Entanglement Renormalization and Wavelets

Glen Evenbly


Entanglement renormalization and wavelets

- real-space renormalization
- quantum circuits
- tensor networks (MERA)
- compact, orthogonal wavelets

Introduction: MERA

Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians

Can be formulated as:
(i) a quantum circuit
(ii) resulting from coarse-graining (entanglement renormalization)

product state
|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩|0⟩

entangled state
|ψ⟩

Circuit
Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians

Can be formulated as:
(i) a quantum circuit
(ii) resulting from coarse-graining (entanglement renormalization)

fixed index can be omitted
Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians

Can be formulated as:
(i) a quantum circuit
(ii) resulting from coarse-graining (entanglement renormalization)

disentanglers (mediate entanglement between blocks)
isometries (truncate blocks)

fixed index can be omitted
Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians

Can be formulated as:
(i) a quantum circuit
(ii) resulting from coarse-graining (entanglement renormalization)

Key properties:
(i) efficiently contractible (for local observables, correlators, etc)
(ii) reproduce logarithmic correction to the area law (for 1D quantum systems)
\[ S_L : \log(L) \]
(iii) reproduce polynomial decay of correlations
(iv) can capture scale-invariance
Introduction: MERA

Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians

Applications:

• numeric study of quantum critical systems
• error-correcting codes (e.g. holographic codes) and topologically ordered systems
• machine learning (convolutional neural networks)
• data compression (multi-resolution analysis and wavelets)
• holography?
Introduction: MERA

Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians.

MERA can be viewed as a tiling of the hyperbolic disk:
Multi-scale Entanglement Renormalization Ansatz (MERA): proposed by Vidal to represent ground states of local Hamiltonians

MERA are useful as a numeric tool for studying ground states of many-body systems... but we lack a deeper conceptual understanding.

Can we understand how MERA can represent the ground state of a lattice CFT? (even if only in a simple example...)

Numerical magic

Optimise tensors (i.e. energy minimization)

input: Hamiltonian

output: Ground state (approximate)

parameters defining disentangler, $u$: 

\[
\begin{bmatrix}
  u_{11} & u_{12} & u_{13} & u_{14} \\
  u_{21} & u_{22} & u_{23} & u_{24} \\
  u_{31} & u_{32} & u_{33} & u_{34} \\
  u_{41} & u_{42} & u_{43} & u_{44}
\end{bmatrix}
\]
**Introduction: Wavelets**

Fourier expansions are ubiquitous in math, science and engineering

- many problems are simplified by expanding in Fourier modes
- smooth functions can be approximated by only a few non-zero Fourier coefficients

Wavelets are a good compromise between real-space and Fourier-space representations

- compact in real-space and in frequency-space
- developed by math and signal processing communities in late 80’s
- applications in signal and image processing, data compression (e.g. JPEG2000 image format)
Introduction: Wavelets

Wavelets are a good compromise between real-space and Fourier-space representations:
- compact in real-space and in frequency-space
- developed by math and signal processing communities in late 80’s
- applications in signal and image processing, data compression (e.g. JPEG2000 image format)

Wavelet basis consists of translations and dilations of the wavelet function:
- is a complete, orthonormal basis
- is a multi-resolution analysis (MRA)

Wavelets are a good compromise between real-space and Fourier-space representations:
- compact in real-space and in frequency-space
- developed by math and signal processing communities in late 80’s
- applications in signal and image processing, data compression (e.g. JPEG2000 image format)
Daubechies wavelets

- scale $z = 4$ wavelet, $g_4$ (with $\frac{N}{16}$ translations)
- scale $z = 3$ wavelet, $g_3$ (with $\frac{N}{8}$ translations)
- scale $z = 2$ wavelet, $g_2$ (with $\frac{N}{4}$ translations)
- scale $z = 1$ wavelet, $g_1$ (with $\frac{N}{2}$ translations)

Daubechies D4 wavelets

- complete, orthonormal basis
- have 2 vanishing moments (orthogonal to constant + linear functions)
- useful for resolving information at different scales

large scale wavelets encode long range (low-frequency) information

small scale wavelets encode short range (high-frequency) information

N-dim vector space
How can we construct wavelets?

