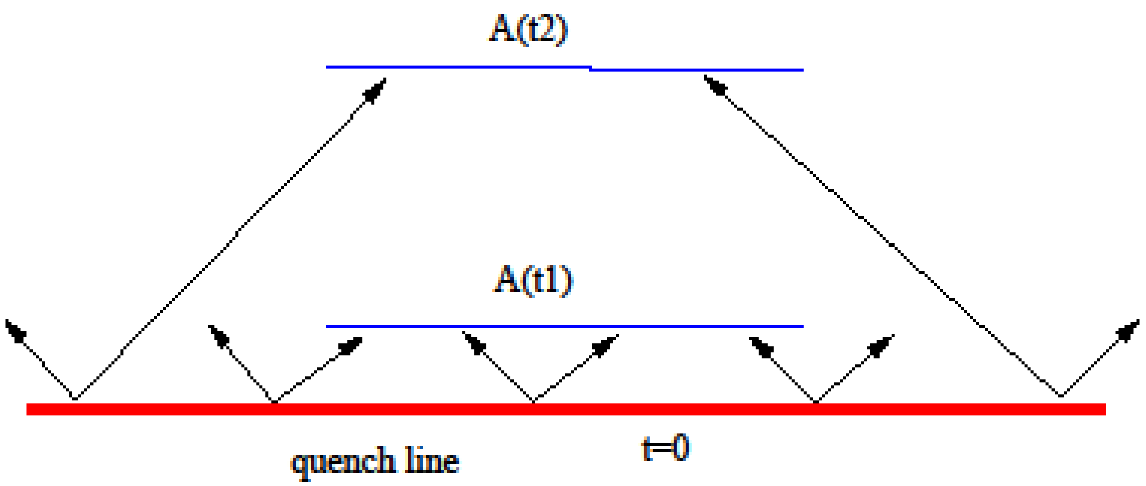
The useful feature of half space is that K is local, universaly given in terms of the stress tensor, for any relativistic QFT.

Early time growth of entropy after a quench

Raul Arias, H.C

$\mathrm{H}{=}\mathrm{H}_0 + \delta(t) \int dx\, \Phi(x)$ Unitary local change in the state

Evolution of entropy of a region in time after the quench Calabrese, Cardy (2005)



$$S(\rho^q(t)) - S(\rho^0(t)) = \Delta S(A(t))$$

Holographic calculations

Balasumramanian et al (2011)
Buchel, Myers, Niekerk (2013)
Abajo, Aparicio,Lopez (2010)
Liu, Suh

Liu, Suh (2014)

Different regimes:

Early growth: t<1/T $\Delta S(t) = \pi \frac{e}{d-1} t^2 A_\Sigma$

Upper bound?

Linear Growth: 1/T< t <R $\Delta S(t) = v_E \, s_{\rm eq} \, t \, A_\Sigma$ $\longrightarrow$ $v_E^{(S)} = \frac{(\eta-1)^{\frac{1}{2}(\eta-1)}}{\eta^{\frac{1}{2}\eta}} \, , \quad \eta \equiv \frac{2(d-1)}{d}$

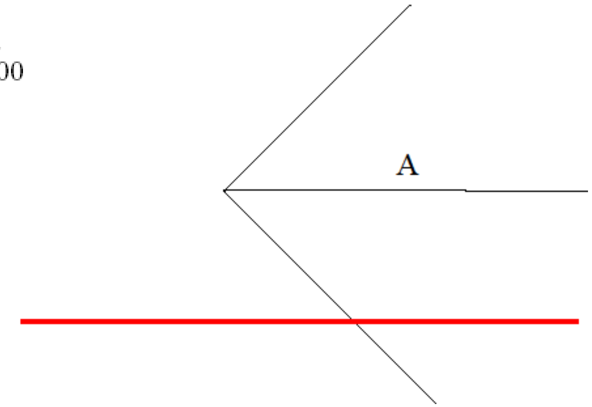
Saturation: t>R $S_A = s_{\rm eq} V_A$

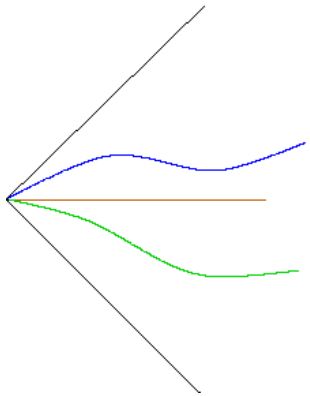
V<1 for relativistic theories
Relative entropy, Hartman, Afkhami-Jeddi
SSA, H.C, H. Liu, M. Mezei

Early time growth: Looks universal and related to relative entropy bound. Proportional to area understood: surface phenomena. Can focus on half space. Why proportional to time square and energy density?

