Renyi-Shannon Entropy and Boundary Field Theory

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Quantum Matter and Information?

Quantum Condensed Matter Physics

Realization of quantum computation/ quantum information processing

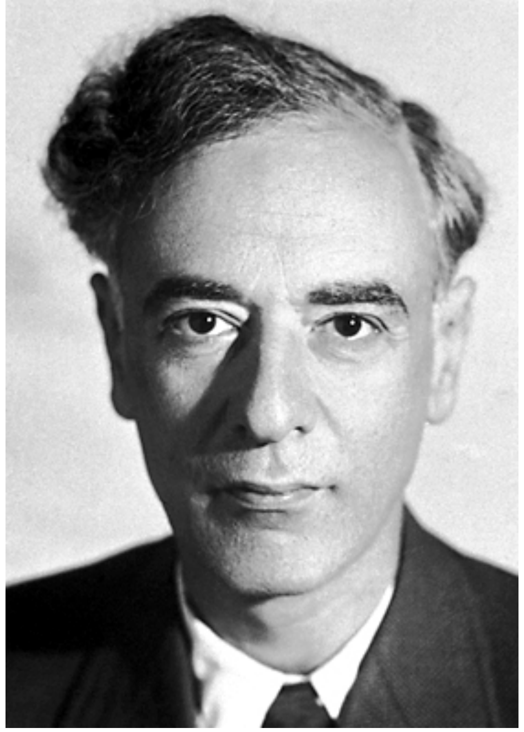
New understanding/ formulation in foundation

Efficient numerical algorithm

(Quantum) Information

How to distinguish phases?

Different orders (or their absence) characterize each phase
— what is “order”?



Landau

Ferromagnet: magnetic order

spontaneously breaks

Z_2 symmetry (Ising model),

SU(2) symmetry (Heisenberg model)…..

Superfluid: off-diagonal long-range order

spontaneously breaks U(1) symmetry

“order” $\supseteq$ Spontaneous Symmetry Breaking

Beyond Landau Paradigm

Many quantum phases, which are distinct but cannot be characterized by a (conventional) SSB have been found

"topological phases"

- quantum Hall states
- Haldane gap phase
- topological insulators/topological superconductors

……

How to characterize them? new tools will be useful!

→ **"information theoretic" measures**

e.g. entanglement entropy

In this talk, I will discuss an
"information theoretic" measure of a quantum state which is different from (but also related to) entanglement entropy.

Based on a collaboration with
Grégoire Misguich and Vincent Pasquier,
also on several earlier works

Rényi-Shannon Entropy

$$|\psi\rangle = \sum_i c_i |i\rangle$$

Define Rényi/von Neumann entropy from the prob. dist.

$$p_i = |c_i|^2$$

$$S_n = \frac{1}{1-n} \log \left(\sum_i {p_i}^n \right)$$

$$S_{\mathrm{vN}} = -\sum_i p_i \log p_i$$
$$= \lim_{n \to 1} S_n$$

This DOES depend on the choice of the basis

Anyway, Rény-Shannon Entropy can be defined for a given quantum state (and choice of basis)

can be used as a new characterization??

XXZ chain in 1+1D

$$\mathcal{H} = \sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z$$

The ground state is critical (gapless) for

$-1 \leqq \Delta < 1$

Effective theory: Tomonaga-Luttinger liquid

(free boson field theory in 1+1D)

$$\mathcal{L} = \frac{g}{4\pi}(\partial_\mu \phi)^2. \qquad \phi \sim \phi + 2\pi R$$

Bethe Ansatz exact solution →

$$\sqrt{g}R = \sqrt{\frac{1}{2\pi}\left(1 - \frac{1}{\pi}\cos^{-1}\Delta\right)}$$

