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Quantum Entanglement of Local Operators in Various CFTs

Masahiro Nozaki

Kadanoff Center for Theoretical Physics, The University of Chicago

1. arXiv:1401.0539 [hep-th]
2. arXiv:1405.5875 [hep-th]
3. arXiv:1405.5946 [hep-th]
4. arXiv:1507.04352[hep-th]
5. arXiv:1512.08132 [hep-th]
6. arxiv:16xx.xxxxx[hep-th]

KADANOFF CENTER
FOR THEORETICAL PHYSICS

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- It is useful to study the distinctive features of various quantum state in condensed matter physics.
- (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT correspondence .(***Gravity*** $\leftrightarrow$ ***Entanglement***)

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.
- (R)EE is expected to be entropy in non-equilibrium system.

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.
- (R)EE is expected to be entropy in non-

It is important to study the properties of (Renyi) entanglement entropy.

Motivation

- Fundamental Property of Entanglement

  Dynamics of Quantum Entanglement

Motivation

- Fundamental Property of Entanglement

  Dynamics of Quantum Entanglement

- **How amount of Quantum Information is stored in Subsystem**

Motivation

- Fundamental Property of Entanglement

Dynamics of Quantum Entanglement

Measuring (Renyi) Entanglement Entropy

- **How amount of Quantum Information is stored in Subsystem**

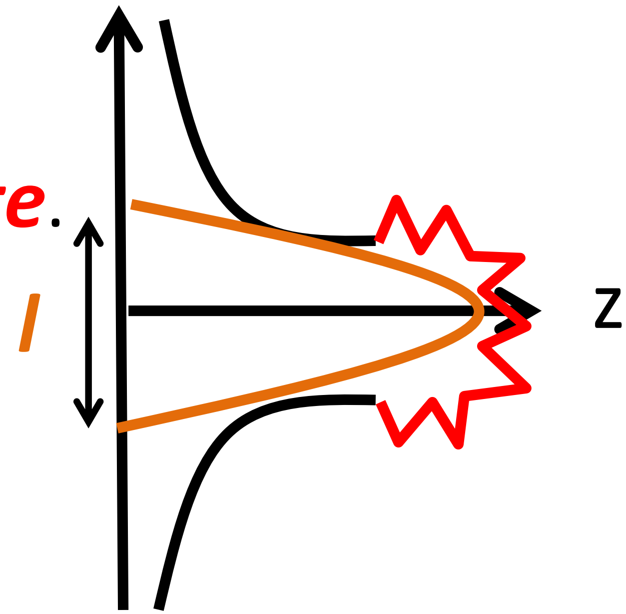
Setup

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

   A half of the total system:

$$x^1 \geq 0$$

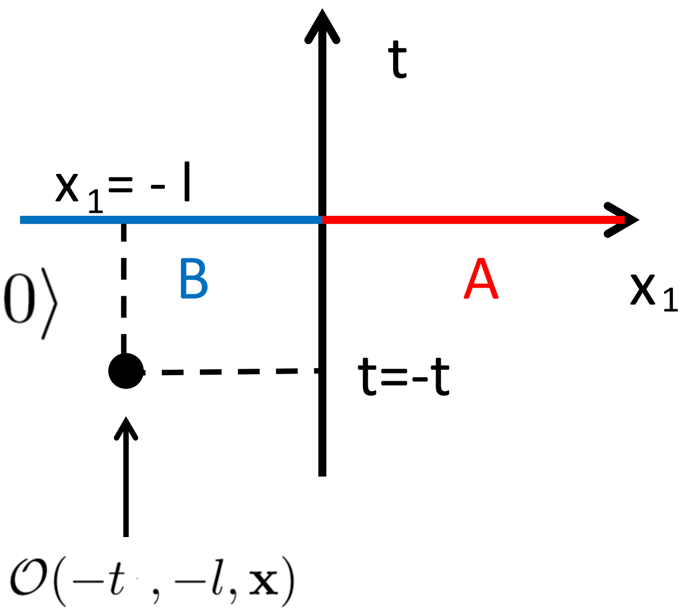


Setup

We study the property of **(R)EE** for

2. A state is defined by acting with
   a local operator on the ground state:

$$|\Psi\rangle = \mathcal{N}\mathcal{O}(t=-t, x^1=-l, \mathbf{x})\,|0\rangle$$



Quantity

We would like to focus on the time evolution of the (R)EE.

We define $\Delta S_A^{(n)}$ the excess of the (R)EE:

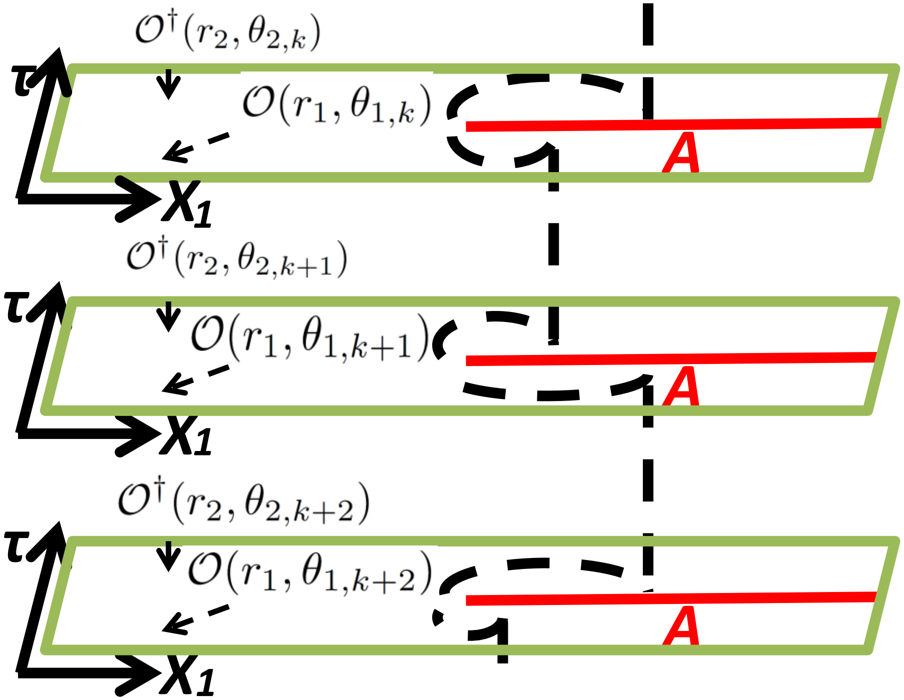
$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G},}$$

$S_A^{(n)Ex}$: (R)EE for $\hat{\rho}_A$ (Reduced Density Matrix for $|\Psi\rangle = \mathcal{N}\mathcal{O}(t, x^i)\,|0\rangle$)

$S_A^{(n)G}$: (R)EE for the ground state

Quantity

Replica Method

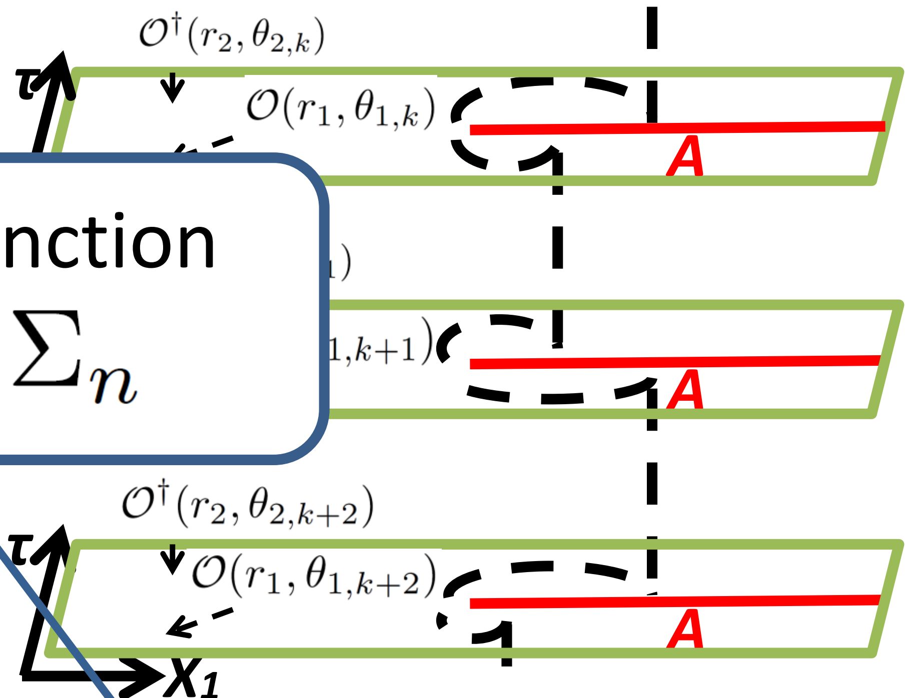


$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\Delta S_A^{(n)} = \frac{1}{1-n}\Big(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\Big).$$