The energy density is constant for t>0
Expectation value of modular Hamiltonian
and relative entropy are very large, increasing with
Volume. State is very different from vacuum

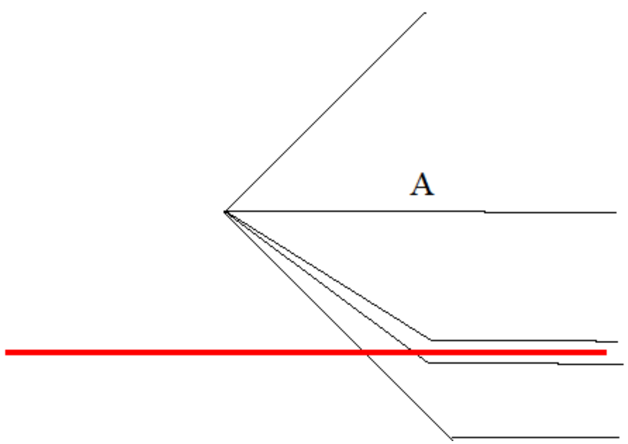
$$K = 2\pi \int_V x^1 T_{00}$$





For two states evolved with the same
time independent local Hamiltonian we dont care
about the choice of Cauchy surface, both DS and DK will be the same

Here however, here the state does change. But the change is an internal unitary that does not change entropy



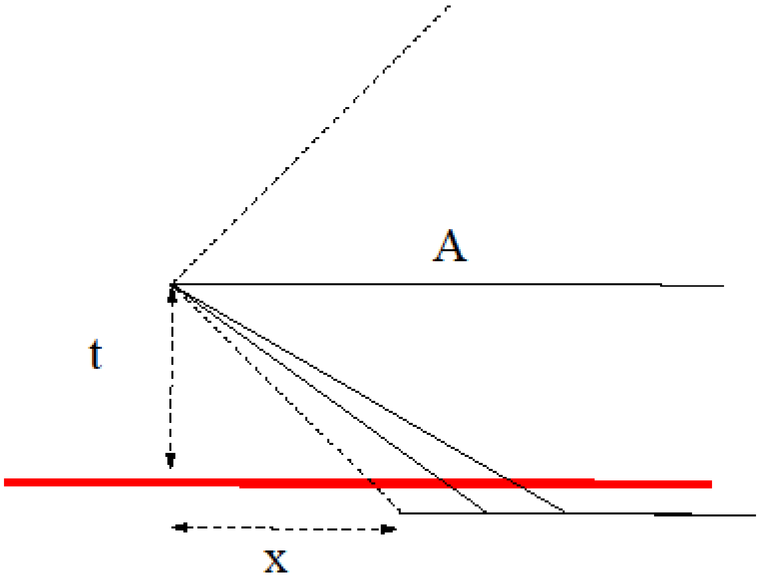
Changing the Cauchy surface for A
(expressing the operators in A in a different surface)
does not change the entropy because the different
states are related by a unitary in the algebra.

However, the state is not the same, and expectation value of K changes because the stress tensor is not conserved

$$K = 2\pi \int_\Sigma d\sigma\, J^\mu \eta_\mu$$

$$j^0 = T^{00}x - T^{01}t \qquad j^1 = T^{10}x - T^{11}t$$

$$\Delta K = \pi(e\, x^2 + p\, t^2) A_\Sigma$$



The minimum is for x=t

$$\boxed{\Delta S \le \pi(e+p)t^2 A_\Sigma =_{CFT} \pi \frac{d}{d-1} e\, t^2 A_\Sigma}$$