- first construct scaling functions (allows recursive construction of functions at different scales)

D4 scaling sequence

\[
\begin{bmatrix}
  h_1 \\
  h_2 \\
  h_3 \\
  h_4 \\
\end{bmatrix} = \begin{bmatrix}
  -0.1294 \\
  0.2241 \\
  0.8365 \\
  0.4830 \\
\end{bmatrix}
\]

scaling function at larger scale defined from a linear combination of scaling functions at previous scale

“Refinement equation”
Daubechies wavelets

How can we construct wavelets?

• first construct scaling functions (allows recursive construction of functions at different scales)

\[
\begin{bmatrix}
    h_1 \\
    h_2 \\
    h_3 \\
    h_4
\end{bmatrix}
= \begin{bmatrix}
    -0.1294 \\
    0.2241 \\
    0.8365 \\
    0.4830
\end{bmatrix}
\]

• wavelets then defined from scaling functions using wavelet sequence

\[
\begin{bmatrix}
    -h_4 \\
    h_3 \\
    -h_2 \\
    h_1
\end{bmatrix}
= \begin{bmatrix}
    -0.4830 \\
    0.8365 \\
    -0.2241 \\
    -0.1294
\end{bmatrix}
\]

N-dim vector space
Daubechies wavelets

**D4** Daubechies wavelets (large scale limit)

orthogonal to **constant** + **linear** functions

**D6** Daubechies wavelets (large scale limit)

orthogonal to **constant** + **linear** + **quadratic** functions

• higher-order wavelets have more vanishing moments (**D2N** Daubechies have **N** vanishing moments)

• higher order may achieve better compression ratios

• many other wavelet families (e.g. Coiflets, Symlets...)
Real-space renormalization and wavelets have many conceptual similarities... … but can one establish a precise connection?

- **classical** multi-scale methods (wavelets) ⇔ **quantum** multi-scale methods (renormalization group and MERA tensor networks)

Free fermion systems:

**Wavelet transform** of fermionic modes precisely corresponds to **Gaussian MERA**

More generally:

MERA can be interpreted as the **generalization of wavelets** from ordinary functions to many-body wavefunctions
Real-space renormalization and wavelets have many conceptual similarities... 
... but can one establish a precise connection?

Applications: 
• better understanding of MERA
• construction of analytic examples of MERA (e.g. for Ising CFT)
• analytic error bounds for MERA?

Applications: 
• design of better wavelets (e.g. for image compression)
Outline: Entanglement renormalization and Wavelets


Introduction

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits

Benchmark calculations from wavelet based MERA

Further application of wavelet – MERA connection
Wavelets for free fermions

Can we expand the ground state of free spinless fermions as wavelets?

$$H_{\text{FF}} = \frac{1}{2} \sum_r \left( \hat{a}_r^\dagger \hat{a}_{r+1} + \text{h.c.} \right)$$

hopping term

first consider plane waves:

$$\hat{a}_r$$

spatial modes

ground state is given by filling in negative energy states (fermi-sea):

$$\left\langle \psi_{\text{GS}} \right| \hat{c}_k^\dagger \hat{c}_k \left| \psi_{\text{GS}} \right\rangle = \begin{cases} 0 & \Lambda_k > 0 \\ 1 & \Lambda_k < 0 \end{cases}$$

Fourier Transform

$$\hat{c}_k = \frac{1}{\sqrt{N}} \sum_r \hat{a}_r e^{-i2\pi kr/N}$$

fourier modes

dispersion relation:

$$\Lambda_k = \cos \left( \frac{2\pi k}{N} \right)$$

$$H_{\text{FF}} = \int_{-\pi}^{\pi} \Lambda_k \hat{c}_k^\dagger \hat{c}_k dk$$

$$\hat{a}_r e^{-i2\pi kr/N}$$

momentum, $k$

Fermi sea
Wavelets for free fermions

Can we expand the ground state of free spinless fermions as wavelets?

\[ \hat{a}_r \]

spatial modes

Wavelet transform

\[ \hat{b}_z = \sum_r g_z \hat{a}_r \]

wavelet modes

Not suitable!

