Universal terms in the Entanglement Entropy of Scale Invariant 2+1 Dimensional Quantum Field Theories

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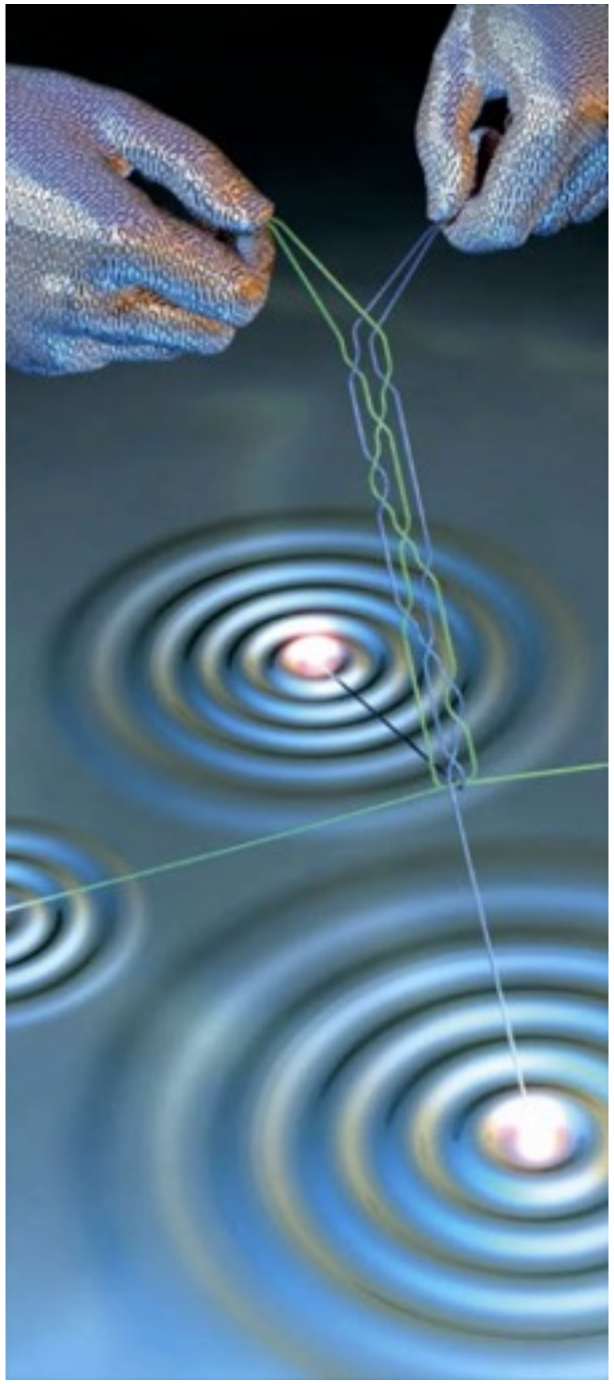
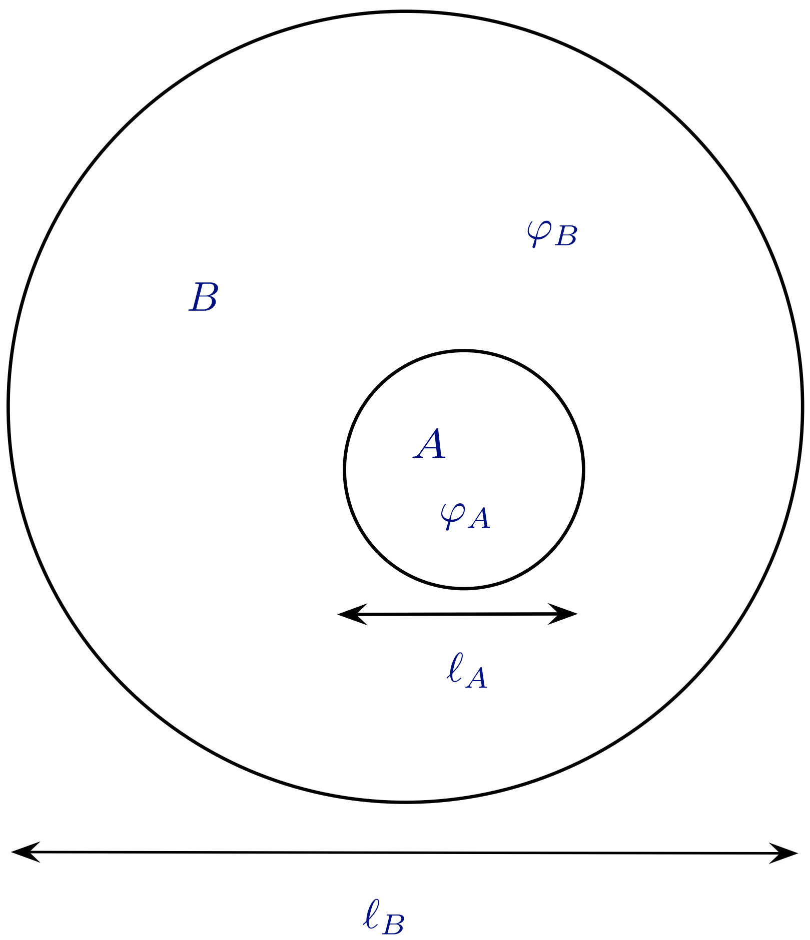
Entanglement