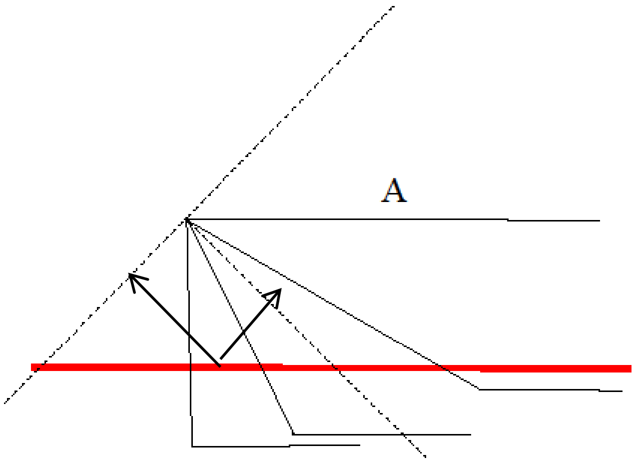
A universal bound (for all times) that explains the t^2 behavior.

Can this be improved? Most probably yes: we still have the freedom to choose a unitary in the null interval segment. It is d times greater than the holographic result.

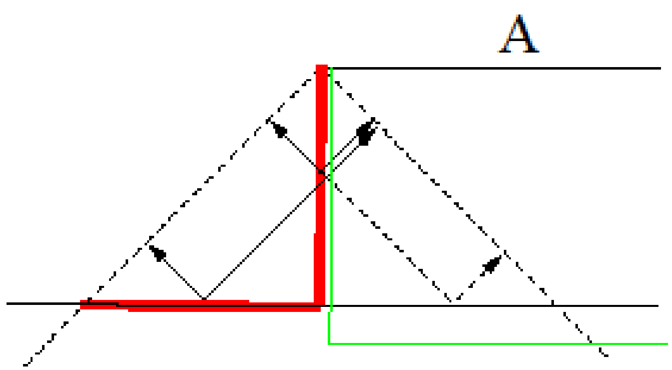
A possible way

How about going down the null line?
This is not a spatial surface anymore. And we
Could be loosing relevant parts of the quench.
But the minimum of K, for the vertical surface,
exactly reproduces the holographic result

$$\Delta K = \pi\, p\, t^2\, A_\Sigma \to \pi\, \frac{e}{d-1}\, t^2\, A_\Sigma$$



Turn to a vertical (unitary) quench that keeps the state on the left
region constant, and eliminate half the horizontal quench. Entropy is the same.

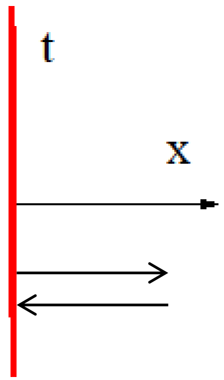


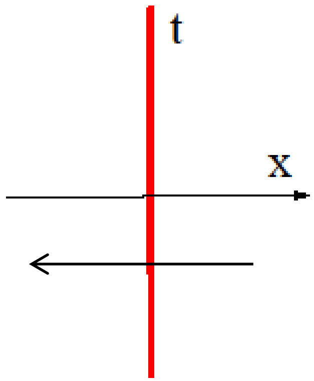
$$\boxed{\Delta S \le \pi\, p\, t^2\, A_\Sigma}$$

Universal bound
on entropy after
a global quench?

Boundary entropy. g-theorem

Gonzalo Torroba, Ignacio Salazar, H.C.





CFT in 1+1 with boundary conditions or an impurity at the origin. Local interaction at x=0 (local Hamiltoninan description)

Thermal entropy. Boundary entropy   Affleck, Ludwig (1991)

$$S(T) = \frac{c\,\pi}{3} T L + \log g(T) \qquad g'(T) \geq 0 \qquad g(\infty) \geq g(0)$$

g-theorem   Friedan, Konechny (2004)

At the fix points conformal boundary condition.   Classified   Cardy (1989)

Entanglement entropy of an interval containing the origin $S(R) = \frac{c}{6} \log(R/\epsilon) + a + \log g(R)$

$$g(R=0) = g(T=\infty) \qquad g(R=\infty) = g(T=0)$$

Calabrese, Cardy (2004)

