Entanglement Holography

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with de Boer, Haehl, Heller & Neiman
Entanglement Entropy

- general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface $\Sigma$ which divides the space into two separate regions
- integrate out degrees of freedom in “outside” region
- remaining dof are described by a density matrix $\rho_A$

\[ S_{EE} = -Tr [\rho_A \log \rho_A] \]

(t = constant)
Holographic Entanglement Entropy:

AdS boundary

AdS bulk spacetime

\[ S(A) = \text{ext}_{V \sim A} \frac{A_V}{4G_N} \]

• 2006 conjecture \( \rightarrow \) many detailed consistency tests
  (Ryu, Takayanagi, Headrick, Hung, Smolkin, RM, Faulkner, . . .)

• 2013 proof (for static geometries) \( \rightarrow \) (Maldacena & Lewkowycz)

(Ryu & Takayanagi)

boundary conformal field theory

Bekenstein-Hawking formula
Holographic Entanglement Entropy:

\[ S(A) = \text{ext } \frac{1}{4G_N} \int_{V \sim A} d^3x \sqrt{h} \]

- 2006 conjecture • many detailed consistency tests
  (Ryu, Takayanagi, Headrick, Hung, Smolkin, RM, Faulkner, . . . )
- 2013 proof (for static geometries)
  (Maldacena & Lewkowycz)
Entanglement Holography:

- building on intuition and experience offered by EE in CFTs (and in AdS/CFT correspondence), propose reorganization of CFT in terms of new nonlocal observables

- find the emergence of a **new auxiliary geometry** as natural framework to describe any CFT – not relying on strong coupling or large # of dof

- may yield new insights into the structure of correlation functions, . . .

- for CFT’s with conventional holographic duals, provides new observables based on extremal surfaces

- may give insight in the nonlocal nature of quantum gravity, bulk reconstruction, . . .

(see also Czech, Lamprou, McCandlish, Mosk & Sully: arXiv:1604.03110)
First Law of Entanglement

• entanglement entropy:  $S(\rho_A) = -\text{tr}(\rho_A \log \rho_A)$

• make a small perturbation of state:  $\tilde{\rho} = \rho_A + \delta \rho$

  \[ \delta S = -\text{tr}(\delta \rho \log \rho_A) - \text{tr}(\rho_A \rho_A^{-1} \delta \rho) + O(\delta \rho^2) \]

  \[ = \text{Tr} (\delta \rho) = 0 \]

  \[ = -\text{tr}(\delta \rho \log \rho_A) + O(\delta \rho^2) \]

• modular (or entanglement) Hamiltonian:  $\rho_A = \exp(-H_A)$

  \[ \delta S_A = \delta \langle H_A \rangle \]

  “1st law” of entanglement entropy

• this is the 1st law for thermal states:  $\rho_A = \exp(-H/T)$
“1\textsuperscript{st} law” of entanglement entropy: \[ \delta S_A = \delta \langle H_A \rangle \]

• generally \( H_A \) is “nonlocal mess” and flow is nonlocal/not geometric

\[
H_A = \int d^{d-1} x \gamma^\mu_1 (x) T_{\mu \nu} + \int d^{d-1} x \int d^{d-1} y \gamma^\mu_2;^{\rho \sigma} (x, y) T_{\mu \nu} T_{\rho \sigma} + \cdots
\]

\[\rightarrow\] hence usefulness of first law is very limited, in general
“1\textsuperscript{st} law” of entanglement entropy: \[ \delta S_A = \delta \langle H_A \rangle \]

- generally \( H_A \) is “nonlocal mess” and flow is nonlocal/not geometric

\[ H_A = \int d^{d-1}x \gamma_1^{\mu\nu}(x) T_{\mu\nu} + \int d^{d-1}x \int d^{d-1}y \gamma_2^{\mu\nu;\rho\sigma}(x, y) T_{\mu\nu}T_{\rho\sigma} + \ldots \]

\[ \implies \text{hence usefulness of first law is very limited, in general} \]

- famous exception: \textbf{Rindler wedge}
- any relativistic QFT in Minkowski vacuum; choose \( \Sigma = (x = 0, t = 0) \)

\[ H_A = 2\pi K \quad \text{boost generator} \]

\[ = 2\pi \int_{A(x>0)} d^{d-2}y \ dx \ [x \ T_{tt}] + c' \]