Density Matrix and Entanglement Entropy



- Pure state in A ∪ B: $\Psi[\varphi_A, \varphi_B]$
- Density Matrix

$$\langle \varphi_A, \varphi_B | \rho_{A\cup B} | \varphi'_A, \varphi'_B \rangle = \Psi[\varphi_A, \varphi_B] \Psi^*[\varphi'_A, \varphi'_B]$$

Observing only A: mixed state with a reduced density matrix ρ_A

$$\langle \varphi_A | \rho_{A\cup B} | \varphi'_A \rangle = \mathrm{tr}_B \rho_{A\cup B}$$

von Neumann Entanglement Entropy: $$S_{vN} = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

Scaling of Entanglement

- We will consider a partition into two simply-connected regions *A* and *B*, such that $l_B \gg l_A \gg \xi \gg a$; ξ is the correlation length (if finite) and *a* is the UV cutoff

- In a massive phase in d space dimensions ξ is finite and the entanglement entropy obeys the **Area Law**

$$S_{vN} = \alpha \left(\frac{\ell_A}{a}\right)^{d-1} + \dots$$

- The prefactor of the Area Law is **not universal** as it depends on the short-distance cutoff a
- In a 1+1 dimensional CFT this law is replaced by the universal form

$$S_{vN} = \frac{c}{3} \ln\left(\frac{\ell}{a}\right) + \dots$$

- In a 2d topological phase it has the scaling $S_{vN}=\alpha\ (l/a)- \gamma+\dots$ (Kitaev and Preskill, Levin and Wen); $\gamma = \ln \mathcal{D}$ ($\mathcal{D}$: effective quantum dimension).

Scaling of Entanglement in General Dimension

- For general *odd* space dimension d the entanglement entropy has an expansion of the form $S_{vN}= \alpha\ (l/a_0)^{d-1}+\alpha'\ (l/a_0)^{d-3}+\ldots+s \ln\ (l/a_0)+\ldots$

- This expression is verified both in free field theories (Casini and Huerta, 2008) and in the AdS/CFT correspondence using the Ryu-Takayanagi ansatz (2006)

- The first term is the Area Law. The coefficients α, α', etc. are non-universal

- The coefficient **s** of the logarithmic term is universal

- In 3+1 dimensional CFTs, the coefficient **s** is expressed in terms of integrals of the entangling surface and of the **central charges a and c**

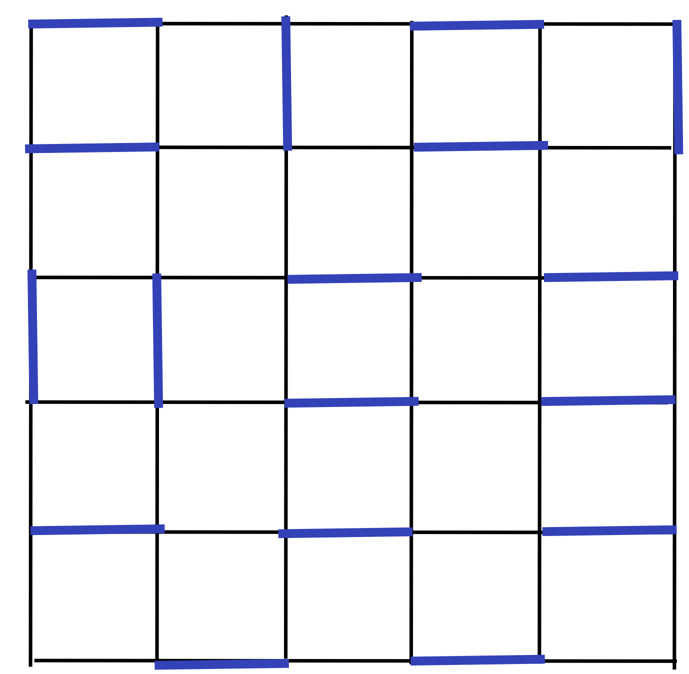
Scaling of Entanglement in 2+1 Dimensions

- Much less is known about the universal terms in the scaling of entanglement in odd space-time dimensions at quantum criticality
- For simply-connected regions with the shape of a **disk** in d=2 dimensions the entanglement entropy has the form of the *F* theorem: $S_{vN}=\alpha\ (l/a)-F+\ldots$, where the constant term *F* is finite and universal (Klebanov, Pufu, Safdi (2011); Casini, Huerta and Myers (2011))
- Here we will discuss the current evidence for universal finite terms in several d=2 quantum critical systems of interest
- We will focus on the case in which the d=2 surface is a 2-torus and the observed regions are cylindrical sections of the torus.