But in general g(T) and g(R) not easily related to each other

A simple example: a free «Kondo» model

$$S=\int dt\int_0^\infty dx\,\left(-i\bar{\psi}\gamma^\mu\partial_\mu\psi+\frac{i}{2}\delta(x)\left[\bar{\chi}\gamma^0\partial_0\chi+m^{1/2}(\bar{\psi}\chi-\bar{\chi}\psi)\right]\right)\qquad \psi=\begin{pmatrix}\psi_+^*\\ \psi_-\end{pmatrix}\ ,\ \chi=\begin{pmatrix}\eta\\ \eta^*\end{pmatrix}$$

$$L_{lattice}=a\sum_{j=0}^{\infty}\left(i\psi_j^*\partial_0\psi_j-\frac{i}{2a}(\psi_j^*\psi_{j+1}-\psi_{j+1}^*\psi_j)\right)+i\eta^*\partial_0\eta-\frac{i}{2}m^{1/2}(\eta^*\psi_0+c.c.)$$

$$H=\sum c_i^\dagger\,M_{ij}\,c_j\qquad\qquad M=\begin{pmatrix}0&\frac{i}{2}m^{1/2}&0&0&\dots\\ -\frac{i}{2}m^{1/2}&0&\frac{i}{2}&0&\dots\\ 0&-\frac{i}{2}&0&\frac{i}{2}&\dots\\ 0&0&-\frac{i}{2}&0&\dots\\ \vdots&\vdots&\vdots&&\ddots\end{pmatrix}$$

$$\psi_+(t,x)=\int dk\,\frac{e^{-ik(t-x)}}{\sqrt{\pi}}d_k\qquad\qquad \{d_k,d_{k'}^\dagger\}=\delta(k-k')$$

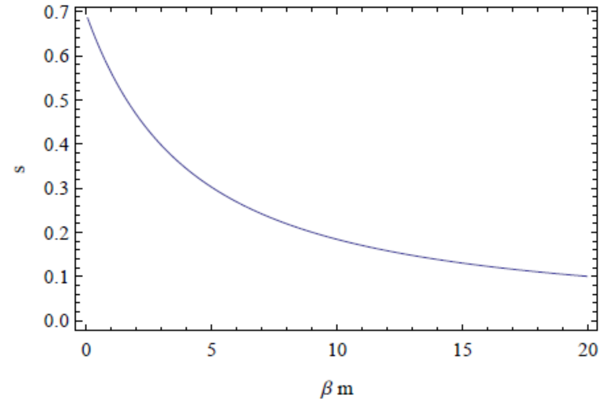
$$\psi_-(t,x)=\int dk\,\frac{e^{-ik(t+x)}}{\sqrt{\pi}}R(k)d_k$$

Momentum dependent reflection coefficient $R(k)=\dfrac{1+i\frac{m}{2k}}{1-i\frac{m}{2k}}=e^{i2\delta(m/k)}$

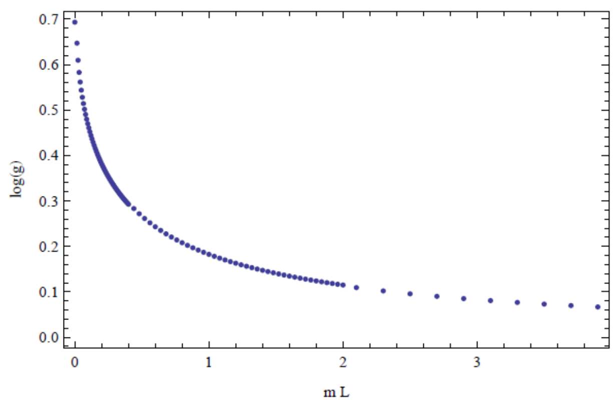
The model interpolates between the conformal boundary conditions

$\psi_+(0)=\psi_-(0)$ UV

$\psi_+(0)=-\psi_-(0)$ IR

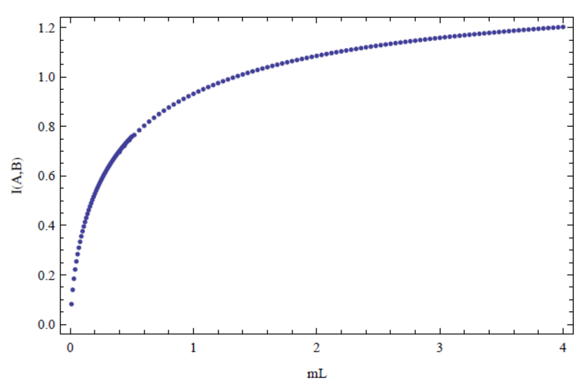


Thermal boundary entropy



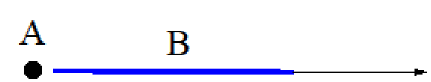
Entanglement entropy. Boundary contribution