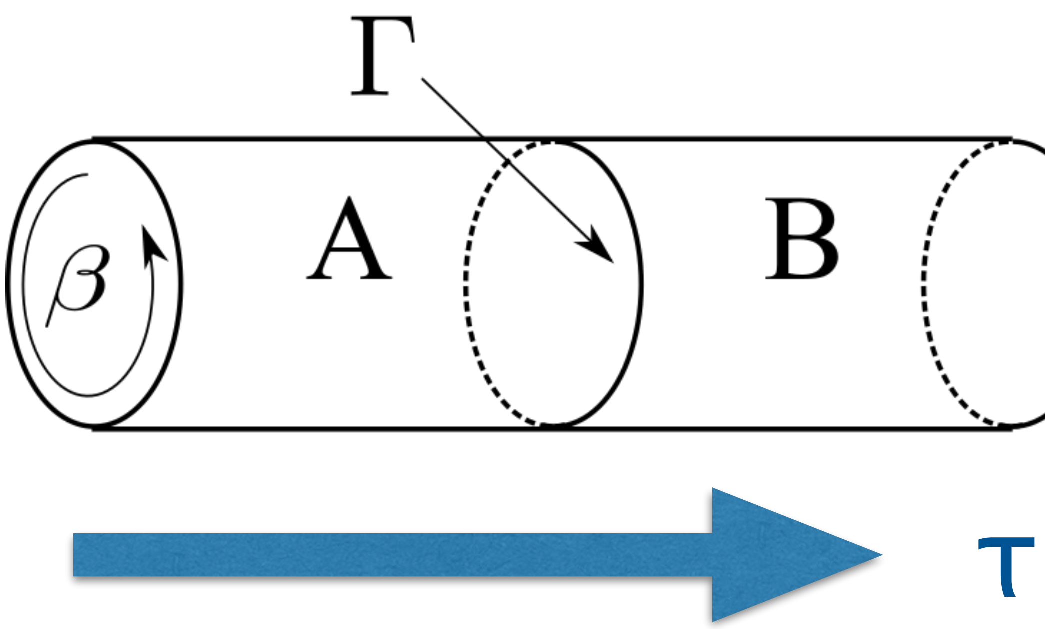
Rényi-Shannon Entropy

of the S=1/2 XXZ chain in the S^z basis:

$$S_j^z \sim \frac{1}{2\pi R}\partial_x \phi + \text{const.}(-1)^j \cos\frac{\phi}{R}$$

S^z basis ⇔ Φ-basis

$$p_i = \frac{1}{z_F}\int \mathcal{D}\phi e^{-S[\phi]}\Big|_{\phi(\tau=0,x)=\phi^{(b)}(x)}$$



$$S_n = \frac{1}{1-n}\log\left[\left(\frac{1}{z_F}\right)^n \int \mathcal{D}\phi^{(b)}(x) \int \prod_{j=1}^{n} \mathcal{D}\phi_j e^{-\sum_j S[\phi_j]}\Big|_{\phi_j(\tau=0,x)=\phi^{(b)}(x)}\right]$$

RSE of XXZ chain

Stephan-Furukawa-Misguich-Pasquier 2009, MO 2010

$$S_n = \frac{1}{1-n}\log\left[\left(\frac{1}{z_F}\right)^n \int \mathcal{D}\phi^{(b)}(x) \int \prod_{j=1}^{n} \mathcal{D}\phi_j e^{-\sum_j S[\phi_j]}\Big|_{\phi_j(\tau=0,x)=\phi^{(b)}(x)}\right]$$

$$= \frac{1}{1-n}\log\left[\left(\frac{1}{z_F}\right)^n \int \prod_{j=1}^{n} \mathcal{D}\phi_j e^{-\sum_j S[\phi_j]}\Big|_{\phi_1(\tau=0,x)=\phi_2(\tau=0,x)=\phi_3(\tau=0,x)=\dots}\right]$$

$$= \text{const.} \times \text{area} \boxed{+ \log(\sqrt{2g}R) + \frac{\log n}{2(1-n)}}$$

universal

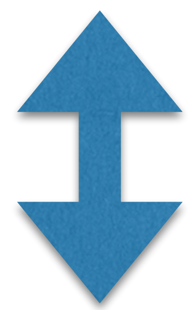
non-universal

"area (=system length) law"

