Rényi Entropy in AdS_3/CFT_2

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Quantum Information in String Theory and Many-body Systems

June 23rd, 2016, Kyoto

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Based on the following works:

- B.C., J.-j. Zhang, arXiv:1309.5453 [hep-th].
- B.C., F. -y. Song and J. -j. Zhang, arXiv:1401.0261 [hep-th].
- B.C., J. Long and J.-j. Zhang, arXiv: 1312.5510 [hep-th].

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- B.C., J.-q. Wu, "Single Interval Rényi Entropy At Low Temperature," arXiv:1405.6254 [hep-th].
- B.C., J.-q. Wu, "Universal relation between thermal entropy and entanglement entropy in CFT," arXiv:1412.0761 [hep-th].
- B.C., J.-q. Wu, "Large Interval Limit of Rényi Entropy At High Temperature," arXiv:1412.0763 [hep-th].
- B.C., J.-q. Wu, "Holographic Calculation for Large Interval Rényi Entropy at High Temperature", arXiv:1506.03206 [hep-th].
- B.C., J.-q. Wu and Z.-c. Zheng, "Holographic Rényi Entropy of Single Interval on Torus: with W symmetry," arXiv:1507.00183 [hep-th].
- B.C., J.-q. Wu, "1-loop Partition Function in AdS_3/CFT_2", arXiv:1509.02062 [hep-th].

- Many other related works ...

AdS_3/CFT_2 correspondence

A new window to study AdS/CFT without resorting to string theory

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R + \frac{2}{l^2})$$

- 3D AdS_3 Einstein gravity is special: No locally dynamical d.o.f
- In 1986, Brown and Heanneaux: there exists boundary d.o.f.
- More precisely they found that under appropriate boundary conditions the asymptotic symmetry group (ASG) of AdS_3 Einstein gravity is generated by two copies of Virasoro algebra with central charges

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- In modern understanding: quantum gravity in AdS_3 is dual to a 2D CFT at AdS boundary

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- However, it is not clear
  1. how to define the quantum AdS_3 gravity?
  2. what is the dual CFT?

Semiclassical AdS_3 gravity

Let us focus on the semiclassical gravity, which corresponds to the CFT at the large central charge limit

$$c = \frac{3l}{2G}$$

- The partition function gets contributions from the saddle points

- For each classical solution, its regularized on-shell action $\propto 1/G \sim c$
- The 1-loop correction comes from the 1-loop determinant of the fluctuations around the solution $\propto O(1)$
- Possibly there are higher loop correction $\propto O(1/c^{l-1})$

Semi-classical AdS_3 Gravity

Semiclassical solutions

$$R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu},$$

- All solutions are locally AdS_3
- More precisely, all classical solutions could be obtained as the quotients of global AdS_3 by the Kleinian group, a discrete subgroup of $PSL(2, C)$

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- It is often convenient to work in Euclideanized signature, in which $AdS_3 \Rightarrow H_3$ with the metric

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- The action of $PSL(2, C)$ on Ω is a Mobius transformation

$$z \to \frac{az+b}{cz+d}, \qquad a, b, c, d \in C, \qquad ad - bc = 1$$

Handle-body solutions

- Among all the solutions, the so-called handle-body solutions are of particular importance, and have been better understood
- The handlebody solution is homeomorphic to a domain enclosed by the closed surface
- The non-handlebody solutions are much less understood (unstable?)

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- For the handlebody solutions, the subgroup Γ is a Schottky group, a finitely generated free group, such that all nontrivial elements are loxodromic

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{pmatrix}, \quad 0 < |p| < 1$$

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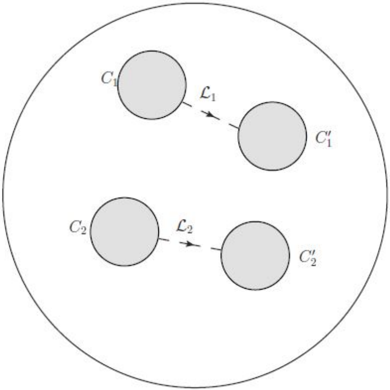
- On the boundary, there is a compact Riemann surface, which could be determined by the Schottky uniformization. Loosely speaking for a genus-g Riemann surface $\mathcal{M}_g$

$$\mathcal{M}_g = \Omega/\Gamma$$

Schottky group



- The loxodromic element $\mathcal{L}_i(\mathcal{L}_i^{-1})$ maps C_i to C_i' such that the outer(inner) part of C_i is mapped to the inner(outer) part of C_i'.
- The elements $\{\mathcal{L}_i\}$ freely generate the Schottky group