Quantum Dimer Models

- Quantum dimer models are simple models of 2d frustrated (or large N) antiferromagnets (Kivelson and Rokhsar, 1988)
- The space of states $|C\rangle$ are the dimer coverings of the lattice, each dimer being regarded as a spin singlet (**valence bond**) of a pair of nearby spins
- The ground states have the RVB form

$$|\Psi_0\rangle = \sum_{\{\mathcal{C}\}} e^{-\beta E[\mathcal{C}]} |\mathcal{C}\rangle$$



- For small enough β, on bipartite lattices (square, honeycomb, etc) these states represent quantum critical points
- On non-bipartite lattices (also for small enough β) they represent $\mathbb{Z}_2$ topological states (deconfined $\mathbb{Z}_2$ gauge theories or Kitaev's Toric Code states)
- For large β they represent ordered (VBS) states

The Quantum Lifshitz Model

- The Quantum Lifshitz Model is the effective continuum theory of the critical Quantum Dimer Model (Ardonne, Fendley, Fradkin, 2004)
- Free compactified boson field theory ($\phi \sim \phi + 2\pi R$) representing the coarse-grained configurations of dimers on a bipartite 2D lattice
- It has dynamic critical exponent z=2
- It describes a quantum critical point between an uniform phase and a spatially modulated phase
- In the uniform phase it becomes a relativistic compactified scalar field, a NG mode.

- Hamiltonian: $$\mathcal{H} = \frac{\Pi^2}{2} + \frac{\kappa^2}{2}\left(\nabla^2\phi\right)^2, \qquad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$$

- Ground State Wave Functional: $$\Psi_0[\phi(\mathbf{x})] = \text{const} \times e^{-\frac{\kappa}{2}\int d^2x \, (\nabla\phi(\mathbf{x}))^2}$$

- The ground state wave functional is conformally invariant

Entanglement Scaling in the QLM

- On **simply connected region** *A* with a smooth boundary the entanglement entropy obeys the scaling $S_{vN}=\alpha\,(l/a)+\gamma+\ldots$, where γ is universal (Fradkin and Moore, 2006)
- If the boundary has corners (cusps), or ends at the edge of the system, S_{vN} has **universal logarithmic terms** (Zaletel, Bardarson and Moore, 2011)
- The Rényi entanglement entropy S_n is computed in terms of ratios of 2D classical partition functions.
- Partition function on an a “book” with 2n sheets.

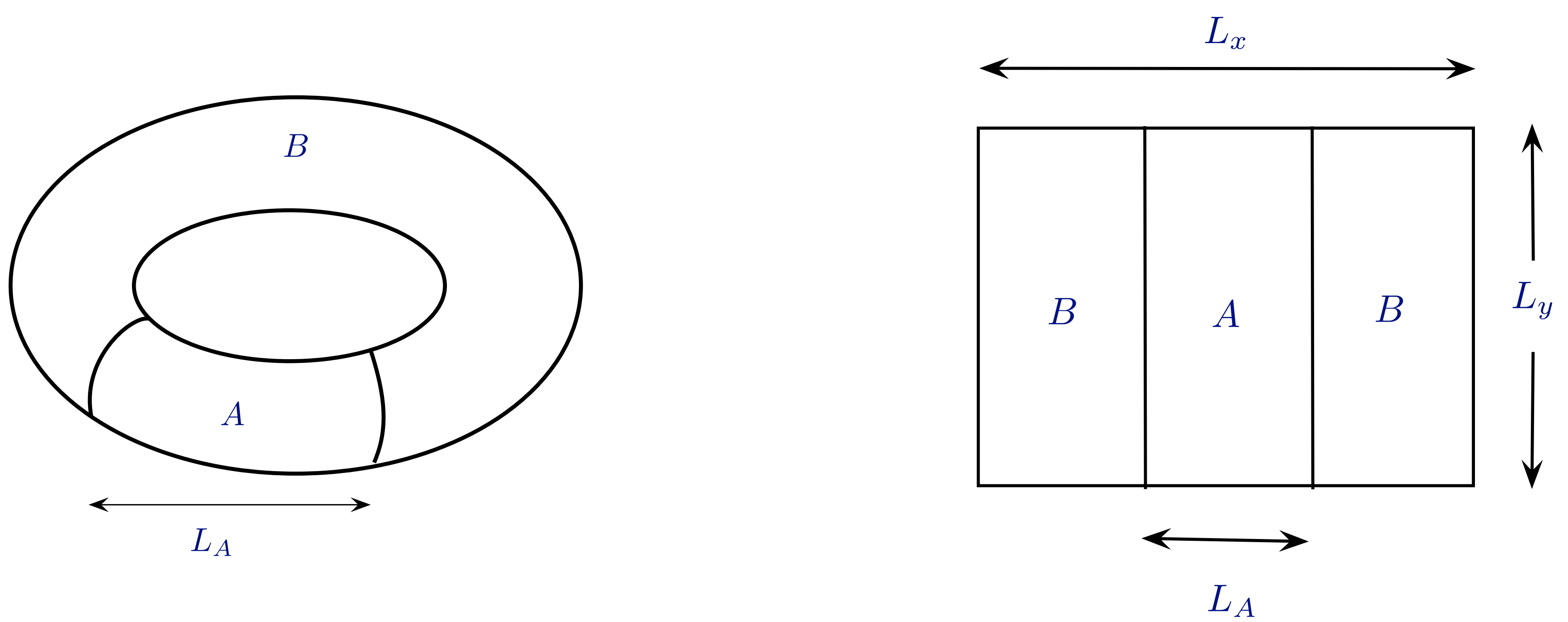
$$\mathrm{tr}\rho_A^n = \frac{Z_n}{Z^n}$$