Quantity

Replica Method



2n-Point Function on Σ_n

$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
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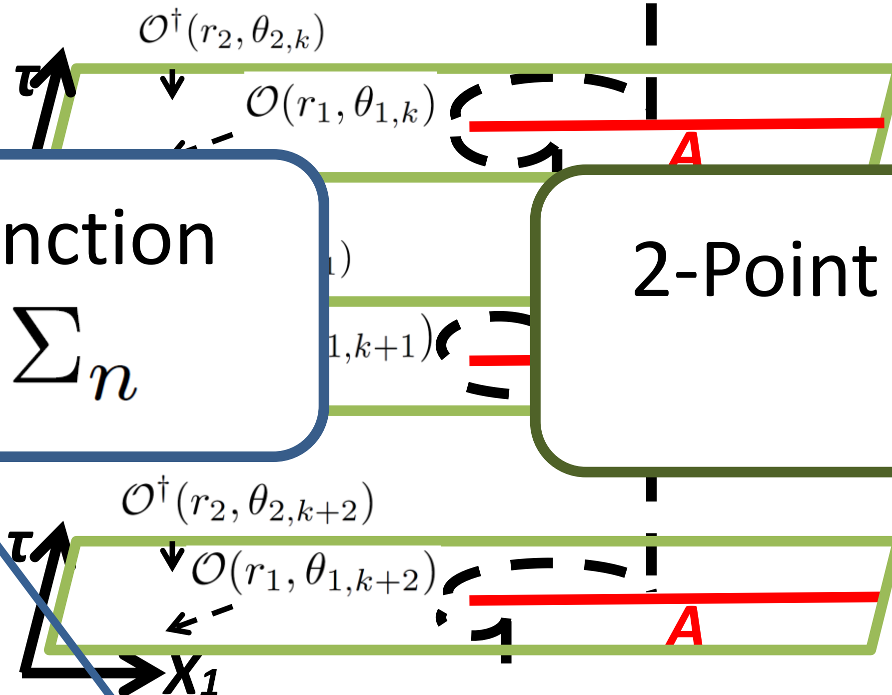
$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Quantity

Replica Method



2n-Point Function on Σ_n

2-Point Function on Σ_1

$$\Delta S_A^{(n)} = \frac{1}{1-n}\Big(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\Big).$$

$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
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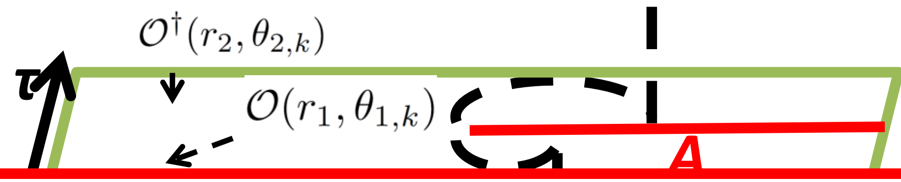
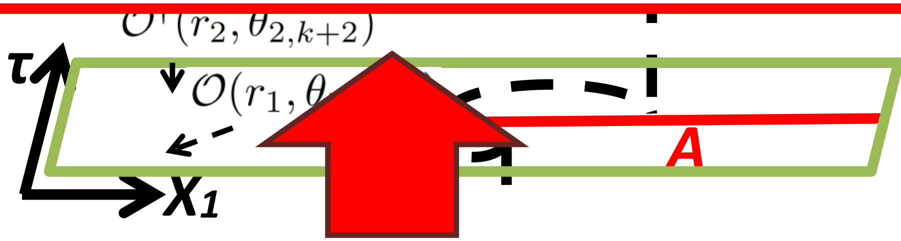
Quantity

Replica Method



This formula holds for any local operators in general QFT in any dimensions.



$$\Delta S_A^{(n)} =$$

$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Field Theory

1. Free massless scalar field theory

2. *U(N)* or *SU(N)* free massless scalar field theory
   in Large N limit

3. Free massless fermionic field theory

4. Charged Renyi Entanglement Entropy (CREE)

5. Maxwell Theory in 4d

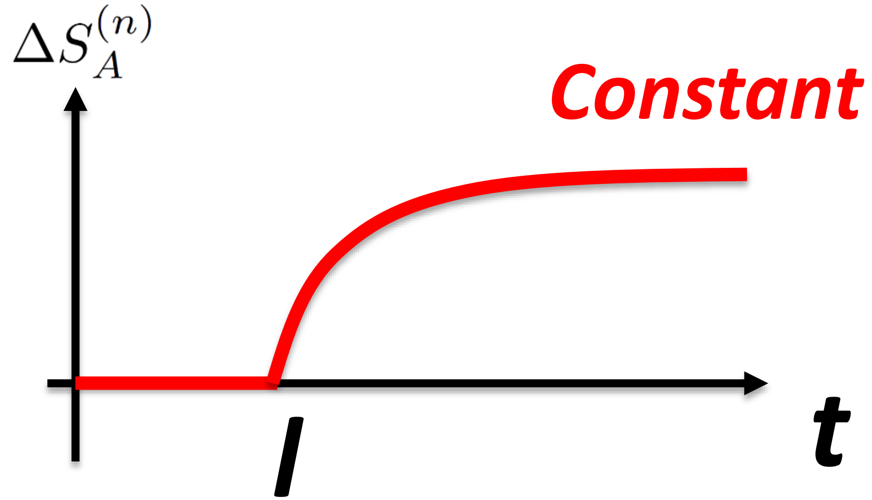
6. Holographic field theory

Free Field Theory

(Massless Scalar)

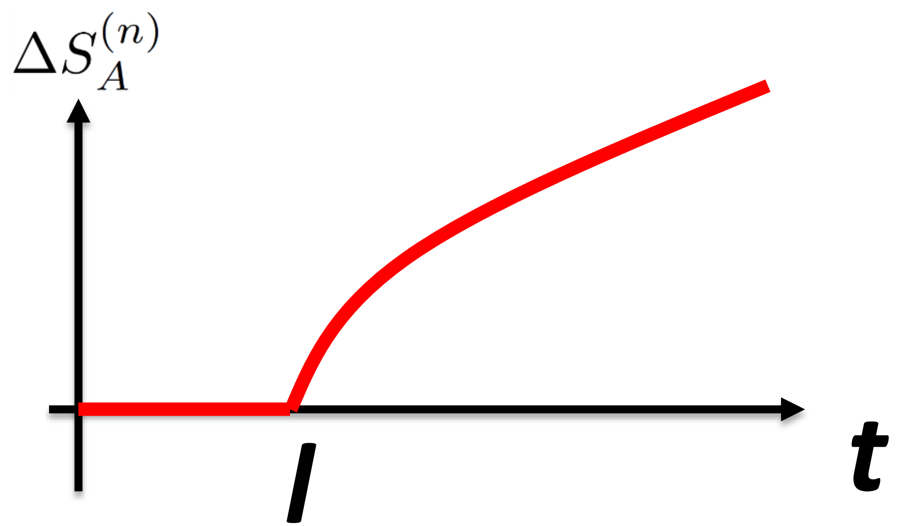
At the late time,

$$\boxed{\Delta S_A^{(n)} \rightarrow Constant.}$$



Holographic Field Theory

$$\boxed{\Delta S_A^{(n)} \sim \log t}$$

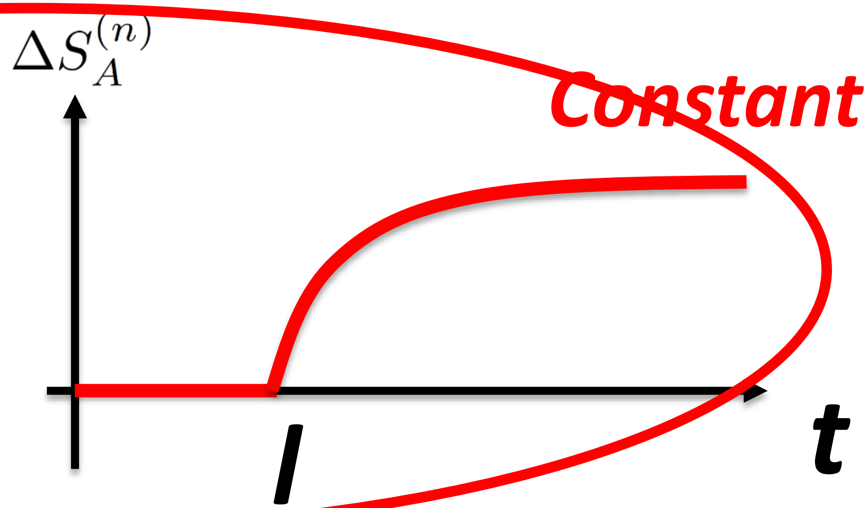


Free Field Theory

(Massless Scalar)