- by causality, \( \rho_A \) and \( H_A \) describe physics throughout domain of dependence \( \mathcal{D} \); eg, generate boost flows (Bisognano & Wichmann; Unruh)
“1st law” of entanglement entropy: \[ \delta S_A = \delta \langle H_A \rangle \]

- **another exception**: CFT in vacuum of d-dim. flat space and entangling surface which is \( S^{d-2} \) with radius \( R \)

\[
H_B = 2\pi \int_B d^{d-1} y \left( \frac{R^2 - |\vec{y}|^2}{2R} \right) T_{tt}(\vec{y}) + c'
\]

(Casini, Huerta & RM; Hislop & Longo)

- generates flow along \( K^\mu \), conformal Killing vector
“1st law” of entanglement entropy:  \[ \delta S_A = \delta \langle H_A \rangle \]

- small excitations of CFT vacuum in d-dim. flat space and entangling surface which is \( S^{d-2} \) with radius \( R \):

\[
\delta S = \delta \langle H_B \rangle = 2\pi \int_B d^{d-1}y \frac{R^2 - |\vec{y}|^2}{2R} \langle T_{tt}(\vec{y}) \rangle
\]
“1st law” of entanglement entropy: \[ \delta S_A = \delta \langle H_A \rangle \]

- small excitations of CFT vacuum in d-dim. flat space and entangling surface which is \( S^{d-2} \) with radius \( R \):

\[
\delta S(R, \vec{x}) = 2\pi \int_B d^{d-1}y \frac{R^2 - |\vec{y} - \vec{x}|^2}{2R} \langle T_{tt}(\vec{y}) \rangle
\]
Entanglement Holography v1.0:

- small excitations of CFT vacuum in d-dim. flat space and entangling surface which is $S^{d-2}$ with radius $R$:

$$\delta S(R, \vec{x}) = 2\pi \int_B d^{d-1}y \left[ \frac{R^2 - |\vec{y} - \vec{x}|^2}{2R} \right] \langle T_{tt}(\vec{y}) \rangle$$

- boundary-to-bulk propagator in d-dim de Sitter space!

(eg, see: Xiao 1402.7080)

$$ds^2 = \frac{L^2}{R^2} \left( -dR^2 + d\vec{x}^2 \right)$$

radius plays role of time
Entanglement Holography v1.0:

• small excitations of CFT vacuum in d-dim. flat space and entangling surface which is $S^{d-2}$ with radius $R$:

$$\delta S(R, \vec{x}) = 2\pi \int_B d^{d-1}y \frac{R^2 - |\vec{y} - \vec{x}|^2}{2R} \langle T_{tt}(\vec{y}) \rangle$$

• boundary-to-bulk propagator in d-dim de Sitter space!

(eg, see: Xiao 1402.7080)

$$ds^2 = \frac{L^2}{R^2} (-dR^2 + d\vec{x}^2)$$

• straightforward to show $\delta S$ satisfies wave equation in $dS_d$

$$\left( \nabla_{dS}^2 - m^2 \right) \delta S = 0 \quad \text{with} \quad m^2 L^2 = -d$$
Entanglement Holography v1.0:

- de Sitter metric: 
  \[ ds^2 = \frac{L^2}{R^2} (-dR^2 + d\bar{x}^2) \]

- wave equation \((\nabla^2_{dS} - m^2) \delta S = 0\) with \(m^2 L^2 = -d\)

\[ \rightarrow \text{2 independent sol's: } \delta S \overset{R \to 0}{\to} F(\vec{x})/R + f(\vec{x}) R^d + \cdots \]

\[ \Delta = -1 \quad \Delta = d \]

- “1st law” solution: 
  \[ \delta S(R, \vec{x}) = 2\pi \int_B d^{d-1}y \frac{R^2 - |\vec{y} - \vec{x}|^2}{2R} \langle T_{tt}(\vec{y}) \rangle \]

\[ \rightarrow F(\vec{x}) = 0 \; ; \; \quad f(\vec{x}) = \frac{\pi \frac{d+1}{2}}{\Gamma \left( \frac{d+3}{2} \right)} \langle T_{tt}(\vec{x}) \rangle \]

\[ \langle T_{tt} \rangle \text{ sets } \delta S \text{ at very small } R \text{ and EE perturbations at larger scales determined by the local Lorentzian propagation into dS geometry} \]

- \(m^2 L^2 = -d\): mass tachyonic! \(\rightarrow\) above precisely removes the “non-normalizable” or unstable modes
New “holographic” coordinate is time-like. Really?