- Z_n: partition function of n copies glued at the boundary or the observed region *A.*
- Z: partition function (norm of the wave function) for one copy.

S_{vN} is expressed in terms of 2D classical free energies (two with Dirichlet BCs at ∂A and ∂B, and one free) and of the contribution of the winding modes, *W(n)*

$$S_{vN}[A] = F_D[A] + F_D[B] - F[A \cup B] - \lim_{n\to 1} \partial_n W(n)$$

Entanglement on a Torus



- 2 torus: entanglement entropy of a cylindrical region has a **universal finite term** (Hsu at al; Hsu and Fradkin; Oshikawa; Stéphan, et al).
- Finite term due to the winding modes of the boson: **long range entanglement**.

Thin "slice" limit: $S_{vN} = 2\alpha \frac{L_y}{a} - \frac{\pi}{24} \frac{L_y}{L_A}$

Thin torus limit: $S_{vN} = 2\alpha \frac{L_y}{a} + 2 \left(\ln \left(\sqrt{8\pi\kappa} R \right) - \frac{1}{2} \right)$

EE in other geometries: Disk and Sphere

- If A is a disk of radius r in the plane with Dirichlet BCs (and for a spherical cap of angle 2θ)

$$\gamma = \ln\left(\sqrt{8\pi\kappa}R\right) - \tfrac{1}{2}$$

- This result should be compared with the -ln (rg^2) dependence found for a compactified relativistic scalar field (a Goldstone boson with dimensionful coupling constant g) by Agon et al.
- Annular region of radii $r_1<r_2$ in the infinite plane (and for two spherical caps after stereographic projection)

$$\text{Thick annulus}: \ \gamma = 2\left[\ln\left(\sqrt{8\pi\kappa}R\right) - \frac{1}{2}\right]$$

$$\text{Thin annulus}: \ \gamma = -\frac{\pi}{24}\left(\frac{2\pi r_1}{r_2 - r_1}\right) + \ln\left(\sqrt{8\pi\kappa}R\right) - \frac{1}{2}$$

Mutual Information

- Mutual information of region A, a disk of radius r_1, with B, the complement of a disk of radius $r_2>r_1$, in the infinite plane

$$I(A,B) = S(A) + S(B) - S(C) = 2\left(\frac{r_1}{r_2}\right)^{1/4\pi\kappa R^2} + \left(\frac{r_1}{r_2}\right)^2$$

- Mutual information of two disks of radii R_A and R_B separated by a large distance r (here $\zeta = R_A R_B/r^2$)

$$I(A,B) = S(A) + S(B) - S(A\cup B) = 2\zeta^{2\Delta_1} + \zeta^{2\Delta_2}$$

$$\Delta_1 = \frac{1}{8\pi\kappa R^2} \text{ and } \Delta_2 = 1$$

scaling dimensions of the vertex operator $\exp(i\phi)$ and $\partial_x\phi$ of the QLM

Torus with general aspect ratio

2 torus with aspect ratio **L_y/L_x** and a cylindrical region *A* with aspect ratio ***$u= L_A/L_x$***, the entanglement entropy is **$S_{vN}=2\alpha\ (L_y/a)+\beta\ J(u)$** (Stéphan et al)

$$J(u) = \log\left(\frac{\lambda}{2}\frac{\eta(\tau)^2}{\theta_3(\lambda\tau)\theta_3(\tau/\lambda)}\frac{\theta_3(\lambda u\tau)\theta_3(\lambda(1-u)\tau)}{\eta(2u\tau)\eta(2(1-u)\tau)}\right)$$