Rényi-Shannon entropy does contain the universal characteristics of the quantum state (TL liquid)

Relation to EE

Entanglement Entropy of "Quantum Lifschitz" state in
2 spatial dimensions



RSE of free boson field theory in 1+1 dimensions
(=Tomonaga-Luttinger Liquid / S=1/2 XXZ chain)

Quantum Lifshitz Field Theory

Ardonne-Fendley-Fradkin 2004

$$\mathcal{S} = \int d^3x \left[\frac{1}{2} (\partial_t \varphi)^2 - \frac{\kappa^2}{2} (\nabla^2 \varphi)^2 \right]$$

e.g. critical point of quantum dimer model on a square lattice

$$H = \int d^2x \, Q^\dagger(\vec{x}) Q(\vec{x})$$

$$Q(x) \equiv \frac{1}{\sqrt{2}} \left(\frac{\delta}{\delta\varphi} + \kappa \, \nabla^2 \varphi \right)$$

Groundstate wavefunction

$$\Psi_0[\varphi] = \frac{1}{\sqrt{\mathcal{Z}}} e^{-\frac{\kappa}{2} \int d^2x \, (\nabla \varphi(x))^2}$$

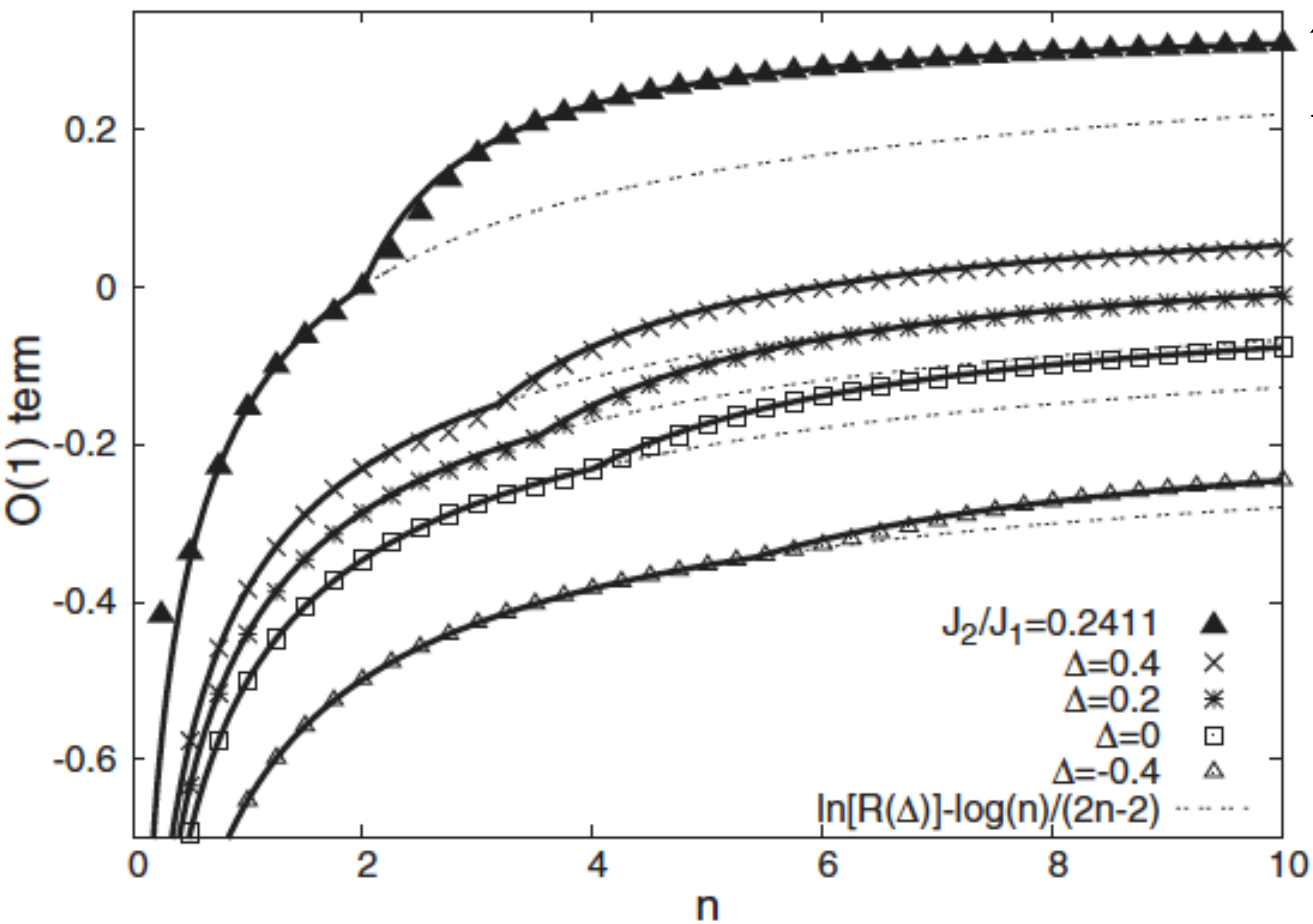
EE of this state in replica formalism

= same formula as RSE of XXZ chain (TL liquid)

Fradkin-Moore 2006, Hsu-Mulligan-Fradkin-Kim 2009

Phase Transition in RSE

Stephan-Misguich-Pasquier 2011



numerical

theory
(discussed earlier)

discrepancy
for $n>n_c$?

Boundary Perturbations

The effective field theory, in general, contains
all the possible perturbations which are not forbidden by the symmetries