Schottky uniformization

- Every compact Riemann surface could be obtained by the Schottky uniformization"Retrosection theorem" by Koebe (1914)
- The Schottky uniformization is determined by a differential equation

$$\psi''(z) + \frac{1}{2} T_{zz}\psi(z) = 0, \qquad (1.1)$$

- Two independent solutions: ψ_1 and ψ_2
- Their ratio $w = \frac{\psi_1}{\psi_2}$ gives the quotient map
- More importantly, T_{zz} is the stress tensor of Liouville CFT. Its explicit form depends on $(3g-3)$ complex accessory parameters with respect to the holomorphic quadratic differentials on the Riemann surface.

On-shell regularized action

- The essential point is that the on-shell regularized bulk action of gravitational configuration in pure AdS_3 gravity is a Liouville type action defined on the fundamental region K. Krasnov (2000), Zograf and Takhtadzhyan (1988)
- More importantly, the dependence of this so-called Zograf-Takhtadzhyan action on the accessory parameters is determined by the differential equation Zograf and Takhtadzhyan (1988)

$$\frac{\partial S_n}{\partial z_i} = -\frac{cn}{6(n-1)}\gamma_i. \tag{1.2}$$

- γ_i are the accessory parameters, being fixed by the monodromy problem of the ordinary differential equation (1.1)
- For a general Riemann surface of high genus, it is a difficult problem to determine this regularized action, even perturbatively
- Nevertheless, for the Riemann surface in computing the Rényi entropy, the problem is simplified due to the replica symmetry
    1. Two-interval case: one cross ratio
    2. Single interval in a torus (finite temperature, finite size)

Rényi entropy in 2D CFT

$$S_A^{(n)} = -\frac{\ln \mathrm{tr}_A \rho_A^n}{n-1}$$

- The partition function on a n-sheeted Riemann surface
- Double-interval case: $\Rightarrow$ genus $(n-1)$ RS
- Single-interval on torus: $\Rightarrow$ genus n RS
- Partition function on a higher genus RS: usually hard to compute

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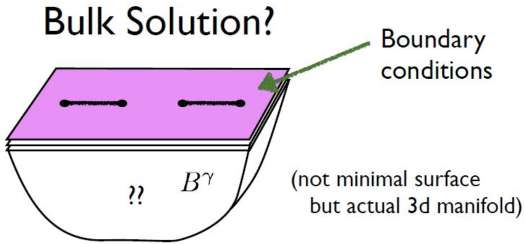
Let's try holographic computation...

HRE in AdS_3: A sketch T. Faulkner 1303.7221



- Find the bulk gravity solutions B^γ such that $\partial B^\gamma = \Sigma_n$
- Assumption: Consider only the handlebody solution, Γ_γ is the Schottky group

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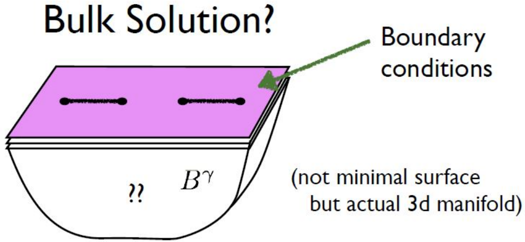

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- For a fixed Riemann surface Σ_n, find its Schottky uniformization
- Extend the uniformization to the bulk to find the gravitational solution
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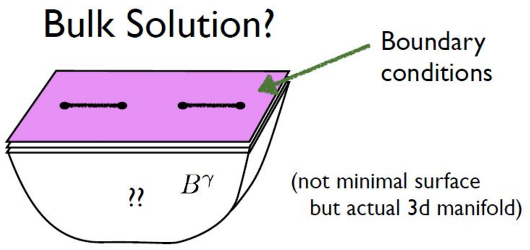