At the late time,

$$\Delta S_A^{(n)} \to Constant.$$



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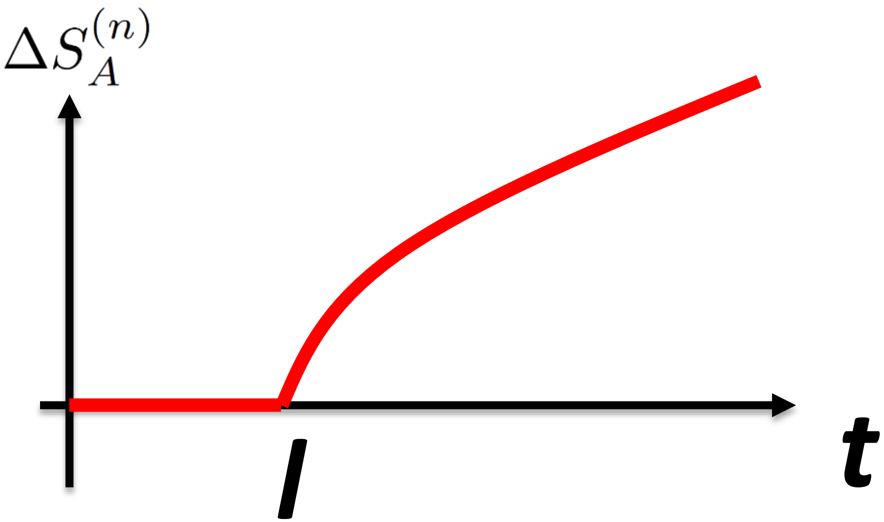
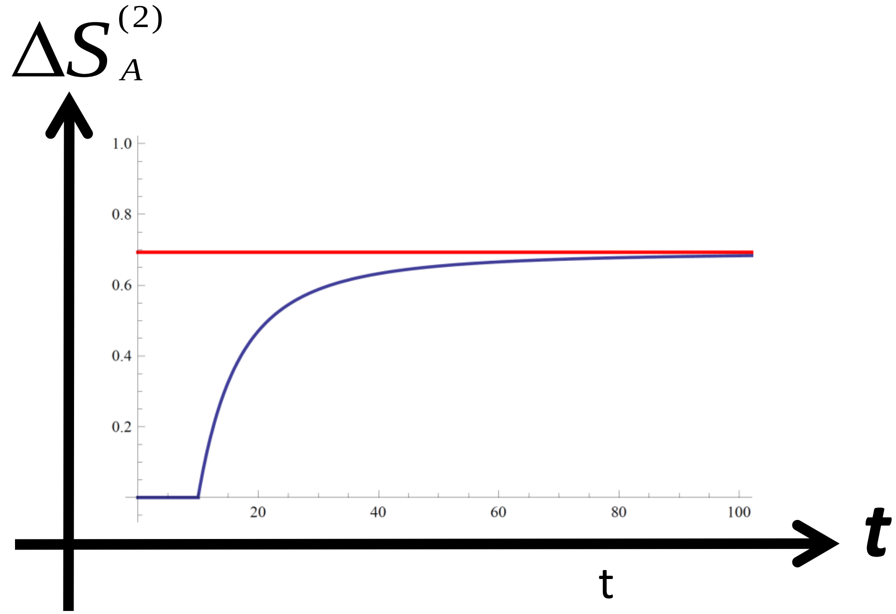


Free massless scalar

Example: $\mathcal{O} = \phi$



Free massless fermion

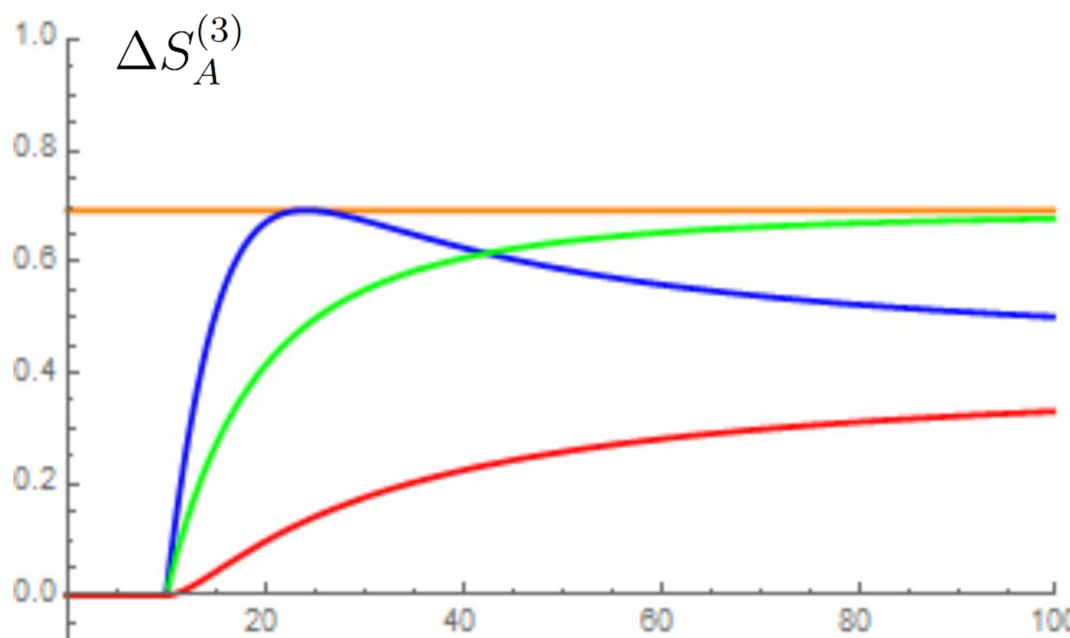
Example: $\mathcal{O} = \psi_a$

Red: $(\gamma^t \gamma^1)_{aa} = 1$

Green: $(\gamma^t \gamma^1)_{aa} = 0$

Blue: $(\gamma^t \gamma^1)_{aa} = -1$

Orange: Log2



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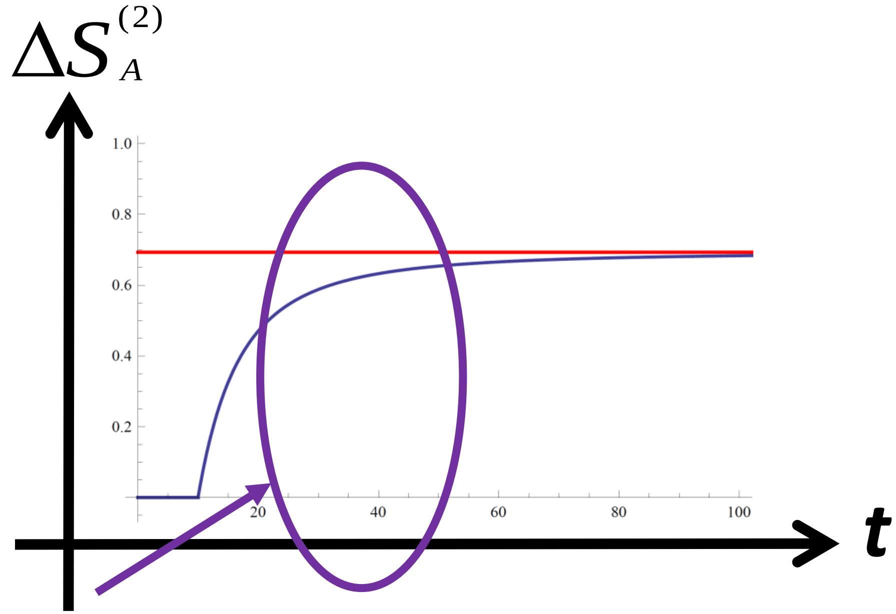

