Bit Threads and Holographic Entanglement

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Based on arXiv:1604.00354 [hep-th], with Michael Freedman

Entropy and area

In semiclassical gravity, entropies are related to surface areas

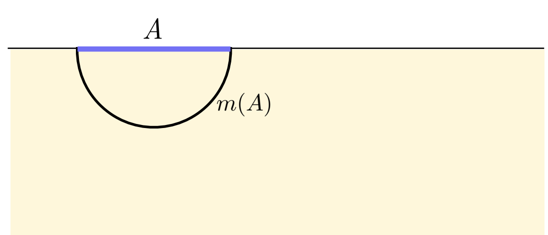
- General surface:
$$S \leq \frac{\text{area}}{4G_{\text{N}}}$$
- Special surfaces (horizon, minimal surface, extremal surface):
$$S = \frac{\text{area}}{4G_{\text{N}}}$$

Ryu-Takayanagi ['06]:

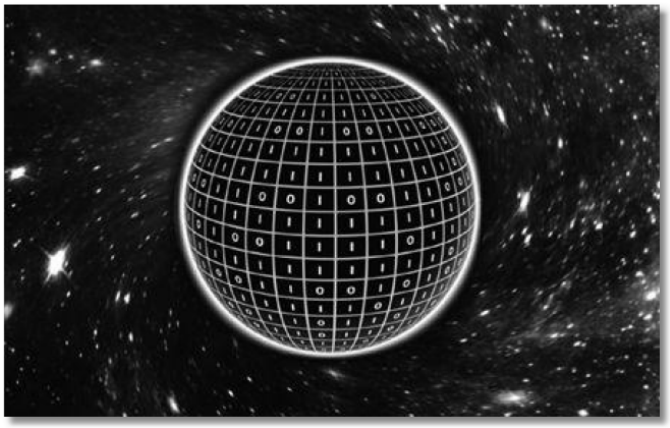
$$S(A) = \frac{1}{4G_N}\,\text{area}(m(A))$$

$m(A) =$ minimal surface homologous to A

(From now on, we set $4G_N = \ln 2 = 1$)



Interpretation



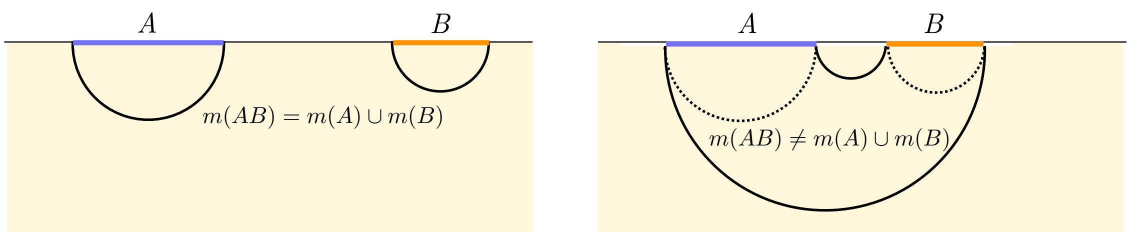
Why?

Naive answer:

> Microstate bits of ρ_A "live" on $m(A)$, one bit per 4 Planck areas

Confusions:

- Under continuous changes in boundary region, minimal surface can jump, for example:



  (Note: ρ_{AB} does not jump; not a conventional exchange-of-dominance phase transition [Headrick '13])

- Important quantities, like condiitonal entropy

$$H(A|B) = S(AB) - S(B)\,,$$

  mutual information

$$I(A:B) = S(A) + S(B) - S(AB)\,,$$

  conditional mutual information

$$I(A:B|C) = S(AB) + S(BC) - S(ABC) - S(C)\,,$$

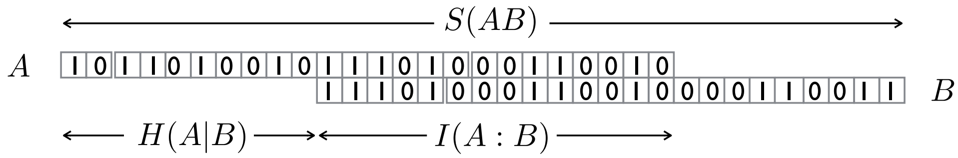
  are given by differences of areas of surfaces passing through different regions of bulk