$\theta_3(z)$: Jacobi theta-function

$\eta(z)$: Dedekind eta-function

For the quantum dimer model at the RK point, $\lambda=2$

$$\tau = i\frac{L_y}{L_x}$$

Scaling of Entanglement in the 2+1 dimensional Ising Model in a Transverse Field

- Inglis and Melko (2013) studied numerically the second Rényi entropy S_2 at the quantum critical point of the 2D TFIM on a 2 torus
- This system is in the universality class of the ϕ^4 Wilson-Fisher fixed point in 2+1 dimensions
- Two different scaling functions:
- $S_2 = 2\alpha\, L_y + \beta \log(\sin(\pi L_x/L))$
- $S_2 = 2\alpha\, L_y + \beta\, J(u) +$ const

The QLM scaling works surprisingly well even though is not relativistic!

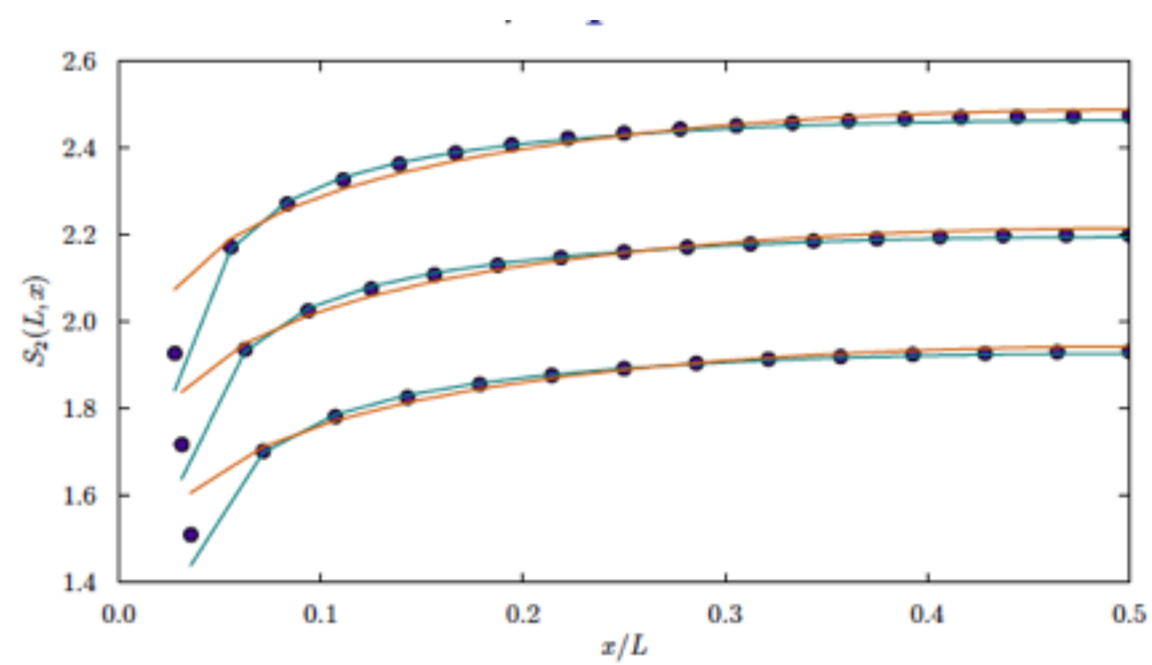
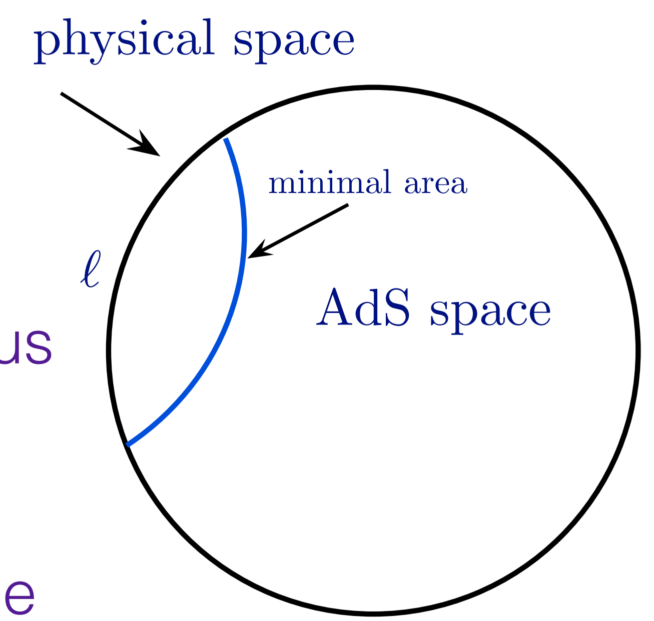


Figure 7. The entanglement entropy of the $L = 28, 32, 36$ systems using two cylinders (figure 1), along with fits to (orange) equation (27) and (teal) equation (28). Notice the lack of any even–odd effect in the entanglement as a function of cut length.