Bulk: all the perturbations are irrelevant in the gapless regime, irrespective of the Renyi parameter *n*

Boundary: *n* replica fields are coupled, and we need to consider possible perturbations w.r.t. "center of mass" field

$$\Phi_0 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi_j$$

Boundary Perturbations

compactification of individual fields: $\phi_j \sim \phi_j + 2\pi R$

replica condition at the boundary: $\phi_1 = \phi_2 = \ldots = \phi_n$

effective compactification of the c.o.m. field

$$\Phi_0 = \frac{1}{\sqrt{n}} \sum_j \phi_j \sim \Phi_0 + 2\pi\sqrt{n}R$$

Possible boundary perturbations:

$$\cos\frac{m\Phi_0}{\sqrt{n}R} \qquad \sin\frac{m\Phi_0}{\sqrt{n}R} \qquad m = 1, 2, 3, \ldots$$

scaling dim. $x = \dfrac{m^2}{ngR^2}$ relevant if < 1

Boundary Phase Transition

m=1 : forbidden by the translation symmetry

$$S_j^z \sim \frac{1}{2\pi R}\partial_x \phi + \mathrm{const.}(-1)^j \cos\frac{\phi}{R}$$

m=2: most relevant perturbation in generic case

relevant if $n > n_c = \frac{2}{gR^2}$

SU(2) AF Heisenberg (Δ=1): $gR^2 = 1$ $n_c = 2$

XY (Δ=0): $gR^2 = \frac{1}{2}$ $n_c = 4$

RSE above n_c

$\cos \frac{2\Phi_0}{\sqrt{n}R}$ relevant⇒ boundary condition is

Φ=const. i.e. “Néel state”

