Universal terms in the Entanglement Entropy of Scale Invariant 2+1 Dimensional Quantum Field Theories

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Entanglement
Density Matrix and Entanglement Entropy

- Pure state in $A \cup B$: $\Psi[\varphi_A, \varphi_B]$ 
- Density Matrix

$$\langle \varphi_A, \varphi_B | \rho_{A \cup B} | \varphi'_A, \varphi'_B \rangle = \Psi[\varphi_A, \varphi_B] \Psi^*[\varphi'_A, \varphi'_B]$$

Observing only $A$: mixed state with a reduced density matrix $\rho_A$

$$\langle \varphi_A | \rho_{A \cup B} | \varphi'_A \rangle = \text{tr}_B \rho_{A \cup B}$$

von Neumann Entanglement Entropy:

$$S_{vN} = -\text{tr} (\rho_A \log \rho_A)$$
Scaling of Entanglement

• We will consider a partition into two simply-connected regions $A$ and $B$, such that $l_B \gg l_A \gg \xi \gg a$; $\xi$ is the correlation length (if finite) and $a$ is the UV cutoff

• In a massive phase in $d$ space dimensions $\xi$ is finite and the entanglement entropy obeys the Area Law

$$S_{vN} = \alpha \left( \frac{l_A}{a} \right)^{d-1} + \ldots$$

• The prefactor of the Area Law is not universal as it depends on the short-distance cutoff $a$

• In a 1+1 dimensional CFT this law is replaced by the universal form

$$S_{vN} = \frac{c}{3} \ln \left( \frac{\ell}{a} \right) + \ldots$$

• In a 2d topological phase it has the scaling $S_{vN} = \alpha \left( \frac{l}{a} \right)^{-\gamma} + \ldots$ (Kitaev and Preskill, Levin and Wen); $\gamma = \ln D$ ($D$: effective quantum dimension).
Scaling of Entanglement in General Dimension

• For general odd space dimension $d$ the entanglement entropy has an expansion of the form $S_{vN} = \alpha (l/a_0)^{d-1} + \alpha' (l/a_0)^{d-3} + \ldots + s \ln (l/a_0) + \ldots$

• This expression is verified both in free field theories (Casini and Huerta, 2008) and in the AdS/CFT correspondence using the Ryu-Takayanagi ansatz (2006)

• The first term is the Area Law. The coefficients $\alpha$, $\alpha'$, etc. are non-universal

• The coefficient $s$ of the logarithmic term is universal

• In 3+1 dimensional CFTs, the coefficient $s$ is expressed in terms of integrals of the entangling surface and of the central charges $a$ and $c$
 Scaling of Entanglement in 2+1 Dimensions

• Much less is known about the universal terms in the scaling of entanglement in odd space-time dimensions at quantum criticality

• For simply-connected regions with the shape of a disk in d=2 dimensions the entanglement entropy has the form of the $F$ theorem: $S_{VN} = \alpha \left( l/a \right) - F + \ldots$, where the constant term $F$ is finite and universal (Klebanov, Pufu, Safdi (2011); Casini, Huerta and Myers (2011))

• Here we will discuss the current evidence for universal finite terms in several d=2 quantum critical systems of interest

• We will focus on the case in which the d=2 surface is a 2-torus and the observed regions are cylindrical sections of the torus.
Quantum Dimer Models

• Quantum dimer models are simple models of 2d frustrated (or large N) antiferromagnets (Kivelson and Rokhsar, 1988)

• The space of states $|C\rangle$ are the dimer coverings of the lattice, each dimer being regarded as a spin singlet (valence bond) of a pair of nearby spins

• The ground states have the RVB form

$$|\Psi_0\rangle = \sum_{\{C\}} e^{-\beta E[C]} |C\rangle$$

• For small enough $\beta$, on bipartite lattices (square, honeycomb, etc) these states represent quantum critical points

• On non-bipartite lattices (also for small enough $\beta$) they represent $\mathbb{Z}_2$ topological states (deconfined $\mathbb{Z}_2$ gauge theories or Kitaev’s Toric Code states)

• For large $\beta$ they represent ordered (VBS) states
The Quantum Lifshitz Model

• The Quantum Lifshitz Model is the effective continuum theory of the critical Quantum Dimer Model (Ardonne, Fendley, Fradkin, 2004)

• Free compactified boson field theory \((\phi \sim \phi + 2\pi R)\) representing the coarse-grained configurations of dimers on a bipartite 2D lattice

• It has dynamic critical exponent \(z=2\)

• It describes a quantum critical point between an uniform phase and a spatially modulated phase

• In the uniform phase it becomes a relativistic compactified scalar field, a NG mode.