• standard wavelets target \( k = 0 \)
• want wavelets that target \( k = \pm \pi/2 \)

D4 daubechies wavelets

real-space

freq-space

\[ g_0(r) \]

\[ |G_0(k)|^2 \]

\[ g_1(r) \]

\[ |G_1(k)|^2 \]

\[ g_2(r) \]

\[ |G_2(k)|^2 \]
Wavelets for free fermions

**Solution:** take symmetric and antisymmetric combination of two copies of D4 daubechies wavelets:

\[ \hat{b}_z^+ = \sum_r l_z \hat{a}_r \]

symmetric wavelets

\[ \hat{b}_z^- = \sum_r h_z \hat{a}_r \]

antisymmetric wavelets

**Wavelet transform**

\[ \hat{b}_z^0 = \sum_{r, \pi/2} \hat{a}_r \]

position, \( r \)

0 \( \pi/2 \) \( \pi \)

momentum, \( k \)

\[ \hat{b}_z^2 = \sum_{h, \pi} \hat{a}_r \]

position, \( r \)

0 \( \pi/2 \) \( \pi \)

momentum, \( k \)

**Symmetric (low freq):**

- \( l_0(r) \)
- \( |L_0(k)|^2 \)
- \( h_0(r) \)
- \( |H_0(k)|^2 \)

**Antisymmetric (high freq):**

- \( l_1(r) \)
- \( |L_1(k)|^2 \)
- \( h_1(r) \)
- \( |H_1(k)|^2 \)

- \( l_2(r) \)
- \( |L_2(k)|^2 \)
- \( h_2(r) \)
- \( |H_2(k)|^2 \)
Wavelets for free fermions

Ground state is approximated by filling in symmetric (low freq.) wavelet modes:

$$|\psi_{GS}\rangle = \prod_z \hat{b}_z^+ |0\rangle$$

- how accurate is this approximation?
- can this be improved? Later!

\[
\begin{align*}
\hat{b}_z^+ &= \sum_r l_z \hat{a}_r & \text{symmetric wavelets} \\
\hat{b}_z^- &= \sum_r h_z \hat{a}_r & \text{antisymmetric wavelets}
\end{align*}
\]

\[
\begin{align*}
l_0(r) & \quad |L_0(k)|^2 \\
l_1(r) & \quad |L_1(k)|^2 \\
l_2(r) & \quad |L_2(k)|^2
\end{align*}
\]

\[
\begin{align*}
h_0(r) & \quad |H_0(k)|^2 \\
h_1(r) & \quad |H_1(k)|^2 \\
h_2(r) & \quad |H_2(k)|^2
\end{align*}
\]

position, $r$  \quad 0  \quad $\pi/2$  \quad $\pi$

momenium, $k$  \quad 0  \quad $\pi/2$  \quad $\pi$
Outline: Entanglement renormalization and Wavelets


Overview

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits

Benchmark calculations from wavelet based MERA

Further application of wavelet – MERA connection
Circuit representation of wavelets

Wavelet transform described by recursion relation:

Recursion relation can be encoded as a (classical) unitary circuit:

Circuit representation of wavelets

Diagrammatic notation:

\[
\begin{align*}
 u(\theta_2) & \rightarrow \quad \begin{array}{c}
 \text{Diagram 1}
 \end{array} \\
 u(\theta_1) & \rightarrow \quad \begin{array}{c}
 \text{Diagram 2}
 \end{array}
\end{align*}
\]

\[
u(\theta) = \begin{bmatrix}
 \cos(\theta) & \sin(\theta) \\
 -\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

Wavelet transform maps from vector of \textbf{N scalars} to vector of \textbf{N scalars}!

Classical circuit here represents \textbf{direct sum} of unitaries (not \textbf{tensor product}!)

\textit{G.E., Steven. R. White, arXiv: 1605.07312 (May `16).}
Circuit representation of wavelets

Diagrammatic notation:

\[
\begin{align*}
    u(\theta_2) & \rightarrow \\
    u(\theta_1) & \rightarrow
\end{align*}
\]

\[
\begin{bmatrix}
    s_1 \\
    s_2 \\
    s_3 \\
    s_4 \\
    s_5 \\
    s_6 \\
    s_7 \\
    s_8 \\
\end{bmatrix} =
\begin{bmatrix}
    \cos(\theta) & \sin(\theta) \\
    -\sin(\theta) & \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
    q_1 \\
    q_2 \\
    q_3 \\
    q_4 \\
    q_5 \\
    q_6 \\
    q_7 \\
    q_8
\end{bmatrix}
\]

Wavelet transform maps from vector of \textbf{N scalars} to vector of \textbf{N scalars}!