Holographic Entanglement on the 2 torus

Chen, Cho, Faulkner and Fradkin (2015) used the **Ryu-Takayanagi ansatz** to obtain the finite term in the **von Neumann entanglement entropy for a cylindrical section of a 2 torus**



$$S_{vN} = \frac{\mathcal{A}}{4G_N}$$

$\mathcal{A}$: area of the minimal AdS surface homologous to the observed region where the QFT lives on the boundary of AdS

Toroidal geometries: minimal surface falls on the "AdS_4 soliton" geometry (Horowitz and Myers)

- The EE is sensitive to which cycle of the torus contracts since the cut is along y
- In the thin torus limit, $L_y \ll L_x$, the EE saturates
- This is natural for fermions with anti-periodic boundary conditions
- In the opposite, **thin slice**, limit we find universal scaling
- The von Neumann EE has the scaling form $S_{vN}=\alpha\, L_y+ \beta\, j(u)$, where j(u) has two different forms depending on whether the aspect ratio of the torus is $L_y/L_x>1$ or $L_y/L_x<1$

Parametric form of j(u) for $L_y/L_x>1$

$$u(\chi) = \frac{3\chi^{1/3}(1-\chi)^{1/2}}{2\pi}\int_0^1 \frac{d\zeta\zeta^2}{(1-\chi\zeta^3)}\frac{1}{\sqrt{P(\chi,\zeta)}} \qquad j(\chi) = \chi^{-1/3}\left(\int_0^1 \frac{d\zeta}{\zeta^2}\left(\frac{1}{\sqrt{P(\chi,\zeta)}}-1\right)-1\right)$$

$$P(\chi,\zeta) = 1-\chi\zeta^3-(1-\chi)\zeta^4$$

Parametric form for j(u) for $L_y/L_x<1$

$$j(\chi) = \chi^{-1/3}\left(\int_0^1 \frac{d\zeta}{\zeta^2}\left(\frac{\sqrt{1-\chi\zeta^3}}{\sqrt{P(\chi,\zeta)}}-1\right)-1\right)$$

$$\frac{L_x}{L_y}u = \frac{3}{2\pi}\chi^{1/3}(1-\chi)^{1/2}\int_0^1 \frac{d\zeta\zeta^2}{\sqrt{1-\chi\zeta^3}}\frac{1}{\sqrt{P(\chi,\zeta)}}$$