Depend on Theory and operators

Free massless fermion

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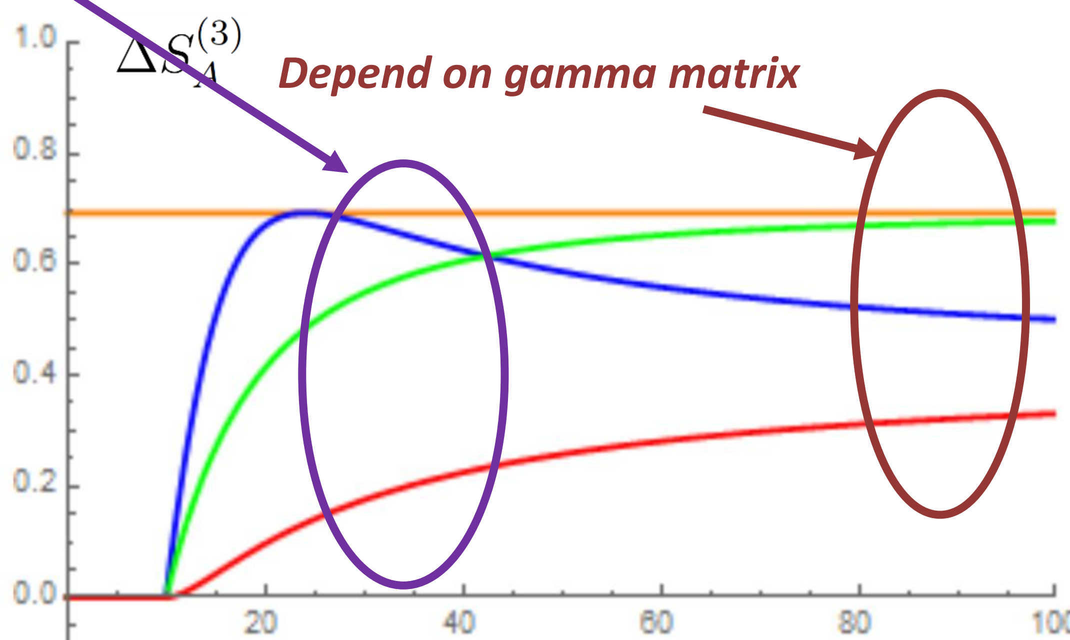

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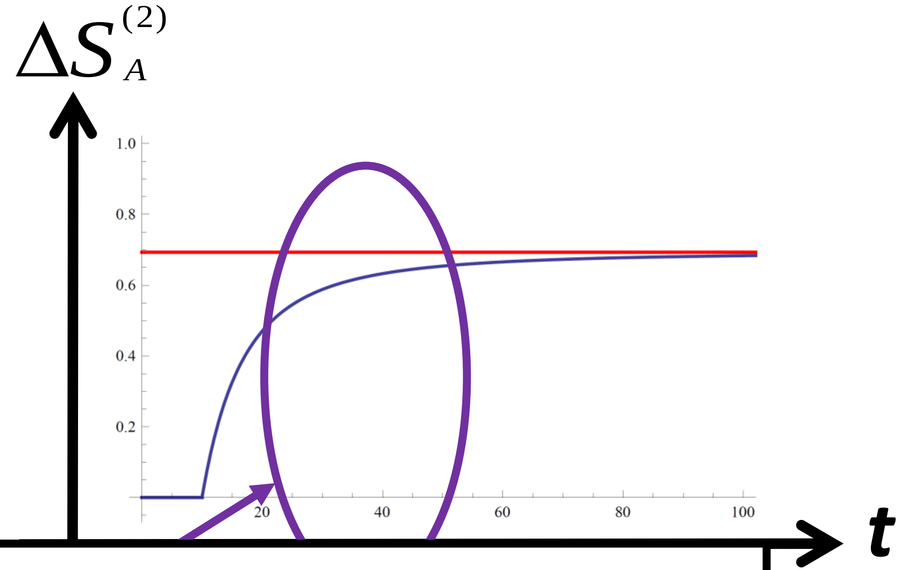

How about Gauge Theory ?

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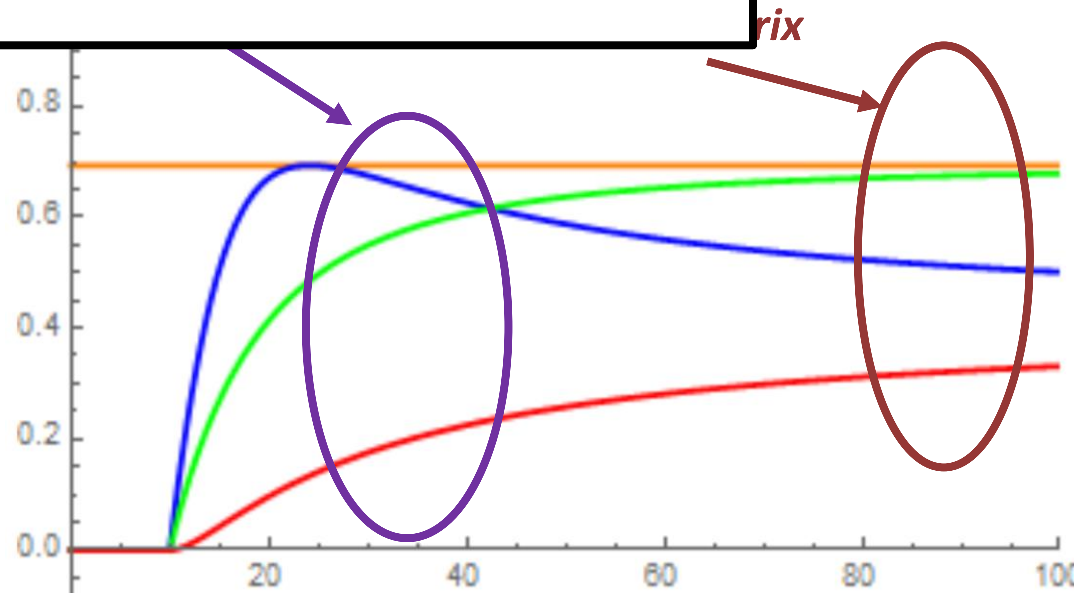

Quasi-Particle

Example: $\Delta S_A^{(n)}$ for ϕ

- Late time value: $\underline{\Delta S_A^{(n)} = \log 2}$

Result by Replica Trick

We assume that late time value comes from entanglement between quasi-particles

Quasi-Particle

We assume that late time value comes from entanglement between quasi-particles.

We define an effective reduced density matrix:

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left[\mathrm{tr}_A\left(\rho_A^e\right)^n\right]$$
$$= \frac{1}{1-n}\log\left[\mathrm{tr}_A\left(\hat{\mathcal{N}}^2\mathcal{O}\left|0\right\rangle\left\langle 0\right|\mathcal{O}^\dagger\right)^n\right]$$

For $\mathcal{O} = \phi$

Decomposition: $\phi = \phi_L^\dagger + \phi_R^\dagger + \phi_L + \phi_R$

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Decomposition: $\phi = \underline{\phi_L^\dagger} + \underline{\phi_R^\dagger} + \underline{\phi_L} + \underline{\phi_R}$

=Left mover which corresponds to the particles included in ***B***

=Right mover which corresponds to the particles included in ***A***

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For $\mathcal{O} = \phi$

Quantization: $\left[\phi_{L,R}, \ \phi_{L,R}^\dagger \right] = 1$

$$\rho_A = \frac{1}{2} diag(1,1)$$

Quasi-Particle

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For $\mathcal{O} = \phi$

Quantization: $\left[\phi_{L,R},\ \phi_{L,R}^\dagger\right] = 1$

$\Rightarrow$ $\rho_A = \frac{1}{2}diag(1,1)$ $\Rightarrow$ $\underline{\Delta S_A^{(n)} = \log 2}$

Consistent

Quasi-Particle

We define an effective reduced density matrix:

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\mathrm{tr}_A \left(\rho_A^e \right)^n \right]$$

$$= \frac{1}{\quad} \log \left[\mathrm{tr}_A \left(\hat{\mathcal{N}}^2 \mathcal{O} \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger \right)^n \right]$$

Fo

How about Gauge Theory ?

Quantization: $\left[\phi_{L,R}, \ \phi^\dagger_{L,R} \right] = 1$

$$\rho_A = \frac{1}{2} diag(1,1)$$

$$\underline{\Delta S_A^{(n)} = \log 2}$$

Consistent

Field Theory

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2. *U(N)* or *SU(N)* free massless scalar field theory in Large N limit

3. Free massless fermionic field theory

4. Charged Renyi Entanglement Entropy (CREE)

5. ***Maxwell Theory in 4d***

6. Holographic field theory

This work will appear on arXiv soon!
Collaborate with Naoki Watamura.