Circuit representation of wavelets

wavelet sequence associated to inverse transforming unit vector (odd sublattice)

scaling sequence associated to inverse transforming unit vector (even sublattice)

Circuit representation of wavelets

Reproduce D4 Daubechies:

\[ \theta_1 = \frac{\pi}{12} \]
\[ \theta_2 = -\frac{\pi}{6} \]

Circuit representation of wavelets

\[
\begin{bmatrix}
 s_1 \\
 s_2 \\
 s_3 \\
 s_4 \\
 s_5 \\
 s_6 \\
 s_7 \\
 s_8 \\
\end{bmatrix}
\xrightarrow{\text{wavelet transform}}
\begin{bmatrix}
 q_1 \\
 q_2 \\
 q_3 \\
 q_4 \\
 q_5 \\
 q_6 \\
 q_7 \\
 q_8 \\
\end{bmatrix}
\]

(can be realised as a classical unitary circuit)

\[
\begin{bmatrix}
 \hat{a}_1 \\
 \hat{a}_2 \\
 \hat{a}_3 \\
 \hat{a}_4 \\
 \hat{a}_5 \\
 \hat{a}_6 \\
 \hat{a}_7 \\
 \hat{a}_8 \\
\end{bmatrix}
\xrightarrow{\text{wavelet transform}}
\begin{bmatrix}
 \hat{b}_1 \\
 \hat{b}_2 \\
 \hat{b}_3 \\
 \hat{b}_4 \\
 \hat{b}_5 \\
 \hat{b}_6 \\
 \hat{b}_7 \\
 \hat{b}_8 \\
\end{bmatrix}
\]

\text{N scalars} \quad \text{N scalars} \quad \text{N fermionic modes} \quad \text{N fermionic modes}

\text{Gaussian MERA!}
MERA for free fermions

- MERA where unitary gates map fermionic modes linearly:

\[
\begin{bmatrix}
\hat{b}_1 \\
\hat{b}_2
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix} \begin{bmatrix}
\hat{a}_1 \\
\hat{a}_2
\end{bmatrix}
\]

- Two sites gates are parameterised by a single angle

Free fermions in 1D:

\[
H_{FF} = \frac{1}{2} \sum_r (\hat{a}_r^\dagger \hat{a}_{r+1} + \text{h.c.})
\]

Can we express the wavelet solution for the ground state as a MERA?
MERA for free fermions

• Take two copies of Gaussian MERA that implement the **D4 Daubechies wavelet** transform
• Combine and then symmetrise
MERA for free fermions

Quantum circuit which (approximately) prepares the ground state of 1D free fermions:

\[ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \]

\[ \theta_1 = \frac{\pi}{12} \]
\[ \theta_2 = -\frac{\pi}{6} \]
MERA for free fermions

\[ L(k) \] has non-zero amplitude for \(|k| > \frac{\pi}{2}\).

\[ \theta_1, \theta_2, \ldots \]

Solution is approximate!
Free fermions at half-filling:

\[
H_{\text{FF}} = \frac{1}{2} \sum_r (\hat{a}_{r+1}^\dagger \hat{a}_r + h.c.) - \mu \sum_r \hat{a}_r^\dagger \hat{a}_r
\]

unitary circuit offers accurate (real-space) approximation to the ground state \(|\psi_{\text{GS}}\rangle\) in terms of:

- ground energy and local observables
- entanglement entropy: \(S_L = \frac{c}{3} \log(L) + \text{const.}\)
- conformal data (scaling dimensions, OPE coefficients, central charge)
- RG flow of the Hamiltonian (flows to gapless fixed point)
\( \pi /4 \) gate creates entangled state in the bulk:

\[
|1\rangle |0\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)
\]

Unitary circuit then `smears out' particles on the boundary.

**single particle wavefunction:**
**MERA for free fermions**

Free fermions at half-filling:

\[ H_{\text{FF}} = \frac{1}{2} \sum_r (\hat{a}_r \dagger \hat{a}_r + \text{h.c.}) - \mu \sum_r \hat{a}_r \dagger \hat{a}_r \]

- **Gate angles:**
  - \( \theta_1 = \pi / 12 \)
  - \( \theta_2 = -\pi / 6 \)

- **Diagram:**
  - **Low-level tensors** generate **short-**ranged entanglement
  - **Higher-level tensors** generate **longer-**ranged entanglement
MERA for free fermions

Is this circuit related to the standard (binary) MERA?
Yes! Just group gates together
**MERA for critical Ising**