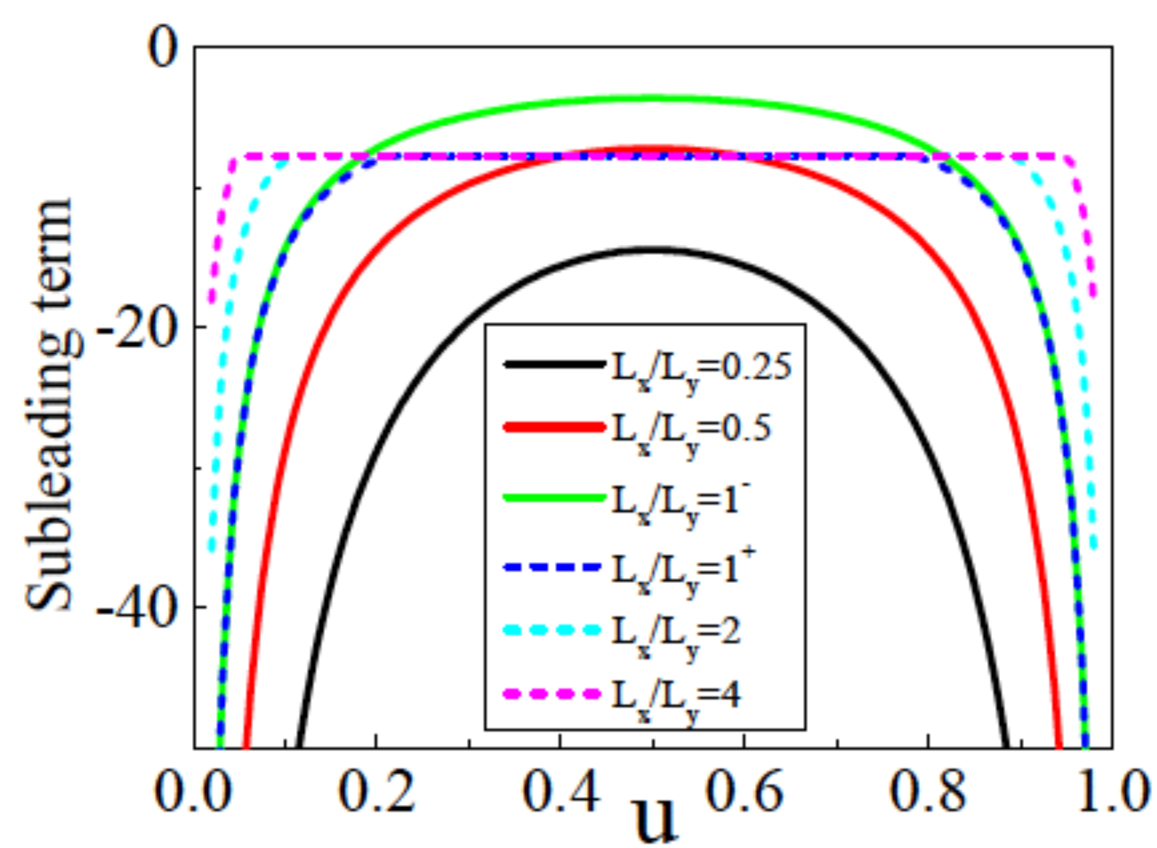
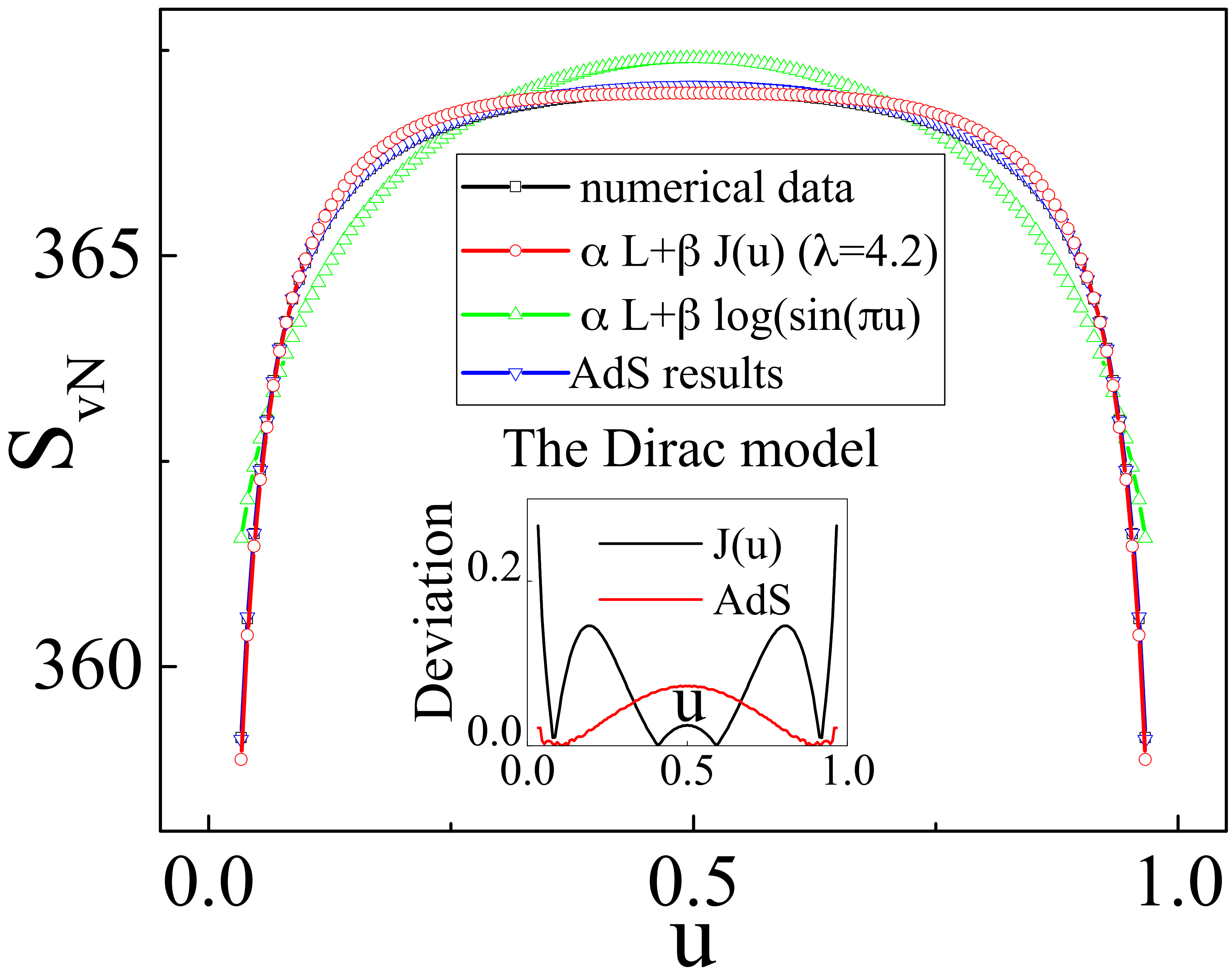


FIG. 2. The subleading term for the minimal surface for various values of L_x/L_y. The solid curves are for $j(u)$ when $L_x \leq L_y$ and the dashed curves are for $\widetilde{j}(u)$ when $L_x > L_y$.