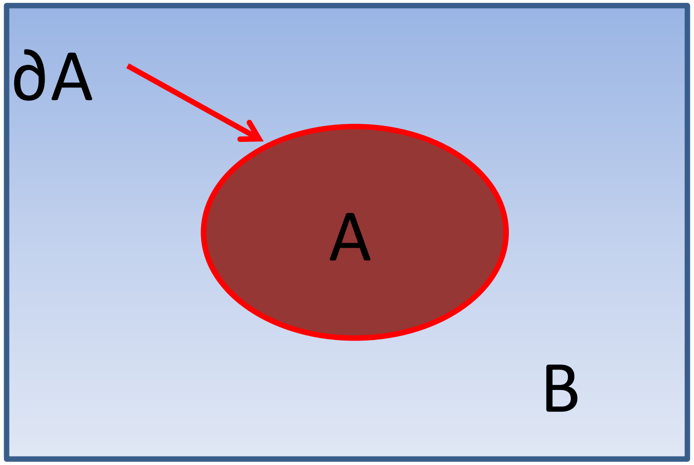
Our Claim

Subtleties : How to divide Hilbert space

⇒ Related to the D.O.F around the entangling surface

⇒ In terms of REE, the terms which depends on ***UV cutoff*** are related to the D.O.F around entangling surface

For $\Delta S_A^{(n)}$, these terms ***are subtracted*** .



on a certain time slice

Our Claim

This work will appear on arXiv soon!
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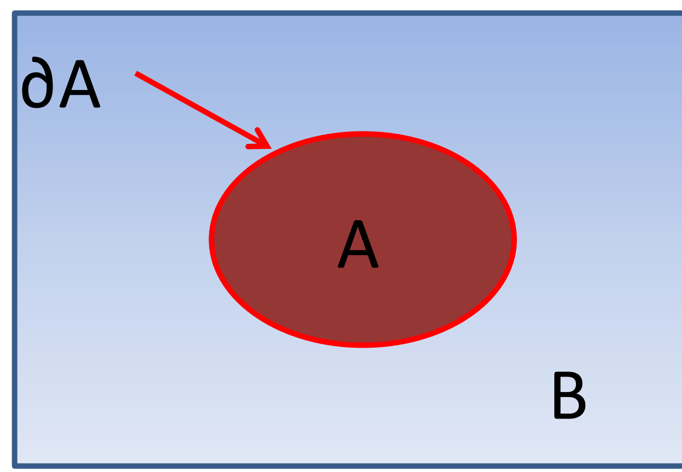
Subtleties : How to divide Hilbert space

Subtleties are negligible !!

In terms of REE, the terms which depends on ***UV cutoff*** are related to the D.O.F around entangling surface



on a certain time slice

For $\Delta S_A^{(n)}$, these terms ***are subtracted*** .

Results.1

We study $\Delta S_A^{(n)}$ for $|\Psi\rangle = \mathcal{N}\mathcal{O}(-t,x)\,|0\rangle$.

$$\mathcal{O} = E_i, B_i, FF, \mathbf{B}\cdot\mathbf{E}, \cdots$$

Time evolution of $\Delta S_A^{(n)}$ shows that ***it is invariant*** under the transformation:

$$E_i \to -B_i$$

$$B_i \to E_i$$

Results.1

Time evolution of $\Delta S_A^{(n)}$ shows that ***it is invariant*** under the transformation:

$$E_i \rightarrow -B_i$$

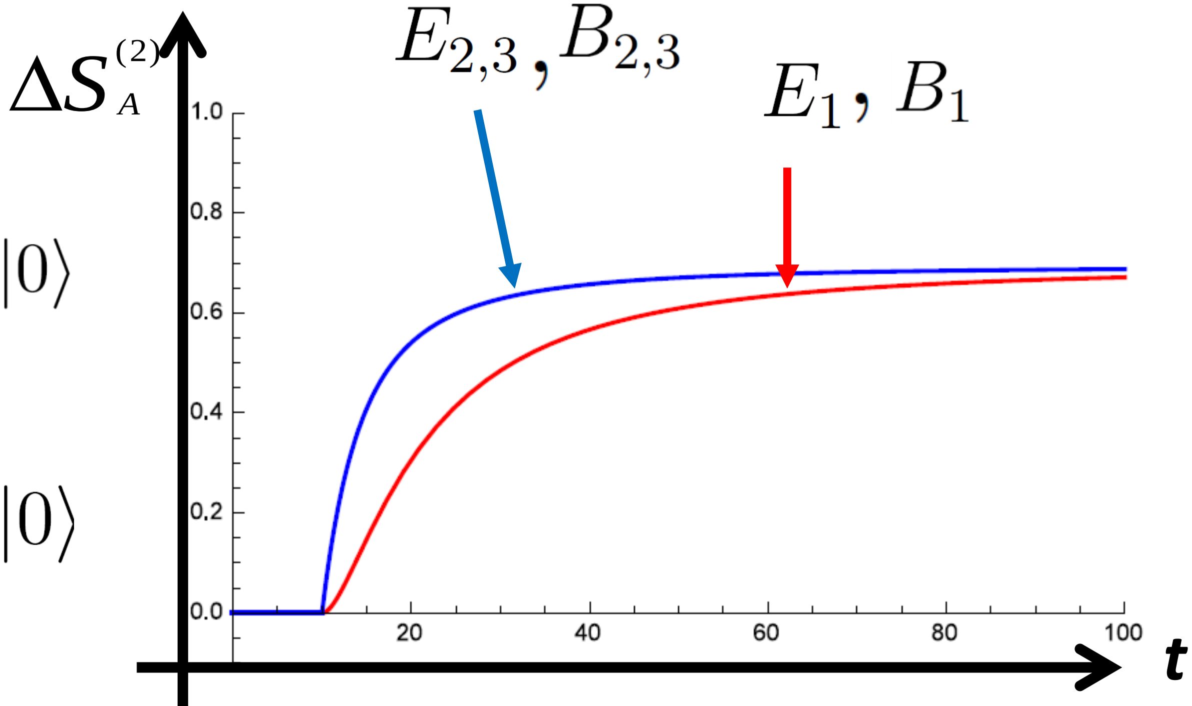
$$B_i \rightarrow E_i$$

Example:

$$|\Psi\rangle = \mathcal{N} E_i(-t,x)\,|0\rangle$$

or

$$\mathcal{N} B_i(-t,x)\,|0\rangle$$



Results.1

Tim

und

$$\Delta S_A^{(n)}$$

$$\boxed{\Delta S_A^{(2)} \text{ for } E_1(E_{2,3}) = \Delta S_A^{(2)} \text{ for } B_1(B_{2,3})}$$

$$E_i \to -B_i$$

$$B_i \to E_i$$

Example:

$$|\Psi\rangle = \mathcal{N} E_i(-t,x)\,|0\rangle$$

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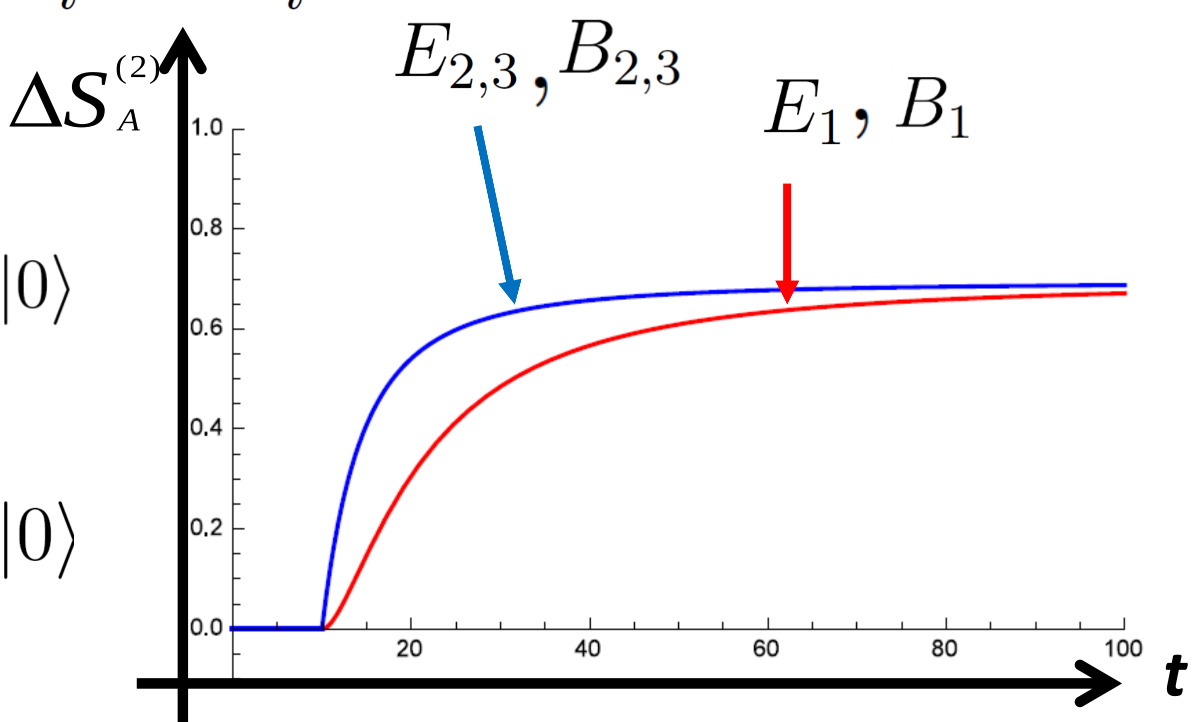