• Hamiltonian: 
  \[ H = \frac{\Pi^2}{2} + \frac{\kappa^2}{2} (\nabla^2 \phi)^2, \quad [\phi(x), \Pi(y)] = i\delta(x - y) \]

• Ground State Wave Functional: 
  \[ \Psi_0[\phi(x)] = \text{const} \times e^{-\frac{\kappa}{2} \int d^2 x \,(\nabla \phi(x))^2} \]

• The ground state wave functional is conformally invariant
Entanglement Scaling in the QLM

- On simply connected region $A$ with a smooth boundary the entanglement entropy obeys the scaling $S_{vN} \propto (l/a)^\gamma + \ldots$, where $\gamma$ is universal (Fradkin and Moore, 2006)
- If the boundary has corners (cusps), or ends at the edge of the system, $S_{vN}$ has universal logarithmic terms (Zaletel, Bardarson and Moore, 2011)
- The Rényi entanglement entropy $S_n$ is computed in terms of ratios of 2D classical partition functions.
- Partition function on an a “book” with 2n sheets.

$$\text{tr} \rho^n_A = \frac{Z_n}{Z^n}$$

- $Z_n$: partition function of $n$ copies glued at the boundary or the observed region $A$.
- $Z$: partition function (norm of the wave function) for one copy.

$S_{vN}$ is expressed in terms of 2D classical free energies (two with Dirichlet BCs at $\partial A$ and $\partial B$, and one free) and of the contribution of the winding modes, $W(n)$

$$S_{vN}[A] = F_D[A] + F_D[B] - F[A \cup B] - \lim_{n \to 1} \partial_n W(n)$$
Entanglement on a Torus

- 2 torus: entanglement entropy of a cylindrical region has a universal finite term (Hsu at al; Hsu and Fradkin; Oshikawa; Stéphan, et al).
- Finite term due to the winding modes of the boson: long range entanglement.

Thin “slice” limit: $S_{vN} = 2\alpha \frac{L_y}{a} - \frac{\pi}{24} \frac{L_y}{L_A}$

Thin torus limit: $S_{vN} = 2\alpha \frac{L_y}{a} + 2 \left( \ln \left( 8\pi K R \right) - \frac{1}{2} \right)$
EE in other geometries: Disk and Sphere

- If $A$ is a disk of radius $r$ in the plane with Dirichlet BCs (and for a spherical cap of angle $2\theta$)

\[ \gamma = \ln \left( \sqrt{8\pi \kappa R} \right) - \frac{1}{2} \]

- This result should be compared with the $-\ln (rg^2)$ dependence found for a compactified relativistic scalar field (a Goldstone boson with dimensionful coupling constant $g$) by Agon et al.

- Annular region of radii $r_1 < r_2$ in the infinite plane (and for two spherical caps after stereographic projection)

\[
\text{Thick annulus: } \gamma = 2 \left[ \ln \left( \sqrt{8\pi \kappa R} \right) - \frac{1}{2} \right] \\
\text{Thin annulus: } \gamma = -\frac{\pi}{24} \left( \frac{2\pi r_1}{r_2 - r_1} \right) + \ln \left( \sqrt{8\pi \kappa R} \right) - \frac{1}{2} \]
Mutual Information

- Mutual information of region A, a disk of radius $r_1$, with B, the complement of a disk of radius $r_2 > r_1$, in the infinite plane

$$I(A, B) = S(A) + S(B) - S(C) = 2 \left( \frac{r_1}{r_2} \right)^{1/4\pi\kappa R^2} + \left( \frac{r_1}{r_2} \right)^2$$

- Mutual information of two disks of radii $R_A$ and $R_B$ separated by a large distance $r$ (here $\zeta = R_A R_B / r^2$)

$$I(A, B) = S(A) + S(B) - S(A \cup B) = 2\zeta^{2\Delta_1} + \zeta^{2\Delta_2}$$

$$\Delta_1 = \frac{1}{8\pi\kappa R^2} \text{ and } \Delta_2 = 1$$ scaling dimensions of the vertex operator $\exp(i\phi)$ and $\partial_x \phi$ of the QLM
Torus with general aspect ratio

2 torus with aspect ratio $\frac{L_y}{L_x}$ and a cylindrical region $A$ with aspect ratio $u = \frac{L_A}{L_x}$, the entanglement entropy is $S_{\text{VN}} = 2\alpha \left(\frac{L_y}{a}\right) + \beta \ J(u)$ (Stéphan et al)

$$J(u) = \log \left( \frac{\lambda}{2} \frac{\eta(\tau)^2}{\theta_3(\lambda \tau) \theta_3(\tau / \lambda)} \frac{\theta_3(\lambda u \tau) \theta_3(\lambda (1 - u) \tau)}{\eta(2u \tau) \eta(2(1 - u) \tau)} \right)$$

$\theta_3(z)$: Jacobi theta-function
$\eta(z)$: Dedekind eta-function

For the quantum dimer model at the RK point, $\lambda = 2$

$$\tau = i \frac{L_y}{L_x}$$
Scaling of Entanglement in the 2+1 dimensional Ising Model in a Transverse Field

- Inglis and Melko (2013) studied numerically the second Rényi entropy $S_2$ at the quantum critical point of the 2D TFIM on a 2 torus
- This system is in the universality class of the $\phi^4$ Wilson-Fisher fixed point in 2+1 dimensions
- Two different scaling functions:
  - $S_2=2\alpha L_y + \beta \log(\sin(\pi L_x/L))$
  - $S_2=2\alpha L_y + \beta J(u) + \text{const}$