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- From AdS_3/CFT_2 correspondence, the classical regularized bulk action should reproduces the leading order partition function on Σ_n.
- Subtlety: For the same Σ_n, there could be more than one B^γ
- In the classical gravity limit, keep only the solution of least action

Two-interval case

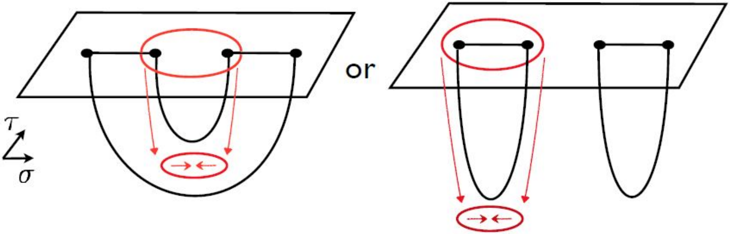
- In this case

$$T_{zz} = \sum_i \frac{\Delta}{(z - z_i)^2} + \frac{\gamma_i}{z - z_i},$$

where

$$\Delta = \frac{1}{2}(1 - \frac{1}{n^2}),$$

- There is only one accessory parameter
- The accessory parameters are determined by requiring trivial monodromy at infinity and on one of two cycles (red one)



Single interval on a torus

$$T_{zz} = \sum_i (\Delta\wp(z - z_i) + \gamma_i \zeta(z - z_i)) + \delta,$$

where $\wp, \zeta$ are the doubly periodic Weierstrass elliptic function and zeta function respectively. Barrella et.al. 1306.4682,BC and J.-q. Wu1604.03644

- Torus: $z \sim z + mL + in\beta \Rightarrow$ thermal circle and spatial circle
- We can set trivial monodromy along one circle and the cycle enclosing two branch points, so that the identification of the other circle gives the generator of Schottky group

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- We can set trivial monodromy along one circle and the cycle enclosing two branch points, so that the identification of the other circle gives the generator of Schottky group
- At high temperature above the Hawking-Page transition, the bulk spacetime is actually a black hole, so the thermal circle is of trivial monodromy.
- At low temperature, the dual bulk is the thermal AdS spacetime, so the spatial circle is of trivial monodromy.

Finite size effect BC and J.q. Wu 1405.6254

For the torus at high temperature, we have to consider the effect of its finite size. The regularized action depends not only on the accessory parameter,

$$\frac{\partial S_n}{\partial z_i} = -\frac{cn}{6(n-1)}\gamma_i,$$

but also on the size of the torus

$$\frac{\partial S_n}{\partial L} = \frac{c}{12\pi}\frac{n}{n-1}\beta(\tilde{\delta} - \tilde{\delta}_{n=1}). \tag{1.3}$$

where $\tilde{\delta}$ includes all the constant contribution in $T(z)$.

Beyond classical action

- Simply speaking, the holographic Renyi entropy(HRE) is given by the classical action of the corresponding gravitational configurations
- The $n \to 1$ limit reproduces the RT formula T. Hartman 1303.6955, T. Faulkner 1303.7221
- It captures only the leading order Rényi entropy in CFT

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- The subleading corrections in 2D CFT being independent of c should correspond to the 1-loop partition function around the configurations
- There are good reasons to consider the quantum correction: mutual information, thermal correction, ...
  e.g. the mutual information satisfies M. Wolf et.al. 0704.3906

$$I(A,B) \geq \frac{|<\mathcal{O}_A \cdot \mathcal{O}_B> - <\mathcal{O}_A><\mathcal{O}_B>|^2}{2|\mathcal{O}_A|^2|\mathcal{O}_B|^2}$$

1-loop correction

For a fixed handle-body solution obtained from the Schottky group, its 1-loop partition function Giombi et.al. 0804.1773, X.Yin 0710.2129

$$Z^{1-loop} = \prod_{\gamma\in\mathcal{P}} \prod_{s} \prod_{m=s}^{\infty} \frac{1}{|1-q_\gamma^m|}. \qquad (1.4)$$

Here the product over s is with respect to the spins of massless fluctuations and $\mathcal{P}$ is a set of representatives of primitive conjugacy classes of the Schottky group Γ. q_γ is defined by writing the two eigenvalues of $\gamma \in \Gamma$ as $q_\gamma^{\pm 1/2}$ with $|q_\gamma| < 1$.