- geometry naturally gives partial ordering of spheres

(ordering of intervals for d=2 discussed by Czech, Lamprou, McCandlish & Sully)
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New “holographic” coordinate is time-like. Really?

- geometry naturally gives partial ordering of spheres
  suggests auxiliary/holographic geometry should be Lorentzian

(ordering of intervals for d=2 discussed by Czech, Lamprou, McCandlish & Sully)
Mapping deSitter ↔ Balls?

- choose one of asymptotic boundaries of dS (eg, $\mathcal{I}^+$) ↔ time slice
- for any point $x$ in bulk and send out future light cone to $\mathcal{I}^+$
- intersects $\mathcal{I}^+$ on a sphere and interior uniquely defines `dual' ball $B_x$

$\mathcal{I}^+ = \{ R = 0, \bar{x} \}$
Mapping deSitter $\leftrightarrow$ Balls?

- choose one of asymptotic boundaries of $dS$ (e.g., $\mathcal{I}^+$) $\leftrightarrow$ time slice
- for any point $x$ in bulk and send out future light cone to $\mathcal{I}^+$
- intersects $\mathcal{I}^+$ on a sphere and interior uniquely defines `dual' ball $B_x$

\[\mathcal{I}^+ = \{ R = 0, \vec{x} \}\]

- proposed “ordering” of spheres = Lorentzian ordering of bulk points
- mapping/dS geometry does not imply local dynamics respecting this structure
Comments:

• same wave equation derived from AdS/CFT correspondence
  
  Nozaki, Numasawa, Prudenziati & Takayanagi: arXiv:1304.7100
  Bhattacharya, Takayanagi: arXiv:1308.3792

• Eg, linearized Einstein eqs in AdS$_4$ implied for holographic EE

\[
\left[ \frac{\partial^2}{\partial R^2} - \frac{1}{R} \frac{\partial}{\partial R} - \frac{3}{R^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] \delta S(t, x, y, R) = 0
\]

• can be recast as d=3 deSitter wave equation:

\[
\left[ -\frac{R^3}{L^2} \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} \right) + \frac{R^2}{L^2} \frac{\partial^2}{\partial x^2} + \frac{R^2}{L^2} \frac{\partial^2}{\partial y^2} + \frac{3}{L^2} \right] \delta S(t, x, y, R) = 0
\]

\text{d’Alembertian on dS}_3 \quad \text{mass term}

• here, we see equation readily extends to any $d$ and follows purely from underlying conformal symmetry
Comments:

• deSitter geometry appears in recent discussions of integral geometry and the interpretation of MERA in terms of AdS$_3$/CFT$_2$

• consider space of intervals $u < x < v$ on time slice of 2d holographic CFT
  space of geodesics on 2d slice of AdS$_3$ ↔ pts in 2d de Sitter
  AdS/CFT

\[
ds^2 = L^2 \frac{du \; dv}{(v - u)^2}
\]

Determine the choice: $L^2 = \frac{c}{3}$

\[
ds^2 = \partial_u \partial_v S_0 \; du \; dv
\]

with $S_0 = \frac{c}{3} \log \frac{v - u}{\delta}$

“hole-ography”:
volume in dS$_2$ = length in AdS$_3$ slice
Entanglement Holography v1.0 – Recap

• EE of excitations of CFT vacuum arranged in novel holographic manner

• $\delta S$ satisfies wave equation in $dS_d$ where scale plays the role of time

$$\left(\nabla^2_{dS} - m^2\right) \delta S = 0 \quad \text{with} \quad m^2 L^2 = -d$$

• $\langle T_{tt} \rangle$ sets $\delta S$ at very small $R$ and EE perturbations at larger scales determined by the local Lorentzian propagation into $dS$ geometry

applies for any CFT in any $d$; relies only on the 1st law of entanglement; does not require strong coupling or large # dof

Question:
Is this only some “kinematic” constraint on entanglement in CFTs?

or

Is there a novel re-organization of CFT where nonlocal observables yield local field theory propagating in $dS$ spacetime?
Question: Other dynamical fields in dS space?
Extension to Higher Spin Charges:

- CFT with conserved symmetric traceless currents $T_{\mu_1 \ldots \mu_s}$ with $s \geq 1$
- modular Hamiltonian is flux of $J^{(2)}_\mu = T_{\mu \nu} K^\nu$ through $B$ where $K^\nu$ is conformal Killing vector that leaves $\partial B$ invariant

  $H_B = \int d\Sigma^\mu J^{(2)}_\mu$

- extends to higher spin charges:

  $\delta Q^{(s)} = \int d\Sigma^\mu J^{(s)}_\mu$ with $J^{(s)}_\mu = T_{\mu_1 \mu_2 \ldots \mu_s} K^\mu_1 \ldots K^\mu_s$

- appear in discussion of modified density matrices

  $\rho_B \sim \exp \left[ - \sum \mu_s \delta Q^{(s)} \right]$  

  $(s \geq 3$: Hijano & Kraus; $s=1$: Belin, Hung etal$)$
Extension to Higher Spin Charges:

- extends to higher spin charges:

\[ \delta Q^{(s)} = \int d\Sigma^\mu J^{(s)}_\mu \quad \text{with} \quad J^{(s)}_\mu = T_{\mu\mu_2...\mu_s} K^{\mu_2} \cdots K^{\mu_s} \]

- on \( t=0 \) slice, yields:

\[ \delta Q^{(s)} = (2\pi)^{s-1} \int_{B} d^{d-1}y \left( \frac{R^2 - |\vec{x} - \vec{y}|^2}{2R} \right)^{s-1} T_{tt...t}(\vec{y}) \]

bdry-to-bulk propagator for deSitter

- \( \delta Q^{(s)} \) satisfies wave equation in \( dS_d \)

\[ (\nabla^2_{dS} - m^2) \delta Q^{(s)} = 0 \]

with

\[ m^2 L^2 = -(s - 1)(d + s - 2) \]

\[ \nabla^2_{dS} = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \left( \frac{1}{R^2} \right) \frac{\partial}{\partial x^\nu} \]

\[ g^{\mu\nu} = \text{diag}(1, -1, -1, ..., -1) \]
**Question:** What about time dependence in CFT?

- so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame

(see also Czech, Lamprou, McCandlish, Mosk & Sully: arXiv:1604.03110)
Question: What about time dependence in CFT?

- so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame
- adopt group theoretic perspective of wave equation:
  - background for spheres on fixed time slice:
    \[ SO(1,d)/SO(1,d-1) \sim d\text{-dim. deSitter space} \]
    - symmetries leaving time slice invariant
    - symmetries leaving sphere invariant
(see also Czech, Lamprou, McCandlish, Mosk & Sully: arXiv:1604.03110)
so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame

• adopt group theoretic perspective of wave equation:

  background for spheres on fixed time slice:

  \[ SO(1, d)/SO(1, d - 1) \simeq d\text{-dim. deSitter space} \]

  symmetries leaving time slice invariant

  symmetries leaving sphere invariant

  background for spheres throughout spacetime:

  \[ SO(2, d)/ [SO(1, d - 1) \times SO(1, 1)] \]

  \[ 2d\text{-dimensional space} \]
Entanglement Holography v2.0:

- so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame
- adopt group theoretic perspective of wave equation:
  - background for spheres throughout spacetime:
    \[ SO(2, d) / [SO(1, d - 1) \times SO(1, 1)] \]
  - \(2d\)-dimensional space
- moduli space of spheres
  - = m.s. of causal diamonds
  - = m.s. of pairs of time-like separated points
    \((y^\mu, x^\mu)\)
    \[ \ell^\mu = \frac{y^\mu - x^\mu}{2} \]
    \[ c^\mu = \frac{y^\mu + x^\mu}{2} \]

(see also Czech, Lamprou, McCandlish, Mosk & Sully: arXiv:1604.03110)
Entanglement Holography v2.0:

• so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame
• adopt group theoretic perspective of wave equation:
  background for spheres throughout spacetime:

\[ SO(2, d) / [SO(1, d - 1) \times SO(1, 1)] \]

\[ \text{2d-dimensional space} \]

\[ \text{signature: } (d, d) \]
so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame

• adopt group theoretic perspective of wave equation:
  background for spheres throughout spacetime:

\[
SO(2, d)/ [SO(1, d – 1) \times SO(1, 1)]
\]

  2\(d\)-dimensional space

  signature: \((d, d)\)

  too many times?!?!