Results.1

Time evolution of $\Delta S_A^{(n)}$ shows that ***it is invariant*** under the transformation:

$$E_i \to -$$

$$B_i \to E$$

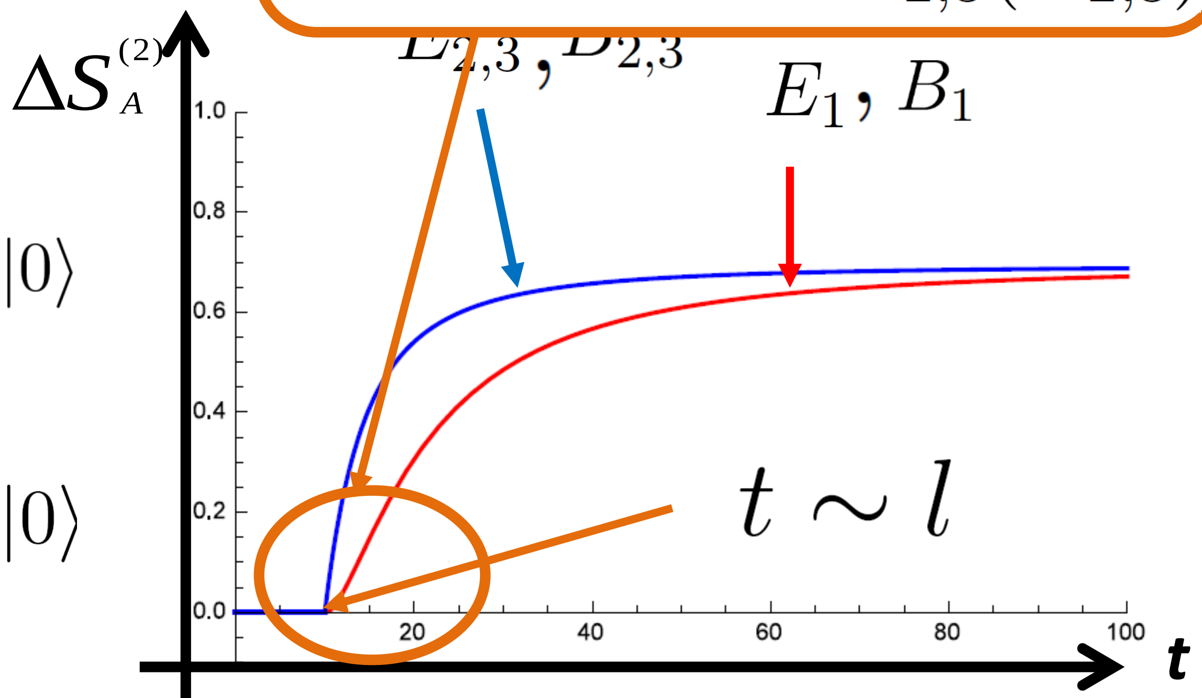
$\Delta S_A^{(2)}$ ***for*** $E_1(B_1)$ ***grows faster than*** $\Delta S_A^{(2)}$ ***for*** $E_{2,3}(B_{2,3})$

Example:

$$|\Psi\rangle = \mathcal{N} E_i(-t,x)\,|0\rangle$$

or

$$\mathcal{N} B_i(-t,x)\,|0\rangle$$



Results.1

Tim

und

$$\Delta S_A^{(2)} \text{ for } E_1(E_{2,3}) = \Delta S_A^{(2)} \text{ for } B_1(B_{2,3})$$

$E_i \to -$

$B_i \to B$

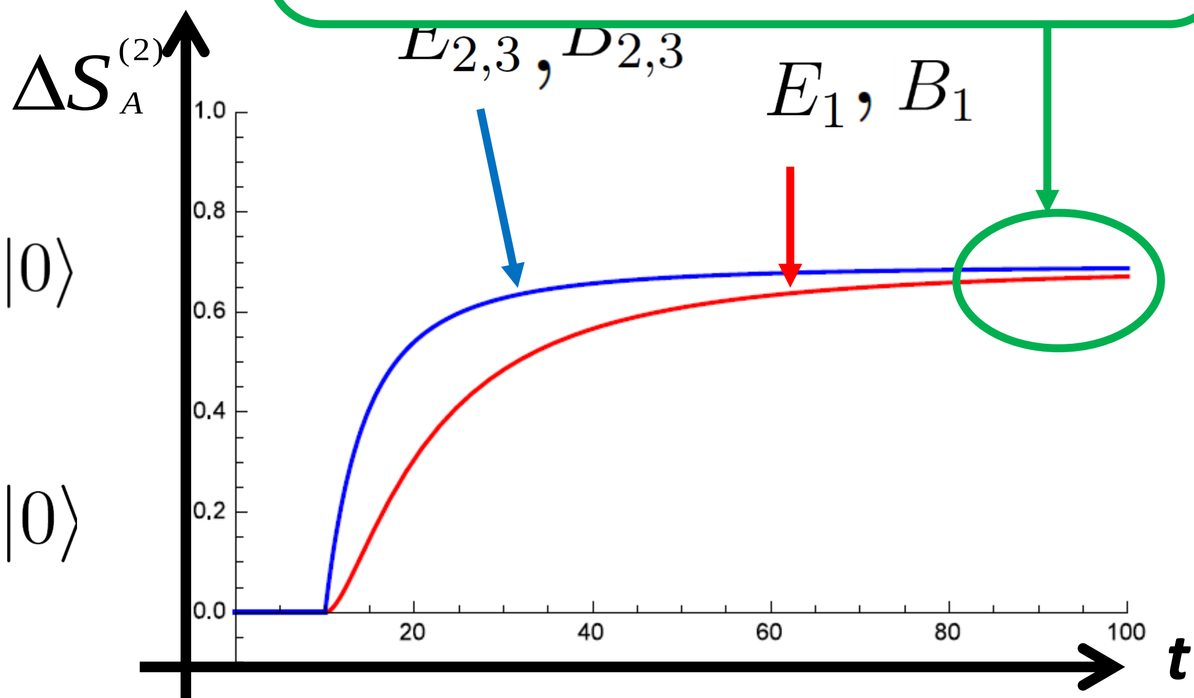
They approach to same number (log 2).

Example:

$$|\Psi\rangle = \mathcal{N} E_i(-t,x)\,|0\rangle$$

or

$$\mathcal{N} B_i(-t,x)\,|0\rangle$$



Results.1

Tim

und

$$\Delta S_A^{(2)} \text{ for } E_1(E_{2,3}) = \Delta S_A^{(2)} \text{ for } B_1(B_{2,3})$$

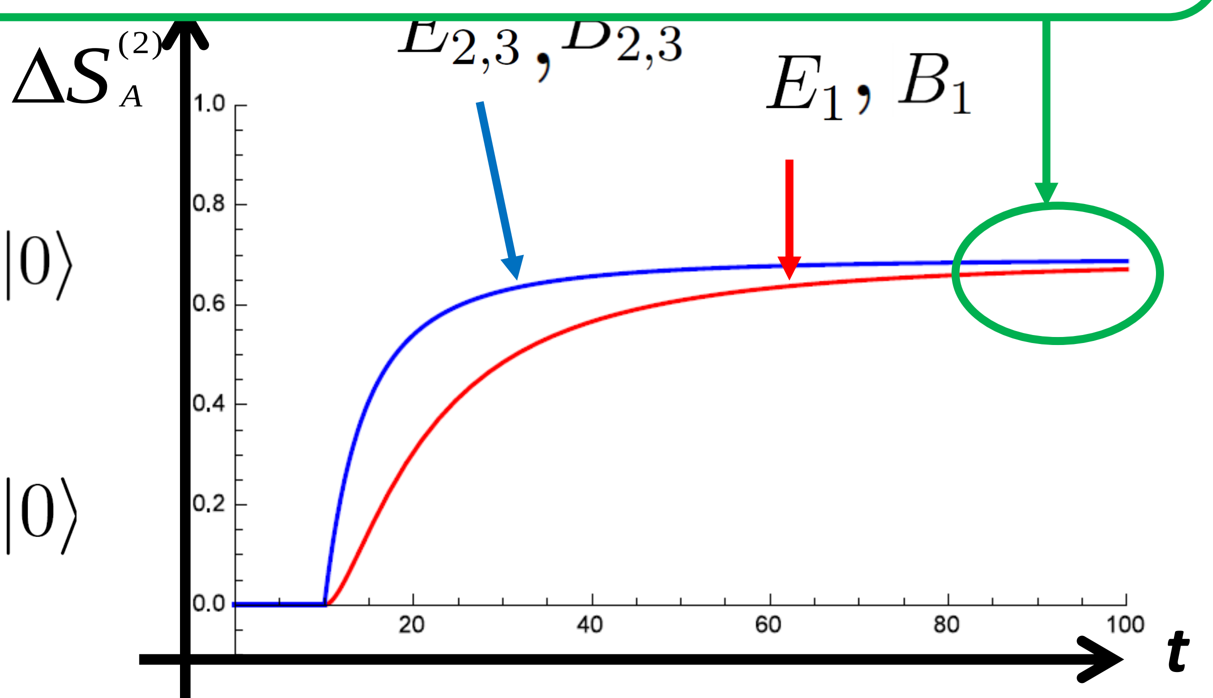
This can be interpreted in terms of scalar quasi-particles $\phi_L^\dagger, \phi_R^\dagger$

Example:

$$|\Psi\rangle = \mathcal{N} E_i(-t,x)\,|0\rangle$$

or

$$\mathcal{N} B_i(-t,x)\,|0\rangle$$



Results. 2 Quasi-particle Interpretation

The late time values of $\Delta S_A^{(n)}$ for operators such as $\mathbf{E}^2, \mathbf{B}^2, E_i E_j, B_i B_j \cdots$, which are ***not*** constructed of both $E_2(E_3)$ and $B_3(B_2)$ can be interpreted in terms of scalar quasi-particles.