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- Find the Schottky group Γ corresponding to $\mathcal{M}_n$
- Generate $\mathcal{P} = \{$non-repeated words up to conjugation$\}$, e.g.

$$\mathcal{P} = \{L_1, L_2, L_1^{-1}, L_2^{-1}, L_1 L_2 \sim L_2 L_1, ...\}$$

- Compute eigenvalues of these words and sum over their contributions
- Difficulty: infinite number of words
- For two intervals with small cross ratio x, only finitely many words contribute to each order in x Barrella et.al. 1306.4682,BC et.al. 1312.5510
- For the single interval on a torus, similar thing happens Barrella et.al. 1306.4682, BC et.al. 1405.6254

CFT computation

Large c CFT

The semiclassical AdS_3 gravity is dual to a large c CFT. The large c CFT has a few attractive features

- Dual to the pure gravity, the vacuum module dominates the contribution
  1. $g_{\mu\nu} \leftrightarrow T_{\mu\nu}$
  2. The study on the conformal block T. Hartman 1303.6955,...
- Conformal block techniques: heavy-heavy-light-light one

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  2. The study on the conformal block T. Hartman 1303.6955,...
- Conformal block techniques: heavy-heavy-light-light one

We only focus on the Virasoro vacuum module, or sometimes the $\mathcal{W}$ vacuum module

General prescription

M. Headrick 1006.0047, P. Calabrese et.al. 1011.5482, BC and J-j Zhang 1309.5453

The replica trick requires us to study a orbifold CFT: $(\mathrm{CFT})_n/Z_n$. When the intervals are short, we have the OPE of the twist operators

$$\sigma(z,\bar{z})\tilde{\sigma}(0,0) = c_n \sum_K d_K \sum_{m,r\geq 0} \frac{a_K^m}{m!}\frac{\bar{a}_K^r}{r!}\frac{1}{z^{2h-h_K-m}\bar{z}^{2\bar{h}-\bar{h}_K-r}}\partial^m\bar{\partial}^r\Phi_K(0,0),$$

with the summation K being over all the independent quasiprimary operators of CFT_n.

Short interval expansion

We are interested in the two-interval case, then

$$\begin{aligned} \mathrm{Tr}\rho_A^n &= \langle \sigma(1+y,1+y)\tilde{\sigma}(1,1)\sigma(y,y)\tilde{\sigma}(0,0)\rangle_C \\ &= c_n^2 x^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \left(\sum_K \alpha_K d_K^2 x^{h_K} F(h_K,h_K;2h_K;x)\right)^2 \end{aligned}$$

where x is the cross ratio and $F(h_K, h_K; 2h_K; x)$ is the hypergeometric function. α_K is the normalization factor of Φ_K, and d_K is the OPE coefficients.

- For a concrete CFT model, the summation should be over all the conformal blocks
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- For a concrete CFT model, the summation should be over all the conformal blocks
- For pure AdS_3 gravity, it is enough to consider the vacuum Verma module
- In the small x limit, to each order only finite number of the quasi-primary operators contribute

Rényi mutual information

- We are interested in the mutual Renyi entropy

$$I^{(n)}_{A_1,A_2} = S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1+A_2}$$

In the following, we write I_n for $I^{(n)}_{A_1,A_2}$.

- The Rényi mutual information can be classified according to the order of the inverse of central charge $\frac{1}{c}$ in the large c limit

$$\begin{aligned} I_n &= \frac{c}{3}(1+\frac{1}{n})\log\frac{y}{\epsilon} + \frac{1}{n-1}\log \mathrm{Tr}\rho_A^n, \\ &= I_n^{LO} + I_n^{NLO} + I_n^{NNLO} + \cdots . \end{aligned}$$

1. $I_n^{LO} \sim \mathcal{O}(c)$ terms
2. $I_n^{NLO} \sim \mathcal{O}(c^0)$ terms
3. $I_n^{NNLO} \sim \mathcal{O}(1/c)$ terms

- After some highly nontrivial summations...