The QLM scaling works surprisingly well even though is not relativistic!
Holographic Entanglement on the 2 torus

Chen, Cho, Faulkner and Fradkin (2015) used the Ryu-Takayanagi ansatz to obtain the finite term in the von Neumann entanglement entropy for a cylindrical section of a 2 torus

\[ S_{vN} = \frac{\mathcal{A}}{4G_N} \]

\( \mathcal{A} \): area of the minimal AdS surface homologous to the observed region where the QFT lives on the boundary of AdS

Toroidal geometries: minimal surface falls on the “AdS_4 soliton” geometry (Horowitz and Myers)

- The EE is sensitive to which cycle of the torus contracts since the cut is along y
- In the thin torus limit, \( L_y \ll L_x \), the EE saturates
- This is natural for fermions with anti-periodic boundary conditions
- In the opposite, thin slice, limit we find universal scaling
- The von Neumann EE has the scaling form \( S_{vN}=\alpha L_y + \beta j(u) \), where \( j(u) \) has two different forms depending on whether the aspect ratio of the torus is \( L_y/L_x>1 \) or \( L_y/L_x<1 \)
Parametric form of \( j(u) \) for \( L_y/L_x > 1 \)

\[
\begin{align*}
    u(\chi) &= \frac{3\chi^{1/3}(1 - \chi)^{1/2}}{2\pi} \int_0^1 \frac{d\zeta \zeta^2}{(1 - \chi \zeta^3)} \frac{1}{\sqrt{P(\chi, \zeta)}} \\
    j(\chi) &= \chi^{-1/3} \left( \int_0^1 \frac{d\zeta}{\zeta^2} \left( \frac{1}{\sqrt{P(\chi, \zeta)}} - 1 \right) - 1 \right)
\end{align*}
\]

\[P(\chi, \zeta) = 1 - \chi \zeta^3 - (1 - \chi) \zeta^4\]

Parametric form for \( j(u) \) for \( L_y/L_x < 1 \)

\[
\begin{align*}
    j(\chi) &= \chi^{-1/3} \left( \int_0^1 \frac{d\zeta}{\zeta^2} \left( \frac{\sqrt{1 - \chi \zeta^3}}{\sqrt{P(\chi, \zeta)}} - 1 \right) - 1 \right) \\
    \frac{L_x}{L_y} u &= \frac{3}{2\pi} \chi^{1/3}(1 - \chi)^{1/2} \int_0^1 \frac{d\zeta \zeta^2}{\sqrt{1 - \chi \zeta^3}} \frac{1}{\sqrt{P(\chi, \zeta)}}
\end{align*}
\]

FIG. 2. The subleading term for the minimal surface for various values of \( L_x/L_y \). The solid curves are for \( j(u) \) when \( L_x \leq L_y \) and the dashed curves are for \( \tilde{j}(u) \) when \( L_x > L_y \).
Tests of scaling in 2d free field fermionic models

- We computed the entanglement entropy for cylindrical regions of a 2 torus for free massless Dirac fermions in 2+1 dimensions and also for free massless “Lifshitz” (QBT) fermions, in both cases with anti-periodic boundary conditions.
- Dirac fermions in 2+1 dimensions are a stable phase and hence an UV fixed point.
- “Lifshitz” fermions in 2+1 dimensions have z=2 and are at the marginal dimension (asymptotically free).
- We used a lattice version of these models and used standard results for the reduced density matrix and the entanglement entropy (Peschel, 2001).
- In the Dirac case we accounted for fermionic doublers.
Scaling of entanglement for free massless Dirac fermions in 2+1 dimensions

The Dirac model

\[ S_{\text{VN}} \]

- numerical data
- \( \alpha L + \beta J(u) (\lambda = 4.2) \)
- \( \alpha L + \beta \log(\sin(\pi u)) \)
- AdS results

Deviation

\[ u \]

0.0 0.5 1.0
Conclusions

• In 2+1 dimensions we expect to find universal finite corrections to the entanglement entropy
• These finite terms are determined by the operator content (compactification radius, scaling dimensions, etc) and have topological origin
• We presented results for von Neumann EE of the QLM on different geometries and for the mutual information
• On cylindrical regions the universal finite terms are given in terms of scaling functions of the aspect ratios of the cylinders and of the 2 torus
• We presented expressions for the entanglement entropy derived from holography on toroidal geometries which fit remarkably well even free massless Dirac fermions (on anti-periodic boundary conditions)
• The finite term derived by Stéphan et al for the QLM fits remarkably well even for Dirac fermions which is a relativistic theory (although the holographic results is better)
• How well does the holographic result fit the 2D Ising model in a transverse field?
Scaling of entanglement for free massless “Lifshitz” fermions in 2+1 dimensions

Figure 8. $S_{VN}$ for the QBT model as a function of $u$. The bipartition geometry is the same as the Dirac model. The inset is the absolute deviation for the fitting function $S_{VN} = \alpha L + \beta J(u)$ with numerical data. The deviation is less than 1% for the whole region of $u$. 