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The late time values of $\Delta S_A^{(n)}$ for operators such as $\mathbf{E}^2, \mathbf{B}^2, E_iE_j, B_iB_j \cdots$, which are ***not*** constructed of both $E_2(E_3)$ and $B_3(B_2)$ can be interpreted in terms of scalar quasi-particles.

The late time values of $\Delta S_A^{(n)}$ for operators such as $FF, \mathbf{B}\cdot\mathbf{E}, \cdots$, which are constructed of both $E_2(E_3)$ and $B_3(B_2)$ ***can not be interpreted in terms of scalar quasi-particles***.

Results. 2 Quasi-particle Interpretation

The late time values of $\Delta S_A^{(n)}$ for operators such as $\mathbf{E}^2, \mathbf{B}^2, E_iE_j, B_iB_j \cdots$, which are ***not*** con

We need electromagnetic quasi-particles

inte

The late time values of $\Delta S_A^{(n)}$ for operators such as $FF, \mathbf{B}\cdot\mathbf{E}, \cdots$, which are constructed of both $E_2(E_3)$ and $B_3(B_2)$ ***can not be interpreted in terms of scalar quasi-particles***.

Results. 2 Quasi-particle Interpretation

Example: Late time value of $\Delta S_A^{(n)}$ for FF

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\mathrm{tr}_A \left(\rho_A^e \right)^n \right]$$

$$\rho_A^e = \frac{1}{192} diag \left(30, 30, 16, 16, 49, 49, 1, 1 \right)$$

Results. 2 Quasi-particle Interpretation

$$E_i = \underline{E_i^{L\dagger}} + \underline{E_i^{R\dagger}} + \underline{E_i^{L}} + \underline{E_i^{R}}\,,$$

$$B_i = \underline{B_i^{L\dagger}} + \underline{B_i^{R\dagger}} + \underline{B_i^{L}} + \underline{B_i^{R}}$$

=Left mover which corresponds to the quasi-particles included in ***B***

=Right mover which corresponds to the quasi-particles included in ***A***

$$E_i^{L,R}\,|0\rangle_{L,R} = B_i^{L,R}\,|0\rangle_{L,R} = 0,$$

$$|0\rangle = |0\rangle_L \otimes |0\rangle_R\,.$$

Quantization

$$\boxed{\begin{aligned}\left[E_i^{L,R}, E_j^{L,R\dagger}\right] &= C\delta_{ij}, \\ \left[B_i^{L,R}, B_j^{L,R\dagger}\right] &= C\delta_{ij},\end{aligned}}$$

C is real number.

same as scalar quasi-particles.

Quantization

$$\left[E_i^{L,R}, E_j^{L,R\dagger}\right] = C\delta_{ij},$$

$$\left[B_i^{L,R}, B_j^{L,R\dagger}\right] = C\delta_{ij},$$

C is real number.

$$\left[E_3^{L,R}, B_2^{L,R\dagger}\right] = X_{R,L}, \quad \left[E_2^{L,R}, B_3^{L,R\dagger}\right] = Y_{R,L},$$

$$X_R = -X_L = Y_L = -Y_R,$$

$$X_{R,L}^2 = Y_{R,L}^2 = \frac{9}{16}C^2.$$

Quantization

$$\left[E_i^{L,R}, E_j^{L,R\dagger}\right] = C\delta_{ij},$$

$$\left[B_i^{L,R}, B_j^{L,R\dagger}\right] = C\delta_{ij},$$

Feature of gauge field

C is real number.

$$\left[E_3^{L,R}, B_2^{L,R\dagger}\right] = X_{R,L}, \quad \left[E_2^{L,R}, B_3^{L,R\dagger}\right] = Y_{R,L},$$

$$X_R = -X_L = Y_L = -Y_R,$$

$$X_{R,L}^2 = Y_{R,L}^2 = \frac{9}{16}C^2.$$

Quantization

$$\left[E_i^{L,R}, E_j^{L,R\dagger}\right] = C\delta_{ij},$$

Electromagnetic fields can have the effect on the late-time structure of entanglement ***differently from*** scalar fields.

C is real number.

$$\left[E_3^{L,R}, B_2^{L,R\dagger}\right] = X_{R,L}, \quad \left[E_2^{L,R}, B_3^{L,R\dagger}\right] = Y_{R,L},$$

$$X_R = -X_L = Y_L = -Y_R,$$
$$X_{R,L}^2 = Y_{R,L}^2 = \frac{9}{16}C^2.$$

Summary

- We check that $\Delta S_A^{(n)}$ respects electric-magnetic duality.
- $\Delta S_A^{(n)}$ for the operators constructed of both electric and magnetic fields can have the effect on the late-time structure of quantum entanglement differently from scalar operator.

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Future directions

▪ Weak Interacting F.T.

▪ Non-relativistic case

▪ Maxwell Theory in general d

▪ Non-local Operator

Propagator-Probability Correspondence

Density matrix:

$$\rho(t) = e^{-iHt} e^{-\epsilon H} \mathcal{O}(x_i) |0\rangle \langle 0| \mathcal{O}(x_i) e^{-\epsilon H} e^{iHt}$$

Smearing Parameter

In the path-integral formalism:

$$\Delta S_A^{(n)} =$$

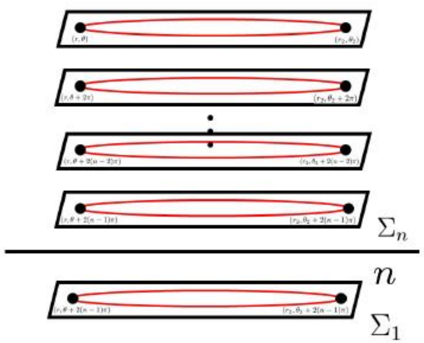
$$\frac{1}{1-n} \Big(\log \left\langle \mathcal{O}^\dagger(r_2, \theta_{2,n}) \mathcal{O}(r_1, \theta_{1,n}) \cdots \mathcal{O}^\dagger(r_2, \theta_{2,1}) \mathcal{O}(r_1, \theta_{1,1}) \right\rangle_{\Sigma_n}$$

$$- n \log \left\langle \mathcal{O}^\dagger(r_2, \theta_{2,1}) \mathcal{O}(r_1, \theta_1) \right\rangle_{\Sigma_1} \Big) .$$

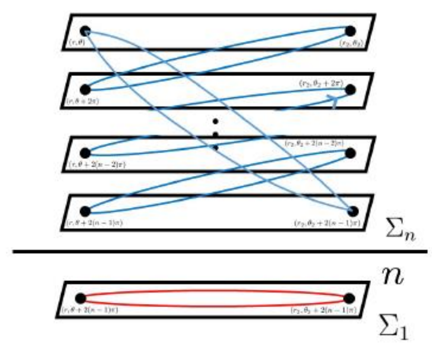
$$\mathcal{O} = \phi^2$$

We take $\varepsilon \to 0$.