Mutual information: leading order

The leading part, being proportional to the central charge c,

$$\begin{aligned} I_n^{tree} = & \frac{c(n-1)(n+1)^2x^2}{144n^3} + \frac{c(n-1)(n+1)^2x^3}{144n^3} \\ & + \frac{c(n-1)(n+1)^2\left(1309n^4-2n^2-11\right)x^4}{207360n^7} \\ & + \frac{c(n-1)(n+1)^2\left(589n^4-2n^2-11\right)x^5}{103680n^7} \\ & + \frac{c(n-1)(n+1)^2(805139n^8-4244n^6-23397n^4-86n^2+188)x^6}{156764160n^{11}} \\ & + (\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}\left(x^9\right) \end{aligned}$$

It matches exactly with the holographic result up to order x^8. M. Headrick 1006.0047, T. Hartman 1303.6955, T. Faulkner 1303.7221

The classical mutual information $(n=1)$ is vanishing when the two intervals are far apart.

Mutual information: next-to-leading order

The NLO part from the vacuum module, being proportional to c^0, is

$$I_n^{NLO} = \frac{(n+1)\left(n^2+11\right)\left(3n^4+10n^2+227\right)x^4}{3628800n^7}$$
$$+\frac{(n+1)\left(109n^8+1495n^6+11307n^4+81905n^2-8416\right)x^5}{59875200n^9}$$
$$+\frac{(n+1)\left(1444050n^{10}+19112974n^8+140565305n^6+1000527837n^4-167731255n^2-14142911\right)x^6}{523069747200n^{11}}$$
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The mutual information is not really vanishing due to the quantum correction

Mutual information: NNLO

Remarkably there is also the NNLO contribution, being proportional to $1/c$,

$$
\begin{aligned}
I_n^{NNLO} \quad = \quad & \frac{(n+1)\left(n^2-4\right)(19n^8+875n^6+22317n^4+505625n^2+5691964)x^6}{70053984000n^{11}c} \\
& \frac{(n+1)\left(n^2-4\right)\left(276n^{10}+12571n^8+317643n^6+7151253n^4+79361381n^2-9428724\right)x^7}{326918592000n^{13}c} \\
& +(\text{the terms proportional to } x^8) + \mathcal{O}\left(x^9\right),
\end{aligned}
$$

This is novel, expected to be confirmed by 2-loop computation in gravity

- When $n = 2$, the two-loop correction is vanishing, as $S^{(2)}$ being genus 1 partition function is 1-loop exact A. Maloney and E. Witten 0712.0155
- When $n > 2$, there are nonvanishing 2-loop corrections Xi Yin, 0710.2129
- Actually there is nonvanishing quantum 3-loop contribution, being proportional to $1/c^2$, for $S^{(n)}, n > 3$.

Our computation weakly indicate (to order x^8) that the m-loop correction ($m \geq 3$) to nth ($n = 2, 3, \cdots, m$) mutual information is vanishing, say

$$I_n^{3-loop} \propto (n^2 - 4)(n^2 - 9) \tag{2.1}$$

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However, the recent study by using sewing construction to compute the partition function of genus-2 RS shows that this can't be true. The nonvanishing terms appear at x^{12}. M. Headrick et.al. 1503.07111

Single interval on a torus

- When the interval is not very large, the Rényi entropy could be computed perturbatively for both high and low temperatures
- At a low temperature T in units of $1/L$, the thermal density matrix could be expanded level by level

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}} = \frac{1}{\mathrm{Tr} e^{-\beta H}} \sum \mid \phi\rangle\langle\phi \mid e^{-\beta E_\phi}$$

- The expansion is respect to $e^{-2\pi\Delta/TL}$, Δ being the dimension of the excitation
- The expansion could be understood in the following way: cut open the torus and insert the complete basis at the cut
- At the low levels, the computations change to multi-point function on a cylinder, via state-operator correspondence

The leading order

On the CFT side, the dominant contribution comes from the vacuum module in the large c limit. T. Hartman (1303.6955),...