Log(g) goes from log(2) in the UV to 0 in the IR



Mutual information between impurity and B. Increasing with B

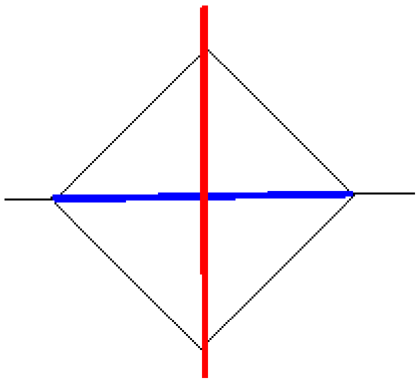
It goes from 0 to 2 log(2)



$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

Is there a g-theorem in terms of entanglement entropy?

Let us try with relative entropy between the state with and without the impurity: little distinguishability?



$$S_{\rm rel}(\rho||\rho^0\otimes\rho^I) = \Delta\langle K\rangle_{\rm field} + \Delta\langle K\rangle_I - \Delta S$$

$$K_{\rm field} = 2\pi\int_{-R}^{R} dx\, \frac{R^2-x^2}{2R} T_{00}$$

$$\left.\begin{aligned} S(R) &= \frac{c}{3}\log(2R/\epsilon) + \log g(R) \\ S(\rho_0) &= \frac{c}{3}\log(2R/\epsilon) \end{aligned}\right\}$$

$$-\Delta S = S_I - \log(g(R))$$

Has the right sign:
Relative entropy is
increasing with size,
g(R) would be decreasing

Stress tensor expectation values vanish in the bulk

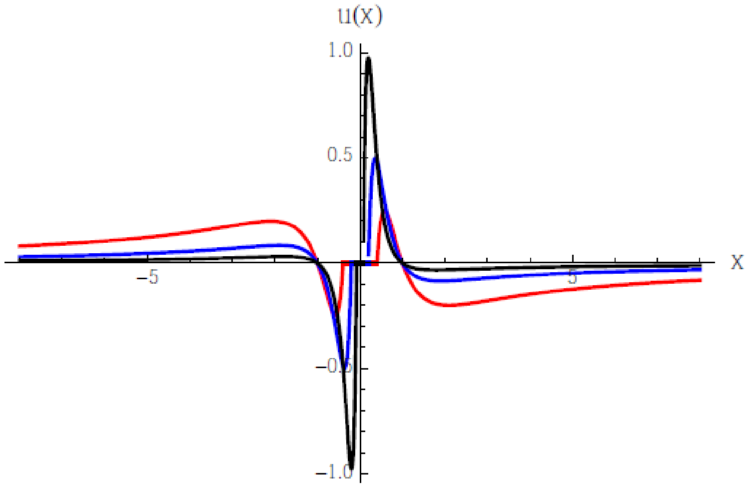
$$\begin{aligned} \langle T_{00}\rangle - \langle T_{11}\rangle &= 0 \\ \partial_0\langle T_{00}\rangle - \partial_1\langle T_{10}\rangle = -\partial_1\langle T_{10}\rangle &= 0 \\ \partial_0\langle T_{01}\rangle - \partial_1\langle T_{11}\rangle = -\partial_1\langle T_{11}\rangle &= 0 \end{aligned} \longrightarrow \Delta\langle K_{\rm field}\rangle = \pi R\int_{-\epsilon}^{\epsilon} dx\,\langle T_{00}(x)\rangle$$

Whatever this is, it is increasing linearly with R, ruining proof of monotonicity of g(R)

Numerics shows relative entropy increases linearly.