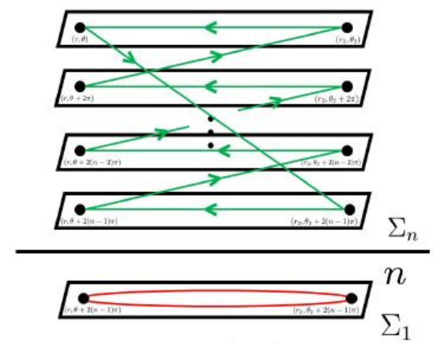
Only three diagrams contribute at the leading order :



$P_1(t)^n$



$P_2(t)^n$



$P_3(t)^n$

$$P_1(t) = \frac{(G(\theta_1 - \theta_2))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

$$P_2(t) = \frac{(G(\theta_1 - \theta_2 + 2\pi))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2) + 2\pi)^2}$$

$$P_3(t) = \frac{2G(\theta_1 - \theta_2)G(\theta_1 - \theta_2 + 2\pi)}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

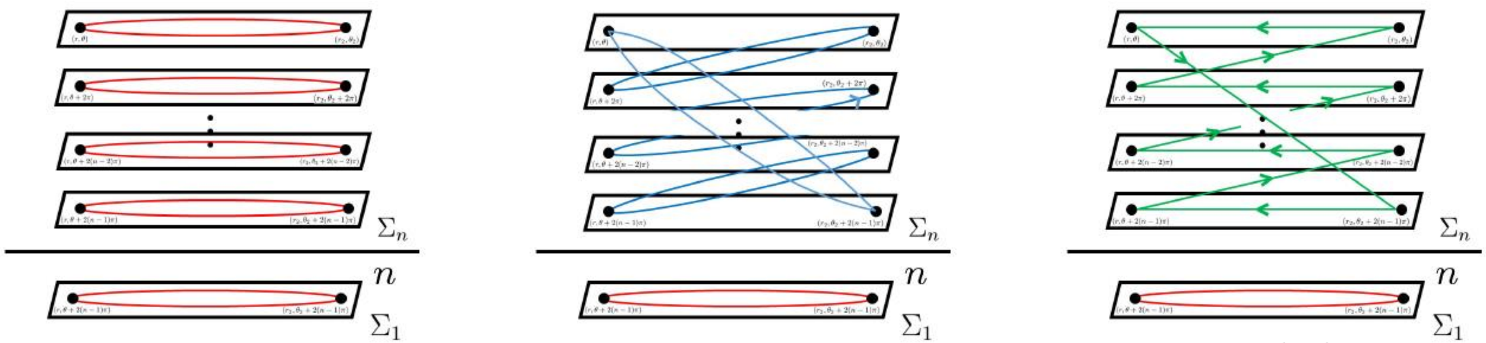
$$G(\theta_1 - \theta_2) = \frac{t+l}{32\pi^2 t \epsilon^2}$$

$$G(\theta_1 - \theta_2 + 2\pi) = \frac{t-l}{32\pi^2 t \epsilon^2}$$

$\mathcal{O} = \phi^2$

We take $\varepsilon \to 0$.

Only three diagrams contribute at the leading order :



$P_1(t)^n$ $\quad$ $P_2(t)^n$ $\quad$ $P_3(t)^n$

$$P_1(t) = \frac{(G(\theta_1 - \theta_2))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

$$P_2(t) = \frac{(G(\theta_1 - \theta_2 + 2\pi))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2) + 2\pi)^2}$$

$$P_3(t) = \frac{2G(\theta_1 - \theta_2)G(\theta_1 - \theta_2 + 2\pi)}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

$$\boxed{P_1(t) + P_2(t) + P_3(t) = 1}$$

Quasi-particles

- Decomposition:

$$\phi(-t,-l,\mathbf{x}) = \phi^{L\dagger}(-t,-l,\mathbf{x}) + \phi^{R\dagger}(-t,-l,\mathbf{x}) + \phi^{L}(-t,-l,\mathbf{x}) + \phi^{R}(-t,-l,\mathbf{x})$$

- C

$$\boxed{P_1 = \frac{f_L^2}{(f_L+f_R)^2}, P_2 = \frac{f_R^2}{(f_L+f_R)^2}, P_3 = \frac{2f_L\cdot f_R}{(f_L+f_R)^2}}$$

$$[\phi^{L}$$

$$[\phi^{R}(-t,-l,\mathbf{x}), \phi^{R\dagger}(-t,-l,\mathbf{x})] = f_R,$$

- Reduced Density matrix:

$$\rho_A(t) = \mathrm{tr}_B\left(\hat{\mathcal{N}}^2\phi^2(-t,-l,\mathbf{x})\left|0\right\rangle\left\langle 0\right|\phi^2(-t,-l,\mathbf{x})\right)$$
$$= P_1(t)\left|0\right\rangle_R\left\langle 0\right|_R + P_2(t)\left|\phi^2\right\rangle_R\left\langle\phi^2\right|_R + P_3(t)\left|\phi\right\rangle_R\left\langle\phi\right|_R$$

Compare

$$P_1(t) = \frac{(G(\theta_1 - \theta_2))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

$$P_2(t) = \frac{(G(\theta_1 - \theta_2 + 2\pi))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2) + 2\pi)^2}$$

$$P_3(t) = \frac{2G(\theta_1 - \theta_2)G(\theta_1 - \theta_2 + 2\pi)}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

=

$$P_1 = \frac{f_L^2}{(f_L + f_R)^2}$$

$$P_2 = \frac{f_R^2}{(f_L + f_R)^2}$$

$$P_3 = \frac{2f_L \cdot f_R}{(f_L + f_R)^2}$$

Compare

$$P_1(t) = \frac{(G(\theta_1 - \theta_2))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2 + 2\pi)^2}$$

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$$P_1 = \frac{f_L^2}{(f_L + f_R)^2}$$

$$P_2 = \frac{f_R^2}{(f_L + f_R)^2}$$

$$P_3 = \frac{2 f_L \cdot f_R}{(f_L + f_R)^2}$$

$$\left[\phi^L(-t, -l, \mathbf{x}), \phi^{L\dagger}(-t, -l, \mathbf{x})\right] = G(\theta_1 - \theta_2),$$

$$\left[\phi^R(-t, -l, \mathbf{x}), \phi^{R\dagger}(-t, -l, \mathbf{x})\right] = G(\theta_1 - \theta_2 + 2\pi)$$

Compare

$$P_1(t) = \frac{(G(\theta_1-\theta_2))^2}{(G(\theta_1-\theta_2)+G(\theta_1-\theta_2)+2\pi)^2}$$
$$P_2(t) = \frac{(G(\theta_1-\theta_2+2\pi))^2}{(G(\theta_1-\theta_2)+G(\theta_1-\theta_2)+2\pi)^2}$$
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=

$$P_1 = \frac{f_L^2}{(f_L+f_R)^2}$$
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$$\left[\phi^L(-t,-l,\mathbf{x}),\phi^{L\dagger}(-t,-l,\mathbf{x})\right] = G(\theta_1-\theta_2),$$
$$\left[\phi^R(-t,-l,\mathbf{x}),\phi^{R\dagger}(-t,-l,\mathbf{x})\right] = G(\theta_1-\theta_2+2\pi)$$

$t \to \infty$

Late time algebra: $\left[\phi_{L,R},\ \phi^\dagger_{L,R}\right] = 1$

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$$P_1(t) = \frac{(G(\theta_1 - \theta_2))^2}{(G(\theta_1 - \theta_2) + G(\theta_1 - \theta_2) + 2\pi)^2}$$
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=

$$P_1 = \frac{f_L^2}{(f_L + f_R)^2}$$
$$P_2 = \frac{f_R^2}{(f_L + f_R)^2}$$

In the same manner,
we derive ***the late-time algebra for gauge theory*** .

$$[\phi^L(-t, -l, \mathbf{x}), \phi^{L\dagger}(-t, -l, \mathbf{x})] = G(\theta_1 - \theta_2),$$
$$[\phi^R(-t, -l, \mathbf{x}), \phi^{R\dagger}(-t, -l, \mathbf{x})] = G(\theta_1 - \theta_2 + 2\pi)$$

$t \to \infty$

Late time algebra: $\left[\phi_{L,R},\ \phi^\dagger_{L,R}\right] = 1$

$$\left[E_i^L(-t,-l,\mathbf{x}),E_j^{L\dagger}(-t,-l,\mathbf{x})\right]=\pi^2\epsilon^4F_{EiEj}^{(n)}\left(\theta_1-\theta_2\right),$$

$$\left[E_i^R(-t,-l,\mathbf{x}),E_j^{R\dagger}(-t,-l,\mathbf{x})\right]=\pi^2\epsilon^4F_{EiEj}^{(n)}\left(\theta_1-\theta_2-2\pi\right)$$

$$\left[B_i^L(-t,-l,\mathbf{x}),B_j^{L\dagger}(-t,-l,\mathbf{x})\right]=\pi^2\epsilon^4F_{BiBj}^{(n)}\left(\theta_1-\theta_2\right),$$

$$\left[B_i^R(-t,-l,\mathbf{x}),B_j^{R\dagger}(-t,-l,\mathbf{x})\right]=\pi^2\epsilon^4F_{BiBj}^{(n)}\left(\theta_1-\theta_2-2\pi\right)$$

$$\left[E_i^L(-t,-l,\mathbf{x}),B_j^{L\dagger}(-t,-l,\mathbf{x})\right]=\pi^2\epsilon^4F_{EiBj}^{(n)}\left(\theta_1-\theta_2\right),$$

$$\left[E_i^R(-t,-l,\mathbf{x}),B_j^{R\dagger}(-t,-l,\mathbf{x})\right]=\pi^2\epsilon^4F_{BiEj}^{(n)}\left(\theta_1-\theta_2-2\pi\right)$$