Free fermions at half-filling

\[ H_{FF} = \frac{1}{2} \sum_r \left( \hat{a}_r^\dagger \hat{a}_r + h.c. \right) \]

Express as 2N majorana fermions

\[ H_{FF} = \sum_r i \left( \tilde{d}_{2r} \tilde{d}_{2r+1} - \tilde{d}_{2r-1} \tilde{d}_{2r+2} \right) \]

Decouple (via local unitaries) into 2 copies of free majorana fermions

\[ H_{FM} = \sum_r i \left( \tilde{d}_r \tilde{d}_{r+1} - \tilde{d}_{r-1} \tilde{d}_r \right) \]

Jordan-Wigner

\[ H_{Ising} = \sum_r \left( -X_r X_{r+1} + Z_r \right) \]

Quantum critical Ising model

Can one get a representation of the ground state of the quantum **critical Ising Model**?
Free fermions at half-filling

\[ H_{FF} = \frac{1}{2} \sum_r (\hat{a}^+_r \hat{a}_r + h.c.) \]

Express as 2N majorana fermions

\[ H_{FF} = \sum_r i(\bar{d}_r \bar{d}_{r+1} - \bar{d}_{r-1} \bar{d}_{r+2}) \]

Decouple (via local unitaries) into 2 copies of free majorana fermions

\[ H_{FM} = \sum_r i(\bar{d}_r \bar{d}_{r+1} - \bar{d}_{r+1} \bar{d}_r) \]

\[ H_{Ising} = \sum_r (-X_r X_{r+1} + Z_r) \]

Quantum critical Ising model
**MERA for critical Ising**

Expressed in Pauli matrices:

**Isometries:**

\[
w_{r,r+1} = \left( \frac{\sqrt{3} + \sqrt{2}}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3} - \sqrt{2}}{4} \right) Z_r Z_{r+1} + i \left( \frac{1 + \sqrt{2}}{4} \right) X_r Y_{r+1} + i \left( \frac{1 - \sqrt{2}}{4} \right) Y_r X_{r+1}
\]

**Disentanglers:**

\[
u_{r,r+1} = \left( \frac{\sqrt{3} + 2}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3} - 2}{4} \right) Z_r Z_{r+1} + \left( \frac{i}{4} \right) X_r Y_{r+1} + \left( \frac{i}{4} \right) Y_r X_{r+1}
\]

Higher order wavelet solutions can also be expressed as a MERA!

---

**Quantum critical Ising model**

\[
H_{\text{Ising}} = \sum_r (-X_r X_{r+1} + Z_r)
\]

---

**Recap:**

1. Ground state of 1D free fermions (or critical Ising model) can be approximated as wavelets.

2. Wavelet solution precisely corresponds to a MERA.
Overview

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits

Benchmark calculations from wavelet based MERA

Further application of wavelet – MERA connection
**MERA for critical Ising**

Expressed in Pauli matrices:

**Isometries:**

\[ w_{r,r+1} = \left( \frac{\sqrt{3}+\sqrt{2}}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3}-\sqrt{2}}{4} \right) Z_r Z_{r+1} + i \left( \frac{1+\sqrt{2}}{4} \right) X_r Y_{r+1} + i \left( \frac{1-\sqrt{2}}{4} \right) Y_r X_{r+1} \]

**Disentanglers:**

\[ u_{r,r+1} = \left( \frac{\sqrt{3}+2}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3}-2}{4} \right) Z_r Z_{r+1} + \left( \frac{i}{4} \right) X_r Y_{r+1} + \left( \frac{i}{4} \right) Y_r X_{r+1} \]

How accurate is the wavelet-based ground state MERA?

**Quantum critical Ising model**

\[ H_{\text{Ising}} = \sum_r (-X_r X_{r+1} + Z_r) \]

<table>
<thead>
<tr>
<th>ground energy</th>
<th>exact:</th>
<th>MERA:</th>
<th>rel. err.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1.27323...</td>
<td>-1.24211</td>
<td>2.4%</td>
</tr>
</tbody>
</table>
Conformal data from MERA

Expressed in Pauli matrices:

Isometries:
\[ w_{r,r+1} = \left( \frac{\sqrt{3}+\sqrt{2}}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3}-\sqrt{2}}{4} \right) Z_r Z_{r+1} + i\left( \frac{1+\sqrt{2}}{4} \right) X_r Y_{r+1} + i\left( \frac{1-\sqrt{2}}{4} \right) Y_r X_{r+1} \]

Disentanglers:
\[ u_{r,r+1} = \left( \frac{\sqrt{3}+1}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3}-1}{4} \right) Z_r Z_{r+1} + i\left( \frac{1}{4} \right) X_r Y_{r+1} + i\left( \frac{1}{4} \right) Y_r X_{r+1} \]

Conformal data from MERA?
Consider coarse-graining local operators...