Regularizing around the impurity (free fermion)

$$A = (-R, -\delta) \cup (\delta, R)$$



$$K = K_{loc} + K_{noloc}$$

$$K_{loc} = 2\pi \int_A dx\, f(x) T_{00}(x)$$

$$f(x) = \frac{(x^2-\delta^2)(R^2-x^2)}{2(R-\delta)(x^2+\delta R)}$$

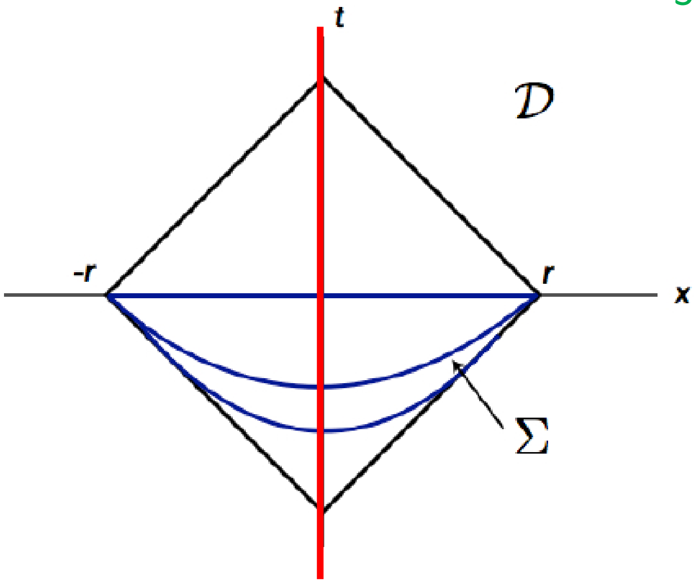
$$K_{\rm noloc} = -\pi i \int_A dx\, u(x)\, \Psi^\dagger(x)\Psi(\bar{x}) \qquad \bar{x} = -\frac{\delta R}{x}$$

$$u(x) = \frac{\delta R}{(R-\delta)} \frac{(R^2-x^2)(x^2-\delta^2)}{x(x^2+\delta R)^2}$$

The subtracted (correlator-vacuum correlator) does not vanish in the limit of small x around x=0 and gives a linearly increasing term in <K>

This relative entropy distinguishes too much!

Let us compare the states on a different Cauchy surfaces
Same idea as for the quench



The entropy is not modifyed because the evolution is unitary inside the diamond (local Hamiltonian evolution for full system)

The modular Hamiltonian on the impurity region vanishes as we move to the null horizon.
Less distinguishability because «high temperature» at the impurity position.

$$K_{CFT} = 2\pi \int_\Sigma ds\, J^\mu\, \eta_\mu$$

$$K_{\rm CFT} = 2\pi \int_0^r dx_+\, \frac{x_+\,(r-x_+)}{r}\, T_{++} + 2\pi \int_0^r dx_-\, \frac{x_-\,(r-x_-)}{r}\, T_{--}$$

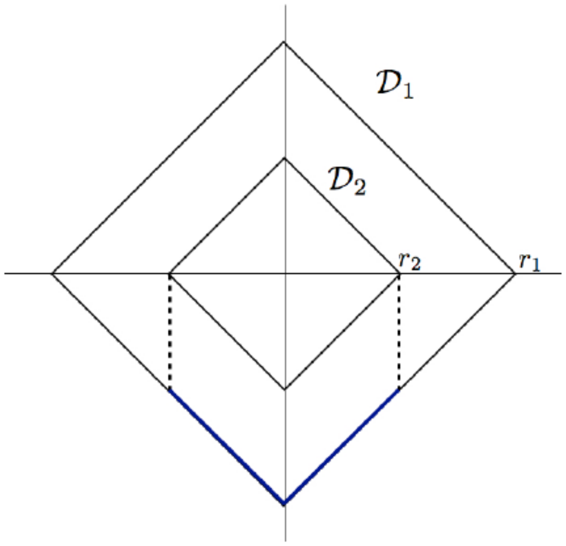
$$x_\pm = (t \pm x + r)/2$$

It is «Rindler like» near the boundaries in null coordinates, and forgets about the value of r in the vecinity of the impurity

$$T_{\pm\pm} \sim \alpha\, \delta'(x_\pm) + \Lambda\, \delta(x_\pm)$$

Eventually there might be
a constant contact term… But
<K> will not change with r