• natural metric:
  need more eoms?!?!

\[
ds^2_\diamond = \frac{4 L^2}{(x - y)^2} \left( -\eta_{\mu\nu} + \frac{2(x_\mu - y_\mu)(x_\nu - y_\nu)}{(x - y)^2} \right) \, dx^\mu \, dy^\nu
\]

\[
= -\frac{L^2}{\ell^2} \left( \eta_{\mu\nu} - \frac{2}{\ell^2} \ell_\mu \ell_\nu \right) \left( dc^\mu \, dc^\nu - d\ell^\mu \, d\ell^\nu \right)
\]
Question: What about time dependence in CFT?

- so far focused on single time slice; natural to consider perturbations of EE for all spheres throughout spacetime on any time slice & any frame
- adopt group theoretic perspective of wave equation:

  background for spheres throughout spacetime:

  \[ SO(2, d)/[SO(1, d - 1) \times SO(1, 1)] \]

  \[ 2d \)-dimensional space \]

  \[ \text{signature: } (d, d) \]

  too many times?!?!?

  need more eoms?!?!

- special case: \( d=2 \)

  \[ SO(2, 2)/[SO(1, 1) \times SO(1, 1)] \]

  \[ = SO(2, 1)/SO(1, 1) \times SO(2, 1)/SO(1, 1) \]

  \[ = dS_2 \times dS_2 \]

(see also Czech, Lamprou, McCandlish, Mosk & Sully: arXiv:1604.03110)
• focus on d=2 CFT where found $dS_2 \times dS_2$
• natural to split $\delta S$ into $\delta S_{\pm} = \text{contributions of left/right-movers}$

eg, in 1st law limit: $\delta S_+ = 2\pi \int d\xi^+ \frac{(x_R^+ - \xi^+)(\xi^+ - x_L^+)}{x_R^+ - x_L^+} \langle T_{++}\rangle(\xi^+)$

$x^\pm = (x \pm t)/\sqrt{2}$

• $\delta S_{\pm}$ propagate on separate $dS_2$ geometries, eg, $ds^2 = L^2 \frac{dx_R^+ dx_L^+}{(x_R^+-x_L^+)^2}$

Need more eoms!?!?
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- focus on $d=2$ CFT where found $dS_2 \times dS_2$
- natural to split $\delta S$ into $\delta S_{\pm} = $ contributions of left/right-movers

eg, in 1st law limit: $\delta S_+ = 2\pi \int d\xi^+ \frac{(x_R^+ - \xi^+)(\xi^+ - x_L^+)}{x_R^+ - x_L^+} \langle T_{++}\rangle(\xi^+)$

$$x^\pm = (x \pm t)/\sqrt{2}$$

- $\delta S_{\pm}$ propagate on separate $dS_2$ geometries, eg, $ds^2 = L^2 \frac{dx_R^+ dx_L^+}{(x_R^+ - x_L^+)^2}$

$$\left(\nabla^2_+ - m_+^2\right) \delta S_+ = 0 \quad \text{with} \quad m_+^2 L^2 = -2$$

implicitly: $$\left(\nabla^2_- - m_-^2\right) \delta S_+ = 0 \quad \text{with} \quad m_-^2 L^2 = 0$$

- $\delta S_{\pm}$ propagate nontrivially on $dS_\pm$ and trivially on $dS_{\mp}$

$$\rightarrow \quad \text{two "standard" second-order wave equations}$$
Question: What about interacting fields?
Beyond 1st Law: \[ \delta S_A \leq \delta \langle H_A \rangle \]

- **specialize:** \( d=2; \) “conformally” excited states

\[
\begin{align*}
w^+ &= f_+(x^+) \quad \text{and} \quad w^- = f_-(x^-) \\
x^\pm &= (x \pm t)/\sqrt{2}
\end{align*}
\]

\[
\langle T_{++} \rangle (x^+) = \frac{c}{12} \left\{ \frac{f'''}{f'} - \frac{3(f'')^2}{2(f')^2} \right\}
\]
Beyond 1\textsuperscript{st} Law: \(\delta S_A \leq \delta \langle H_A \rangle\)

**specialize:** \(d=2; \) “conformally” excited states

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w^+ = f_+(x^+) \quad \text{and} \quad w^- = f_-(x^-) \quad \quad x^\pm = (x \pm t)/\sqrt{2}
\]

\[\langle T_{++}\rangle(x^+) = \frac{c}{12} \left\{ \frac{f^{''''}}{f_+} - \frac{3(f'_+)^2}{2(f'_+)^2} \right\} \]