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The leading order terms are proportional to the central charge

$$
\begin{aligned}
S_n^{\rm LO} \quad = \quad & \frac{c(1+n)}{n}\Bigg\{\frac{1}{12}\log\sin^2\frac{\pi l}{L} + const \\
& -\frac{1}{9}\frac{(n^2-1)}{n^2}\Bigg(\sin^4(\frac{\pi l}{L})e^{-\frac{4\pi}{TL}} + 4\sin^4(\frac{\pi l}{L})\cos^2(\frac{\pi l}{nL})e^{-\frac{6\pi}{TL}} \\
& +\Bigg(\frac{-11-2n^2+1309n^4}{11520n^4}\cos(\frac{8\pi l}{L}) - \frac{-11+28n^2+119n^4}{1440n^4}\cos(\frac{6\pi l}{L}) \\
& -\frac{77-346n^2+197n^4}{2880n^4}\cos(\frac{4\pi l}{L}) - \frac{-77+436n^2+433n^4}{1440n^4}\cos(\frac{2\pi l}{L}) \\
& +\frac{-77+466n^2+907n^4}{1152n^4}\Bigg)e^{-\frac{8\pi}{TL}}\Bigg) + O(e^{-\frac{10\pi}{TL}})\Bigg\}
\end{aligned}
$$

It is in perfect agreement with the HRE. BC and J-q. Wu,1405.6254

Next-to-leading order

In the CFT side, the next-to-leading terms are independent of c

$$
\begin{aligned}
S_n^{\text{1-loop}} &= -\frac{2n}{n-1}\left(\frac{1}{n^4}\frac{\sin^4(\frac{\pi l}{L})}{\sin^4\frac{\pi l}{nL}}-1\right)e^{-\frac{4\pi}{TL}} \\
&-\frac{2n}{n-1}\left(\frac{4}{n^4}(\frac{\sin\frac{\pi l}{L}}{\sin\frac{\pi l}{nL}})^4\cos^2\frac{\pi l}{L}-\frac{8}{n^5}(\frac{\sin\frac{\pi l}{L}}{\sin\frac{\pi l}{nL}})^5\cos\frac{\pi l}{nL}\cos\frac{\pi l}{L}\right. \\
&\left.-\frac{4}{n^6}(\frac{\sin\frac{\pi l}{L}}{\sin\frac{\pi l}{nL}})^4\sin^2\frac{\pi l}{L}+\frac{5}{n^6}(\frac{\sin\frac{\pi l}{L}}{\sin\frac{\pi l}{nL}})^6-1\right)e^{-\frac{6\pi}{TL}} \\
&+(\text{order } e^{-\frac{8\pi}{TL}} \text{ terms})+O(e^{-\frac{10\pi}{TL}}). \qquad (2.2)
\end{aligned}
$$

It is in perfect match with the 1-loop correction to HRE. BC and J-q. Wu,1405.6254, BC, J.-q. Wu and Z.-c. Zheng 1507.00183

The NNLO result is hard to compute, but it could be read in the short interval limit BC, J-b. Wu and J.-j Zhang, 1606.05444

Entanglement entropy

The entanglement entropy could be read easily

$$\begin{aligned} S_{EE} &= c\left(\frac{1}{6}\log\sin^2\frac{\pi l}{L} + const\right) \\ &+8(1-\frac{\pi l}{L}\cot(\frac{\pi l}{L}))e^{-\frac{4\pi}{TL}} + 12(1-\frac{\pi l}{L}\cot(\frac{\pi l}{L}))e^{-\frac{6\pi}{TL}} + O(e^{-\frac{8\pi}{TL}}) \end{aligned}$$

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- The thermal corrections to EE are independent of c
- Correspondingly, they are captured by the quantum corrections in the bulk
- This is not true for RE, in which the thermal corrections appear even in the leading order