- MERA has bounded causal width (3 sites for binary MERA)
- Local operators coarse-grained through the causal cone

Quantum critical Ising model
\[ H_{\text{Ising}} = \sum_r (-X_r X_{r+1} + Z_r) \]
Conformal data from MERA

Local operator is coarse-grained into new local operator

Scaling superoperator, \( S \)

Scaling operators are eigen-operators of \( S \)

\[
S \left( \phi_\alpha \right) = 2^{-\Delta_\alpha} \phi_\alpha
\]

\( \phi_\alpha \) scaling operators

\( \Delta_\alpha \) scaling dimensions
Conformal data from MERA

Lowest order solution, $\chi = 2$ MERA

**Isometries:**

$$w_{r, r+1} = \left( \frac{\sqrt{3}+\sqrt{2}}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3}-\sqrt{2}}{4} \right) Z_r Z_{r+1} + i\left( \frac{1+\sqrt{2}}{4} \right) X_r Y_{r+1} + i\left( \frac{1-\sqrt{2}}{4} \right) Y_r X_{r+1}$$

**Disentanglers:**

$$u_{r, r+1} = \left( \frac{\sqrt{3}+2}{4} \right) I_r I_{r+1} + \left( \frac{\sqrt{3}-2}{4} \right) Z_r Z_{r+1} + \left( \frac{i}{4} \right) X_r Y_{r+1} + \left( \frac{i}{4} \right) Y_r X_{r+1}$$

<table>
<thead>
<tr>
<th>$I$</th>
<th>$\sigma$</th>
<th>$\varepsilon$</th>
<th>Exact</th>
<th>MERA: D4 wavelets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.140</td>
</tr>
<tr>
<td>0.125</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.136</td>
</tr>
<tr>
<td>1.125</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1.150</td>
</tr>
<tr>
<td>1.125</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2.113</td>
</tr>
<tr>
<td>2.125</td>
<td>2.125</td>
<td>2.125</td>
<td>2.125</td>
<td>2.113</td>
</tr>
<tr>
<td>2.125</td>
<td>2.125</td>
<td>2.125</td>
<td>2.125</td>
<td>2.131</td>
</tr>
</tbody>
</table>

- Scaling dimensions of primary fields and some descendants are reproduced
- Integer scaling dimensions reproduced exactly
Conformal data from MERA

Quantum critical Ising model
\[ H_{\text{Ising}} = \sum_r (-X_r X_{r+1} + Z_r) \]

Expressed in Pauli matrices:

**Isometries:**
\[ w_{r,r+1} = \left( \frac{\sqrt{3} + \sqrt{2}}{4} \right) I_{r,r+1} + \left( \frac{\sqrt{3} - \sqrt{2}}{4} \right) Z_{r} Z_{r+1} + i \left( \frac{1 + \sqrt{2}}{4} \right) X_r Y_{r+1} + i \left( \frac{1 - \sqrt{2}}{4} \right) Y_r X_{r+1} \]

**Disentanglers:**
\[ u_{r,r+1} = \left( \frac{\sqrt{3} + 2}{4} \right) I_{r,r+1} + \left( \frac{\sqrt{3} - 2}{4} \right) Z_{r} Z_{r+1} + \left( \frac{i}{4} \right) X_r Y_{r+1} + \left( \frac{i}{4} \right) Y_r X_{r+1} \]

**Conformal data from MERA?**
Consider coarse-graining local operators...

What about **non-local** scaling operators?