Relative entropy is increasing
On the null Cauchy surface

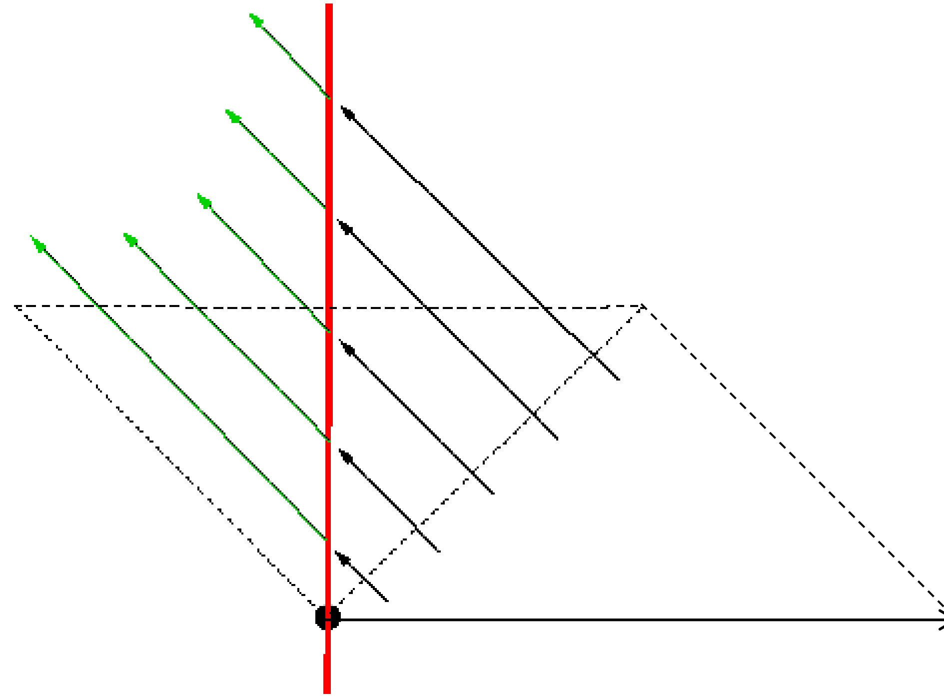


$$S_{\rm rel}(\rho||\rho^0 \otimes \rho^I) = \Delta\langle K\rangle_{\rm field} + \Delta\langle K\rangle_I - \Delta S$$

$$S_{\rm rel}(\rho||\rho^0 \otimes \rho^I) = {\rm cons} - \log(g(r))$$

$\boxed{g'(r) \leq 0}$ $\boxed{g(0) \geq g(\infty)}$

In the model there is no constant contribution of the field <K> and log(g(0)) coincides with the impurity entropy



$$C = \begin{pmatrix} 1/2 & \langle \eta \psi_+^\dagger(x) \rangle \\ \langle \eta \psi_+^\dagger(y) \rangle^* & C_0(x,y) \end{pmatrix}$$

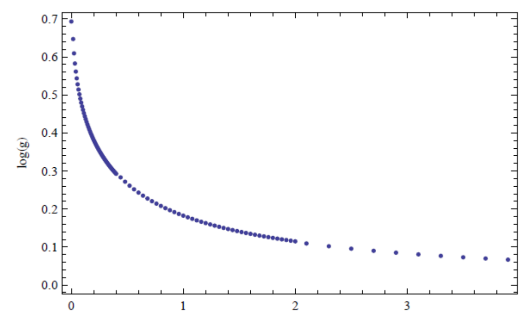
$$S_{rel}(\rho|\rho_0) = -\log \frac{g(r)}{g(0)} + S_{rel}(\rho_{imp}|\rho_{imp}^0)$$

$$S_{rel}(\rho|\rho_0) = -\log \frac{g(r)}{g(0)}$$

$$S_{\mathrm{rel}}(\rho|\rho^0) = I(\mathrm{imp}, \mathrm{field})$$

$$S = 2\int_1^\infty d\lambda \, \frac{\log\det(\lambda^{-1/2}(1+(\lambda-1)C))}{(\lambda-1)^2}$$

$$a(s,y) = \int_0^y dz \, \frac{iy^{1/2}}{(2\pi)^{3/2}} \frac{e^{z/2} Ei(-z/2)}{z^{1/2}(y-z)^{1/2}} e^{-is\log(z/(y-z))}$$



$$\log g(R) = 2\int_1^\infty d\lambda \, \frac{1}{(\lambda-1)^2} \log\left(\frac{1}{2}(\lambda^{1/2}+\lambda^{-1/2}) - \frac{(\lambda-1)^2}{\lambda^{1/2}} \int_{-\infty}^{\infty} ds \, \frac{|a(s,2mR)|^2}{1+\frac{1}{2}(\lambda-1)(1+\tanh(\pi s))} \right)$$

Conclusions:

Relative entropy can be useful to bound entropy differences between two states even if the states are very distinguishable provided a unitary operator in the algebra converts one of the states to one near to the other.