High temperature case

- In the high temperature case, one may "quantize" the theory along the spacial direction rather than the thermal direction
- In other words, the spacial direction and the thermal direction exchange the role and there is a modular transformation

$$L \to i\beta, \qquad \beta \to iL,$$

relating the two cases
- As a result, we have the density matrix

$$\rho \propto e^{-LH} = e^{-2\pi(L/\beta)(L_0+\tilde{L}_0-\frac{c}{12})}$$

- This is in accord with the holographic computation

Large interval limit

$$
\begin{aligned}
S_{EE} &= c\left(\frac{1}{6}\log\sin^2\frac{\pi l}{L} + const\right) \\
&+8\left(1-\frac{\pi l}{L}\cot\left(\frac{\pi l}{L}\right)\right)e^{-\frac{4\pi}{TL}} + 12\left(1-\frac{\pi l}{L}\cot\left(\frac{\pi l}{L}\right)\right)e^{-\frac{6\pi}{TL}} + O\left(e^{-\frac{8\pi}{TL}}\right)
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The large interval limit $l \to L$ is singular

Such singular behavior exists for other CFT. For example, the thermal correction of a primary operator to EE takes a universal form J. Cardy and C.P. Herzog 1403.0578

$$
\begin{aligned}
\delta S_n &= \frac{g}{1-n}\left(\frac{1}{n^{2\Delta-1}}\frac{\sin^{2\Delta}\left(\frac{\pi l}{L}\right)}{\sin^{2\Delta}\left(\frac{\pi l}{nL}\right)} - n\right)e^{-2\pi\Delta/TL} + o(e^{-2\pi\Delta/TL}) \\
\delta S_{EE} &= 2g\Delta\left(1-\frac{\pi l}{L}\cot\left(\frac{\pi l}{L}\right)\right)e^{-2\pi\Delta/TL} + o(e^{-2\pi\Delta/TL}), \quad (2.3)
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It is singular in the limit $l \to L$. We needs a different way to compute the Rényi entropy in the large interval limit.

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Rényi entropy in the large interval limit.

We got inspiration from the holographic computation

Large interval: holographic result

- The entanglement entropy of single interval at high temperature is

$$S_{EE} = \frac{c}{3} \log \sinh(\pi T l) \tag{2.4}$$

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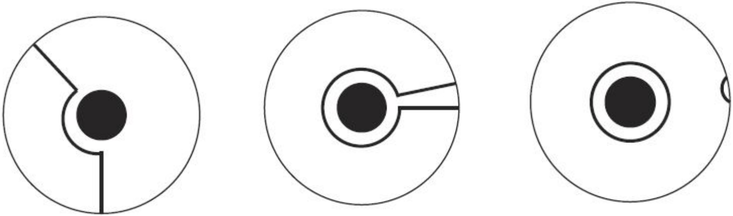
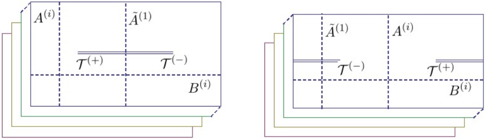
$$S_{EE} = \frac{c}{3} \log \sinh(\pi T l) \tag{2.4}$$

- From holographic point of view, it is given by the geodesic in the BTZ background ending on the interval
- However, it is only true when the interval is not very large



- When the interval is very large, the disconnected curve gives smaller length T. Azeyanagi et.al. 0710.2956

Our proposal in CFT BC and J.-q. Wu 1412.0763



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- This requires us to study the twist sector of the CFT carefully

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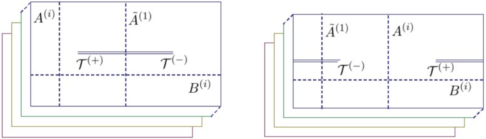

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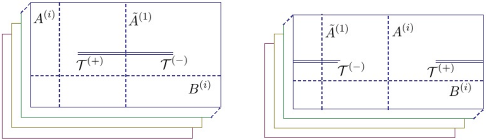

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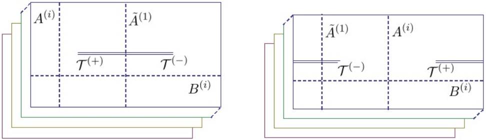

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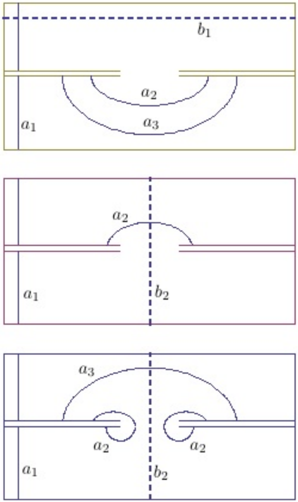
$$S_{thermal} = S_{EE}(L-\epsilon) - S_{EE}(\epsilon)$$