  Let's recall their information-theoretic meaning

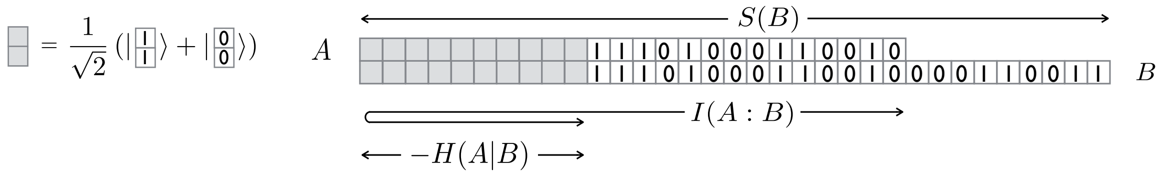
  Classical:

  $H(A|B) = \#$ of (independent) bits belonging purely to A

  $I(A:B) = \#$ shared with B

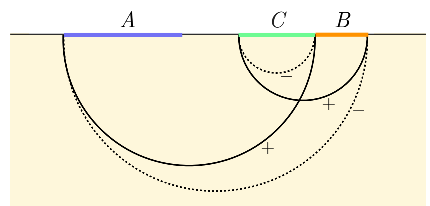


Quantum: Entangled pair of bits contributes 2 to $I(A:B)$, -1 to $H(A|B)$
Can lead to $H(A|B) < 0$



What do differences between areas of surfaces, passing through different parts of bulk, have to do with redundancy, entanglement, etc. between bits of A and B?

What does holographic proof of strong subadditivity have to do with monotonicity of correlations?



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To answer these questions, I will present a new formulation of RT

- Does not refer to minimal surfaces; these are demoted to a calculational device
- Suggests a new way to think about the connection between spacetime geometry and information

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Max flow-min cut

(Originally on graphs, in context of network theory; continuous version [Federer '74, Strang '83, Nozawa '90])

Consider a Riemannian manifold with boundary

Define a *flow* as a vector field v s.t. $\nabla \cdot v = 0$, $|v| \leq 1$
Let A be a subset of boundary

For any surface m homologous to A,

$$\int_A v = \int_m v \leq \text{area}(m)$$

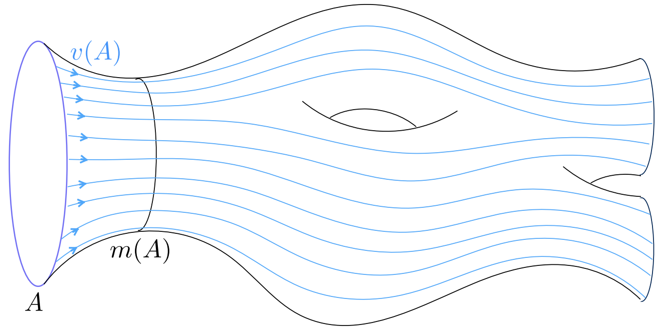
Strongest bound is min cut:

$$\max_v \int_A v \leq \min_{m \sim A} \text{area}(m)$$

Max flow-min cut theorem says there are no other obstructions to increasing flux:

$$\max_v \int_A v = \min_{m \sim A} \text{area}(m)$$

$v(A) :=$ maximizer

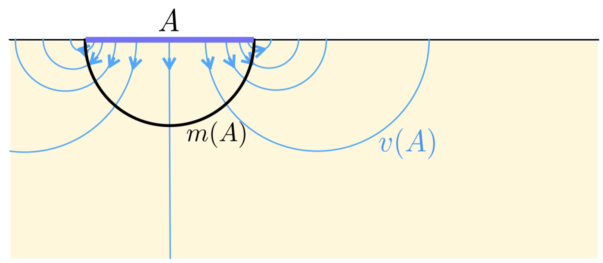


Notes:

- On $m(A)$, $v(A) =$ unit normal vector; elsewhere, $v(A)$ is highly non-unique
- Unlike min cut, max flow is a linear programming problem
- A flow can be thought of as a set of oriented threads (flow lines) with transverse density $= |v| \leq 1$

RT version 2.0:

$$\begin{aligned} S(A) &= \max_v \int_A v \\ &= \text{max \# of threads coming out of } A \end{aligned}$$



Each thread has cross section of 4 Planck areas

Automatically incorporates homology condition & global minimization

Threads can end on A^c or horizon
Each thread carries one independent bit of ρ_A, either entangled with A^c or in a mixed state

Threads can also return to A, but those don't count

Minimal surface does not play a fundamental role, but acts as bottleneck limiting number of threads

Naturally implements holographic principle: entropy is area because bits are carried by one-dimensional objects

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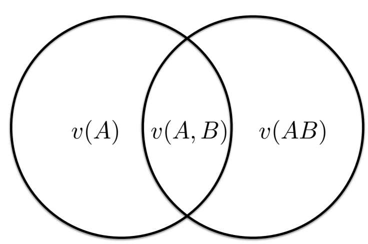
Threads & information

Now we address conceptual puzzles with RT raised before

First, it can be shown that $v(A)$ changes continuously with A, even when $m(A)$ jumps

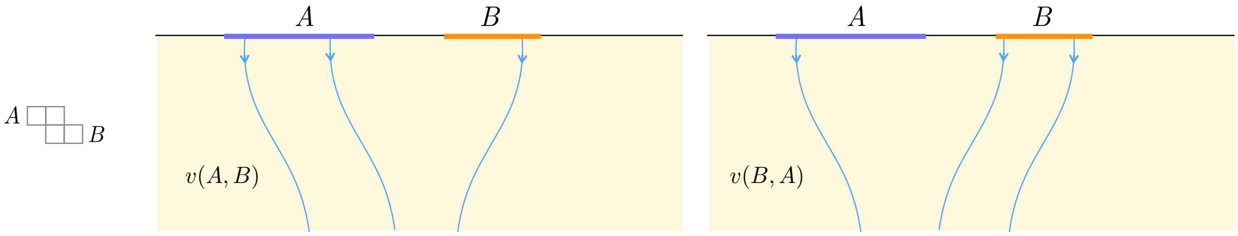
Now consider two regions A, B

We can maximize flux through A *or* B
If $S(AB) < S(A) + S(B)$, then we cannot simultaneously maximize through both
But we *can* always maximize through A *and* AB (nesting property)
Call such a flow $v(A, B)$



Example 1: $S(A) = S(B) = 2$, $S(AB) = 3 \Rightarrow I(A:B) = 1$, $H(A|B) = 1$

Maximizing on AB, we can also maximize on *either* A *or* B

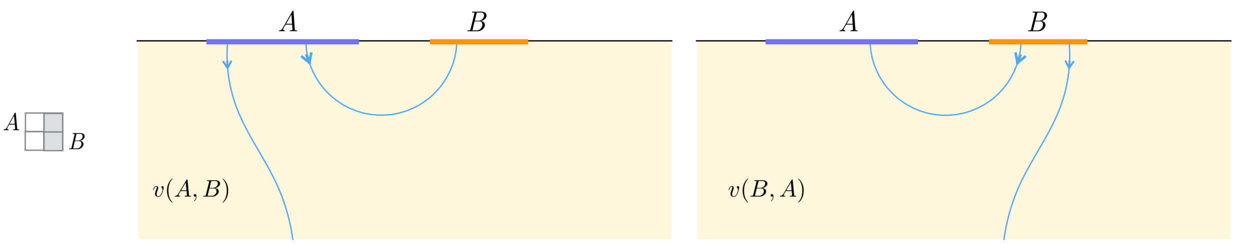


Lesson 1:

- Threads that are stuck on A represent bits unique to A
- Threads that can be moved between A & B represent correlated pairs of bits

Example 2: $S(A) = S(B) = 2$, $S(AB) = 1 \Rightarrow I(A:B) = 3$, $H(A|B) = -1 \Rightarrow$ entanglement!