**recall:**

\[
S = \lim_{n \to 1} \frac{1}{1-n} \log \text{tr} \rho^n = \lim_{n \to 1} \frac{1}{1-n} \log \langle \sigma_n \sigma_{-n} \rangle
\]

correlator of local primaries

**evaluate change of entropy under local conformal transformations**

\[
S(w^+_L, w^-_L; w^+_R, w^-_R) = S_+(f_+; w^+_L, w^+_R) + S_-(f_-; w^-_L, w^-_R)
\]

with \( S_+(f_+; w^+_L, w^+_R) = \frac{c}{12} \log \frac{(f_+(w^+_R) - f(w^+_L))^2}{\delta^2 f'_+(w^+_R)f'_+(w^+_L)} \)

(Holzhey, Larsen & Wilczek; Calabrese & Cardy)
Beyond 1st Law: \( \delta S_A \leq \delta \langle H_A \rangle \)

- define: \( \delta S_+ (w_L^+, w_R^+) = S_+ (f_+; w_L^+, w_R^+) - S_+ (f_+(z) = z; w_L^+, w_R^+) \)
- for finite shift of state, find nonlinear wave equation:

\[
\nabla_+^2 \delta S_+ = V'(\delta S_+) \quad \text{with} \quad V'(\delta S_+) = \frac{c}{6L^2} \left[ \exp \left( -\frac{12 \delta S_+}{c} \right) - 1 \right]
\]

\[
\text{expected } m^2 \text{ for } d=2 \quad \Rightarrow \quad -\frac{2}{L^2} \delta S_+ + \frac{12}{cL^2} \delta S_+^2 + \cdots
\]

(also implicitly: \( \nabla_-^2 \delta S_+ = 0 \))
Beyond 1st Law: \( \delta S_A \leq \delta \langle H_A \rangle \)

• define: \( \delta S_+ (w_L^+, w_R^+) = S_+ (f_+; w_L^+, w_R^+) - S_+ (f_+ (z) = z; w_L^+, w_R^+) \)

• for finite shift of state, find nonlinear wave equation:
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  \]
  
  expected \( m^2 \) for \( d=2 \)
  
  \[
  = -\frac{2}{L^2} \delta S_+ + \frac{12}{cL^2} \delta S_+^2 + \cdots
  \]
  
  interactions suppressed by central charge

(also implicitly: \( \nabla^2_- \delta S_+ = 0 \))

• **local** dynamics on auxiliary geometry!!

(see also: Beach, Lee, Rabideau & Van Raamsdonk: arXiv:1604.05308)
Beyond 1st Law: \( \delta S_A \leq \delta \langle H_A \rangle \)

- define: \( \delta S_+(w^+_L, w^+_R) = S_+(f_+; w^+_L, w^+_R) - S_+(f_0; w^+_L, w^+_R) \)
- for finite shift of state, find nonlinear wave equation:

\[
\nabla^2_+ \delta S_+ = V'(\delta S_+) \quad \text{with} \quad V'(\delta S_+) = \frac{c}{6L^2} \left[ \exp \left( -\frac{12 \delta S_+}{c} \right) - 1 \right]
\]

expected \( m^2 \) for \( d=2 \):

\[
= -\frac{2}{L^2} \delta S_+ + \frac{12}{c L^2} \delta S^2_+ + \cdots
\]

interactions suppressed by central charge

(also implicitly: \( \nabla^2_- \delta S_+ = 0 \))

- **local** dynamics on auxiliary geometry!!

- choosing alternate reference state produces coordinate transformation on \( dS_2 \) geometry with \( \tilde{w}^+_R = f_0(w^+_R) \) and \( \tilde{w}^+_L = f_0(w^+_L) \)

Beyond 1\textsuperscript{st} Law: \( \delta S_A \leq \delta \langle H_A \rangle \)

- d=2 higher spin CFT (use CS theory with 3d gauge fields & use Wilson line prescription for EE)
  
  \( \text{(deBoer & Jottar; Ammon, Castro & Iqbal; Hijano & Kraus, ...)} \)

\[
\nabla^2 \delta S + \frac{c}{6} - \frac{c}{6} \exp\left(-12 \frac{\delta S}{c}\right) \cosh\left(72 \frac{\delta Q^{(3)}}{c}\right) = 0
\]

\[
\nabla^2 \delta Q^{(3)} + \frac{c}{12} \exp\left(-12 \frac{\delta S}{c}\right) \sinh\left(72 \frac{\delta Q^{(3)}}{c}\right) = 0
\]