Tests of scaling in 2d free field fermionic models

- We computed the entanglement entropy for cylindrical regions of a 2 torus for free massless Dirac fermions in 2+1 dimensions and also for free massless “Lifshitz” (QBT) fermions, in both cases with anti-periodic boundary conditions
- Dirac fermions in 2+1 dimensions are a stable phase and hence an UV fixed point
- “Lifshitz” fermions in 2+1 dimensions have z=2 and are are at the marginal dimension (asymptotically free)
- We used a lattice version of these models and used standard results for the reduced density matrix and the entanglement entropy (Peschel, 2001)
- In the Dirac case we accounted for fermionic doublers

Scaling of entanglement for free massless Dirac fermions in 2+1 dimensions



Conclusions

- In 2+1 dimensions we expect to find universal finite corrections to the entanglement entropy
- These finite terms are determined by the operator content (compactification radius, scaling dimensions, etc) and have topological origin
- We presented results for von Neumann EE of the QLM on different geometries and for the mutual information
- On cylindrical regions the universal finite terms are given in terms of scaling functions of the aspect ratios of the cylinders and of the 2 torus
- We presented expressions for the entanglement entropy derived from holography on toroidal geometries which fit remarkably well even free massless Dirac fermions (on anti-periodic boundary conditions)
- The finite term derived by Stéphan et al for the QLM fits remarkably well even for Dirac fermions which is a relativistic theory (although the holographic results is better)
- How well does the holographic result fit the 2D Ising model in a transverse field?

Scaling of entanglement for free massless "Lifshitz" fermions in 2+1 dimensions

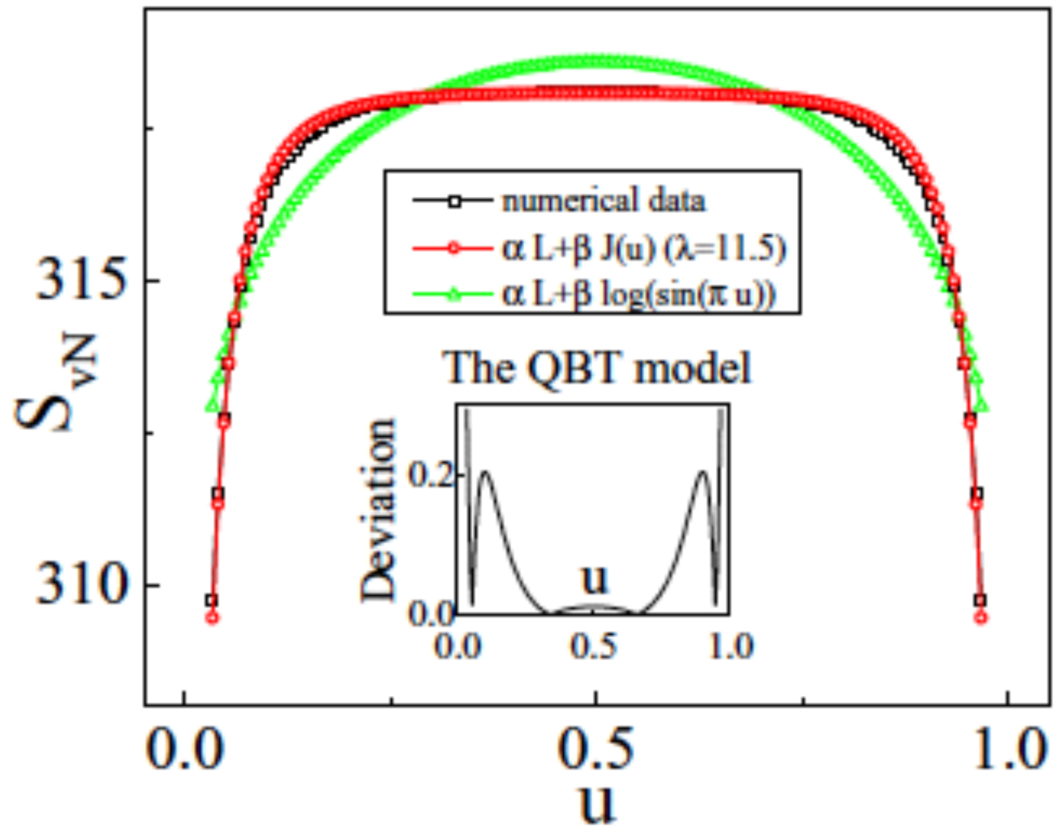


Figure 8. S_{vN} for the QBT model as a function of u. The bipartition geometry is the same as the Dirac model. The inset is the absolute deviation for the fitting function $S_{\mathrm{vN}} = \alpha L + \beta J(u)$ with numerical data. The deviation is less than 1% for the whole region of u.