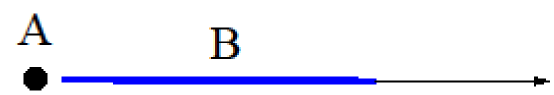
A universal bound on entropy grow after a global quench $\Delta S \leq \pi(e+p)t^2 A_\Sigma$

Probably can be improved to $\Delta S \leq \pi\, p\, t^2\, A_\Sigma$
saturated by holographic calculations
Where this comes from geometrically? This is not simply DS=DH for spheres which has been understood geometrically.

g-theorem for entanglement entropy. First of the list g, c, F, a, theorems to have a clear QIT explanation: decrease in distinguishability as we eliminate operators of the algebra.

It is important in the proof relativistic CFT in 1+1. More dimensions, massive fields?

Mutual information between
Impurity and B. Increasing with B
It is relative entropy of state in AB
With state in A times state in B
Similar to previous, but does not use
the state without impurity



$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

$$S(A \cup B) = \log\, g(r) + S_{\text{no imp}}(B)$$

$$S(B) = \log\, \tilde{g}(r) + S_{\text{no imp}}(B)$$

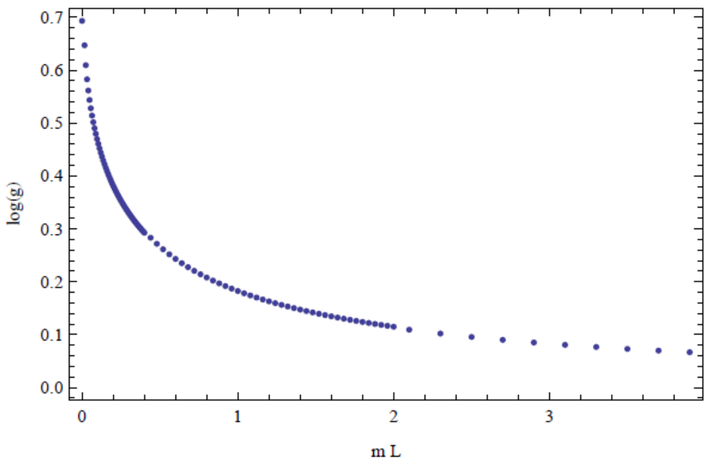
$$I(A,B) = I(r) = S(A) + \log\, \tilde{g}(r) - \log\, g(r)$$

$$S(A) = \log\, g(0) - \log\, \tilde{g}(0)$$

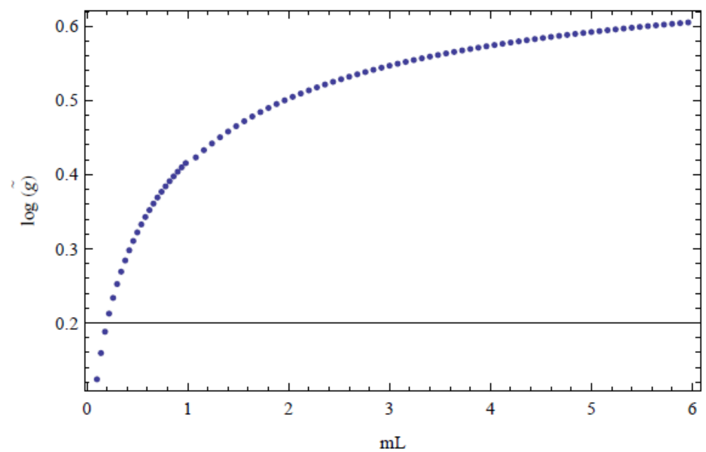
$$S(A) = \log\, \tilde{g}(\infty) - \log\, g(\infty)$$

$$S(A) = \log(g(0)/g(\infty)) \geq 0$$

$$I(\infty) = 2\log(g(0)/g(\infty)) \geq I(0) = 0$$



log(g)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
1
2
3
m L



0.2
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