- Specifically those that come with a string of Z’s (correspond to fermionic operators)
Conformal data from MERA

Local operator (with string) is coarse-grained into new local operator (with string)

Scaling superoperator $S$, (non-local operators)

$$S(\hat{\phi}_\alpha) = 2^{-\Delta}\hat{\phi}_\alpha$$

$(\text{non-local})$ Scaling superoperator $\hat{\phi}_\alpha$ (non-local) scaling operators

Scaling dimensions $\Delta\alpha$
Conformal data from MERA

Lowest order solution, $\chi = 2$ MERA

Isometries: $\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad...
Conformal data from MERA

How to extract **OPE coefficients** from MERA?
Consider fusion of two scaling operators...

\[
\eta_{\alpha\beta} = \sum_{\gamma} C_{\alpha\beta\gamma} \phi_{\gamma}
\]

OPE coefficients
Conformal data from MERA

How to extract OPE coefficients from MERA? Consider fusion of two scaling operators...

| $|0\rangle$ | $|0\rangle$ | $|0\rangle$ | $|0\rangle$ | $|0\rangle$ | $|0\rangle$ |
|---|---|---|---|---|---|
| $w$ | $w$ | $w$ | $w$ | $w$ | $w$ |
| | $u$ | $u$ | $u$ | $u$ | $u$ |
| | | $\phi_\alpha$ | $\phi_\beta$ | | |

$\eta_{\alpha\beta}$

**Exact**

<table>
<thead>
<tr>
<th>$C_{\varepsilon,\sigma,\sigma}$</th>
<th>$C_{\varepsilon,\mu,\mu}$</th>
<th>$C_{\psi,\mu,\sigma}$</th>
<th>$C_{\bar{\psi},\mu,\sigma}$</th>
<th>$C_{\varepsilon,\psi,\psi}$</th>
<th>$C_{\varepsilon,\bar{\psi},\psi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-0.5</td>
<td>0.5 - 0.5i</td>
<td>0.5 + 0.5i</td>
<td>1i</td>
<td>-1i</td>
</tr>
</tbody>
</table>

**MERA D4 wavelets**

<table>
<thead>
<tr>
<th>$C_{\varepsilon,\sigma,\sigma}$</th>
<th>$C_{\varepsilon,\mu,\mu}$</th>
<th>$C_{\psi,\mu,\sigma}$</th>
<th>$C_{\bar{\psi},\mu,\sigma}$</th>
<th>$C_{\varepsilon,\psi,\psi}$</th>
<th>$C_{\varepsilon,\bar{\psi},\psi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.458</td>
<td>-0.420</td>
<td>0.571 - 0.571i</td>
<td>0.571 + 0.571i</td>
<td>1.23i</td>
<td>-1.23i</td>
</tr>
</tbody>
</table>
Conformal data from MERA

**Central charge from MERA?**
Many ways to do this (based on scaling of entanglement entropy)...

1. Compute entanglement entropy of different blocks length \( L \) and use formula:
   \[
   S_L = \frac{c}{3} \log(L) + \text{const.}
   \]

2. Compute entanglement contribution (per scale) to the density matrix for half-infinite system

<table>
<thead>
<tr>
<th>exact</th>
<th>MERA</th>
</tr>
</thead>
<tbody>
<tr>
<td>c = 0.5</td>
<td>0.495</td>
</tr>
</tbody>
</table>

![Central charge from MERA diagram]
**MERA for critical Ising**

Expressed in Pauli matrices:

**Isometries:**

\[
W_{r,r+1} = \left( \frac{\sqrt{3}+i\sqrt{2}}{4} \right) I_{r}I_{r+1} + \left( \frac{\sqrt{3}-i\sqrt{2}}{4} \right) Z_{r}Z_{r+1} + i \left( \frac{1+i\sqrt{2}}{4} \right) X_{r}Y_{r+1} + i \left( \frac{1-i\sqrt{2}}{4} \right) Y_{r}X_{r+1}
\]

**Disentanglers:**

\[
U_{r,r+1} = \left( \frac{\sqrt{3}+2}{4} \right) I_{r}I_{r+1} + \left( \frac{\sqrt{3}-2}{4} \right) Z_{r}Z_{r+1} + \left( \frac{i}{4} \right) X_{r}Y_{r+1} + \left( \frac{i}{4} \right) Y_{r}X_{r+1}
\]

Quantum critical Ising model

\[
H_{\text{Ising}} = \sum_{r} (-X_{r}X_{r+1} + Z_{r})
\]

Wavelet based MERA does a remarkably good job of encoding the Ising CFT! (considering its simplicity...)

---

**Diagram:**

[Diagram showing the structure of a MERA tree with wavelet (w) and disentangler (u) nodes.]
Higher order solutions

Is there a systematic way to generate better approximations to the ground state?