One thread leaving A *must* go to B, and vice versa



Lesson 2:

- Threads connecting A & B (and switch orientation) represent entangled pairs of bits

Conditional entropy:

$$\begin{aligned} H(A|B) &= S(AB) - S(B) \\ &= \int_{AB} v(B,A) - \int_B v(B,A) \\ &= \int_A v(B,A) \end{aligned}$$

Mutual information:

$$\begin{aligned} I(A:B) &= S(A) - H(A|B) \\ &= \int_A v(A,B) - \int_A v(B,A) \end{aligned}$$

Subadditivity is clear

Max flows can be defined without regulator, as flows that cannot be augmented

Regulator-free definition of mutual information:

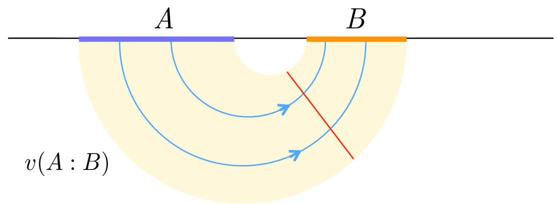
$$I(A:B) = \int_A (v(A,B) - v(B,A))$$

Define flow

$$v(A:B) = \frac{1}{2}(v(A,B) - v(B,A))$$

which goes from A to B through entanglement wedge $r(AB)$
Implies

$$\frac{1}{2} I(A:B) \leq \text{area}\,(r(AB)\ \text{bottleneck})$$



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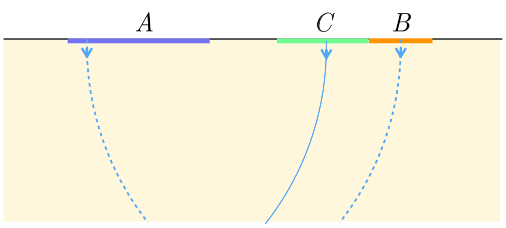
Given lessons above, freedom to move threads around on A indicates correlations with A; freedom to add loops that begin and end on A indicates entanglement within A.

Conditional mutual information:



$$\begin{aligned}
I(A:B|C) &= H(A|C) - H(A|BC) \\
&= \int_A v(C,A,B) - \int_A v(C,B,A) \\
&= (\text{max on } A) - (\text{min on } A),\ \text{while maximizing on } C\ \&\ ABC \\
&= \text{moveable between } A\ \&\ B,\ \text{while maximizing on } C\ \&\ ABC \\
&= (\text{moveable between } A\ \&\ BC) - (\text{moveable between } A\ \&\ C) \\
&= I(A:BC) - I(A:C)
\end{aligned}$$

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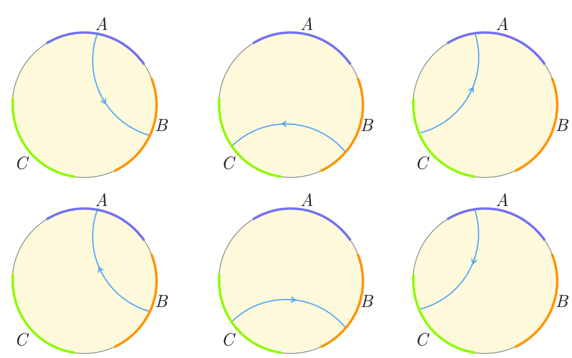
Strong subadditivity is clear .

Exercise for reader: Find flow-based proof of "monogamy of mutual information" inequality [Hayden-Hedarick-Maloney '11],

$$I(A:BC) \geq I(A:B) + I(A:C)$$

and higher inequalities [Bao et al. '15]
Cannot be proved using just nesting property (see e.g. 4-party GHZ); indicates some new property of flows

Do bit threads indicate that bipartite entanglement is privileged?
Possible to represent 3-party GHZ state



Covariant flows

To appear (with Veronika Hubeny)

Flow version of Hubeny-Rangamani-Takayanagi ['07] covariant entanglement entropy formula:

Define a flow as a vector field v (in the full spacetime) s.t. $\nabla \cdot v = 0$ and *integrated* norm bound:
$\forall$ timelike curve C and unit normal vector field u on it,

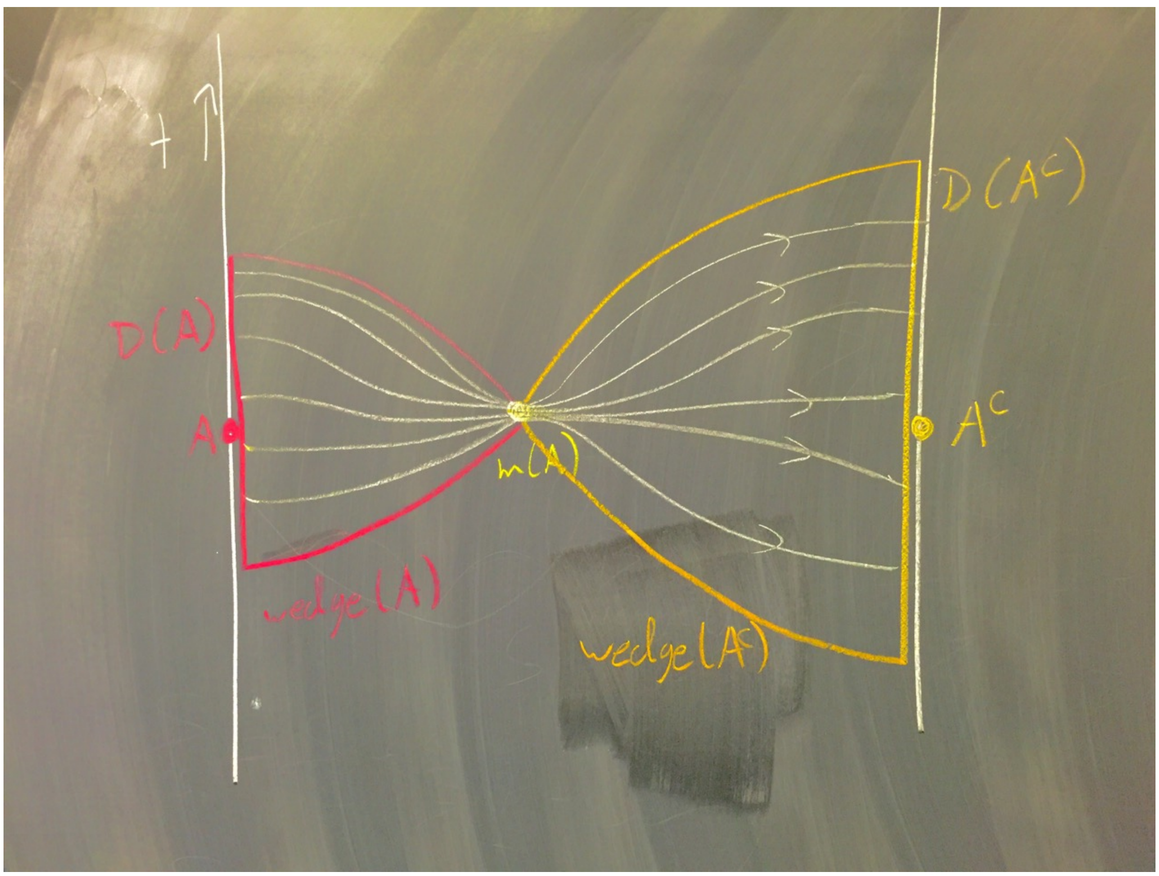
$$\int_C ds\, u \cdot v \leq 1$$

In other words, any observer sees over their lifetime a total of at most 1 thread per 4 Planck areas

HRT version 2.0:

$$S(A) = \max_v \int_{D(A)} v \qquad (D(A) = \text{boundary causal domain of } A)$$

Max flow $v(A)$ stays inside entanglement wedge, squeezes out through extremal surface



Bit threads can lie on a common Cauchy slice (equivalent to maximin [Wall '12]), or spread out in time