- Next, we studied the large interval Rényi entropy at high temperature in the context of AdS_3/CFT_2 correspondence

HRE: large interval limit BC and J.-q. Wu, 1506.03206

- Different gravitational configurations
- Different set of monodromy conditions
- Among n cycles of trivial monodromy
  1. One cycle which goes across the branch cut for n times
  2. The other $n-1$ independent cycles enclosing the complementary interval
- Both the classical and 1-loop contributions are in good agreements with CFT results



Semi-classical picture

- Rényi entropy opens a new window to study the AdS_3/CFT_2 correspondence
- AdS_3/CFT_2 correspondence at semi-classical level: the leading order partition function on a general (higher genus) RS is captured by the on-shell regularized gravity action, which reduces to the ZT action
- Compelling evidence from the holographic computation of Rényi entropy suggests that this is true

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- AdS_3/CFT_2 correspondence at semi-classical level: the leading order partition function on a general (higher genus) RS is captured by the on-shell regularized gravity action, which reduces to the ZT action
- Compelling evidence from the holographic computation of Rényi entropy suggests that this is true
- Moreover, the next-to-leading order partition function is given by the 1-loop partition function in the bulk, this can be proved

Goal

Prove the 1-loop correction from dual CFT:

$$Z^{1-loop} = \prod_{\gamma \in \mathcal{P}} \prod_{s} \prod_{m=s}^{\infty} \frac{1}{|1 - q_\gamma^m|}.$$

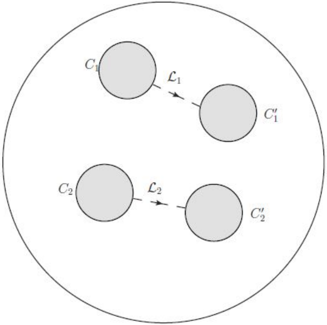
Not only for the configurations appearing in the computation of HRE, but for any handle-body solutions.

Sewing prescription

It can be computed using the sewing construction, following Segal's approach to CFT. It is defined to be the summation of $2g$-point functions on the Riemann sphere M.R. Gaberdiel et.al. 1002.3371

$$Z_g = \sum_{\phi_i,\psi_i\in\mathcal{H}} \prod_{i=1}^{g} G^{-1}_{\phi_i\psi_i} \langle \prod_{i=1}^{g} \phi_i[C_i]\psi_i[C_i']\rangle_D,$$

ϕ_i, ψ_i are the states in the Hilbert space $\mathcal{H}$, and $\phi_i[C_i]$ denote the states associated with the boundary circle C_i. $G_{\phi\psi}$ is the metric on the space of the states



Partition funciton on Σ_g

Via state-operator correspondence, the states can be transformed to the vertex operators inserted at the fixed points. With the vertex operators, the partition function is changed to the summation over $2g$-point functions of the vertex operators inserted at $2g$ fixed points

$$Z_g = \sum_{\phi_i,\psi_i\in\mathcal{H}} \prod_{i=1}^{g} G^{-1}_{\phi_i\psi_i} \langle \prod_{i=1}^{g} V(U(\gamma_i)p_i^{L_0}\phi_i, a_i) V(U(\gamma_i\hat{\gamma})\psi_i, r_i)\rangle,$$

This prescription could be applied to any CFT, but is most effective to read the next-to-leading terms in the large c CFT. The large c limit brings simplifications and make the graviton effectively "free" at this level. The details can be found in 1509.02062.

Large c CFT

- For pure AdS_3 quantum gravity, it is the vacuum conformal module in the dual CFT which dominate the contribution
- The other modules in the dual CFT?

Large c CFT

- For pure AdS_3 quantum gravity, it is the vacuum conformal module in the dual CFT which dominate the contribution
- The other modules in the dual CFT?
- What's the CFT dual of quantum AdS_3 gravity? E. Witten 1988, S. Carlip 050302, A. Maloney and E. Witten 0712.0155,H. Verlinde et.al. 1412.5205,...
- ...

Thanks for your attention!