Yes! Use higher order wavelets (which correspond to circuits with a greater depth of unitary gates in each layer)

How does MERA with many layers of unitaries relate to standard (binary) MERA?

Four free parameters in the ansatz

\[ [\theta_1, \theta_2, \theta_3, \theta_4] \]

Wavelets have larger support (more compact in momentum space)
Higher order solutions

MERA with two layers of disentanglers:

Binary MERA of larger bond dimension
Quantum critical Ising model

\[ H_{\text{Ising}} = \sum_r (-X_r X_{r+1} + Z_r) \]

MERAs for critical Ising

- Higher order wavelets = larger bond dimension MERA
- Higher order wavelets offer systematic improvement in accuracy

<table>
<thead>
<tr>
<th>MERA:</th>
<th>exact:</th>
<th>rel. err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 parameter</td>
<td>-1.27323...</td>
<td>2.4%</td>
</tr>
<tr>
<td>(\chi = 2)</td>
<td>-1.24211</td>
<td>0.4%</td>
</tr>
<tr>
<td>3 parameter</td>
<td>-1.26773</td>
<td>0.02%</td>
</tr>
<tr>
<td>(\chi = 8)</td>
<td>-1.27296</td>
<td></td>
</tr>
<tr>
<td>5 parameter</td>
<td>-1.27296</td>
<td></td>
</tr>
<tr>
<td>(\chi = 16)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How accurately can a MERA of finite bond dimension \(\chi\) approximate the ground state of a CFT? Analytic bounds?
Real-space renormalization and wavelets have many conceptual similarities...  
... but can one establish a precise connection?

Applications:  
• better understanding of MERA  
• construction of analytic examples of MERA (e.g. for Ising CFT)  
• analytic error bounds for MERA?  
• design of better wavelets (e.g. for image compression)
Outline: Entanglement renormalization and Wavelets


Overview

Wavelet solution to free fermion model

Representation of wavelets as unitary circuits and MERA

Benchmark calculations from wavelet based MERA

Further application of wavelet–MERA connection
Wavelets for image compression

transform to wavelet basis

truncation: keep only largest 2% of coefficients

inverse transform

peak signal to noise: PSNR: 37.0 dB

- This is the key part of JPEG2000 format, and many other standards for image, audio and video compression
Wavelets for image compression

Can we do better?


Desirable properties of wavelets:
- Orthogonality? Yes
- Symmetric? No
- Compression ratio? Okay

Daubechies

Coiflets

CDF wavelets

near orthogonal

bad

JPEG2000
**Application: wavelet design**

**Things learned in the context of tensor networks / MERA:**

- how to construct circuits with different forms and scaling factors
- incorporation of spatial and global internal symmetries
- optimization of networks!

---


(i) Binary MERA

(ii) Ternary MERA:

(iii) Modified Binary MERA:
Design a family of symmetric / antisymmetric wavelets based upon ternary unitary circuits:
Wavelets for image compression

JPEG2000 wavelets

image
transform to wavelet basis

truncate (keep only largest 2% of coefficients)

inverse transform
compressed image

PSNR: 37.0 dB

new scale-3 wavelets

PSNR: 37.4 dB
Wavelets for image compression

Can we do better?


Desirable properties of wavelets

<table>
<thead>
<tr>
<th>Wavelet Type</th>
<th>Orthogonality</th>
<th>Symmetry</th>
<th>Compression Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daubechies</td>
<td>yes</td>
<td>no</td>
<td>okay</td>
</tr>
<tr>
<td>Coiflets</td>
<td>yes</td>
<td>near symmetric</td>
<td>good</td>
</tr>
<tr>
<td>CDF wavelets</td>
<td>near orthogonal</td>
<td>yes</td>
<td>good</td>
</tr>
<tr>
<td>JPEG2000</td>
<td>yes</td>
<td>yes</td>
<td>bad</td>
</tr>
<tr>
<td>New scale-3</td>
<td>yes</td>
<td>yes</td>
<td>good</td>
</tr>
</tbody>
</table>
Summary

Real-space renormalization and wavelets have many conceptual similarities...
... but can one establish a precise connection?

Applications:  
- better understanding of MERA
- construction of analytic examples of MERA (e.g. for Ising CFT)
- analytic error bounds for MERA?

Applications:  
- design of better wavelets (e.g. for image compression)