\((+/−\text{ indices are suppressed})\)

- theory of two interacting scalar fields with \textbf{local} interactions

- appears to be related to Toda theory with same SL(3,R) symmetry
Beyond conserved currents:

- motivated by first law, define observables:

\[ \delta Q(O; x, y) = C_O \int_{D(x, y)} d^d \xi \left( \frac{(y - \xi)^2 (\xi - x)^2}{-(y - x)^2} \right)^{\frac{1}{2}} \left( \Delta_O d \right) \langle O(\xi) \rangle \]

integrate over entire causal diamond \( D(x, y) \)
Beyond conserved currents:

• motivated by first law, define observables:

\[
\delta Q(\mathcal{O}; x, y) = C_\mathcal{O} \int_{D(x, y)} d^d\xi \left( \frac{(y - \xi)^2(\xi - x)^2}{-(y - x)^2} \right)^{\frac{1}{2}(\Delta_\mathcal{O} - d)} \langle \mathcal{O}(\xi) \rangle
\]

• satisfies wave equation of moduli space:

\[
(\nabla^2 - m^2) \delta Q(\mathcal{O}) = 0 \quad \text{with} \quad m^2 L^2 = \Delta_\mathcal{O}(d - \Delta_\mathcal{O})
\]

• reduces to known “charges” for conserved higher spin currents

• resummation of OPE contributions of \( \mathcal{O} \) and all descendants

\[ \rightarrow \text{conformal blocks} \quad \text{(Czech, Lamprou, McCandlish, Mosk & Sully)} \]

• for holographic CFTs, bulk dual given by integral of extremal surface

\[
\delta Q_{\text{holo}}(\mathcal{O}; x, y) = \frac{C_\mathcal{O}}{8\pi G_N} \frac{\Gamma \left( \frac{\Delta_\mathcal{O} + 2 - d}{2} \right) \Gamma \left( \frac{\Delta_\mathcal{O}}{2} \right)}{\Gamma \left( \Delta_\mathcal{O} - \frac{d}{2} \right)} \int_{B(x, y)} d^{d-1}u \sqrt{h} \phi(u)
\]
Beyond conserved currents:

- motivated by first law, define observables:

\[
\delta Q(O; x, y) = C_O \int_{D(x,y)} d^d \xi \left( \frac{(y-\xi)^2(\xi-x)^2}{-(y-x)^2} \right)^{\frac{1}{2}(\Delta_O-d)} \langle O(\xi) \rangle 
\]

- satisfies wave equation of moduli space:

\[
(\nabla_O^2 - m_O^2) \delta Q(O) = 0 \quad \text{with} \quad m_O^2 L^2 = \Delta_O(d - \Delta_O)
\]

- need more eoms!?!?

\[
\Gamma_{abcd}(x, y) \delta Q(O; x, y) = C_O \int_{D(x,y)} d^d \xi \left( \frac{(y-\xi)^2(\xi-x)^2}{-(y-x)^2} \right)^{\frac{1}{2}(\Delta_O-d)} \langle [\Gamma_{abcd}(\xi), O(\xi)] \rangle
\]

where \(J_{ab} = \) conformal generators with \(a, b = -, 0, 1, \ldots, d - 1, d\)

- these constraints are not all independent; left with

\[
12 \Gamma_{-d\mu\nu} = 2\{M_{\mu\nu}, D\} - \{P_\mu, Q_\nu\} + \{Q_\mu, P_\nu\}
\]
Conclusions:

• EE of excitations of CFT vacuum arranged in novel “holographic” way

• $\delta S$ satisfies wave equation on moduli space of causal diamonds

$$\left(\nabla^2 - m^2\right) \delta S = 0 \quad \text{with} \quad m^2 L^2 = -2d$$

applies for any CFT in any $d$; relies only on the 1st law of entanglement; does not require strong coupling or large # dof

• extends to a variety of other nonlocal observables, as well as an interacting theory on moduli space for two dimensions

Question:

Is this only some “kinematic” constraint on entanglement in CFTs?

or

Is there a novel re-organization of CFT where nonlocal observables yield local field theory propagating in auxiliary spacetime?
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Still lots to explore!!