$$S_{n>n_c} = \frac{1}{1-n} \log\left[2(p_{max})^n\right]$$

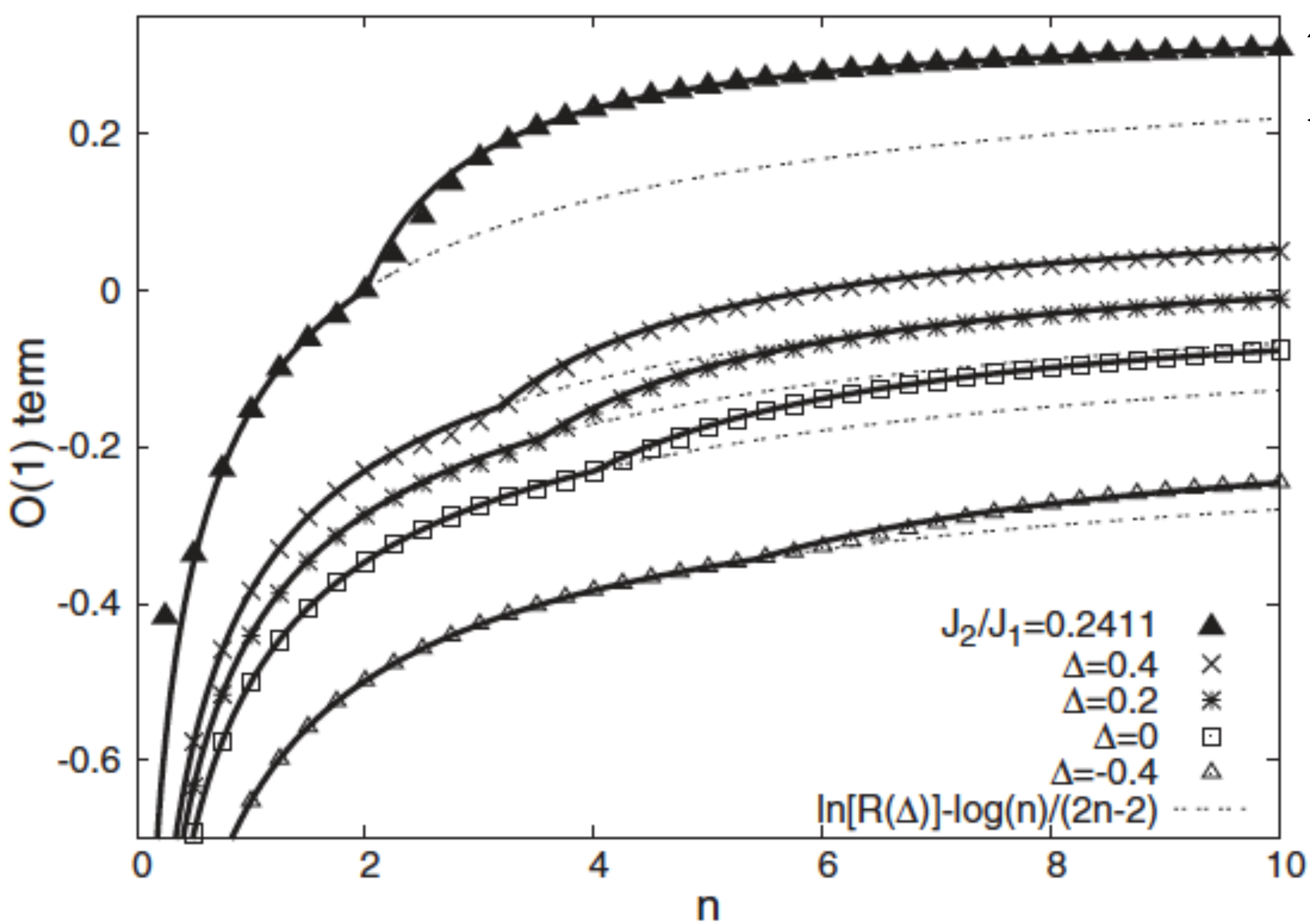
$$\begin{aligned} p_{max} &= |\langle + - + - + - \ldots |\Psi_0\rangle|^2 \\ &= \frac{Z_D}{Z} = \frac{1}{\sqrt{2g}R} \end{aligned}$$

Phase Transition in RSE

Stephan-Misguich-Pasquier 2011

theory
$n>n_c$

theory
$n<n_c$



RSE of XXZ model in 2+1D

$$\mathcal{H} = \sum_{\langle j,k \rangle} S_j^x S_k^x + S_j^y S_k^y + \Delta S_j^z S_k^z$$

On a bipartite lattice (square etc.) there is no geometric frustration; the ground state has a long-range (Néel) order

$|\Delta|>1$: Z_2 symmetry is spontaneously broken

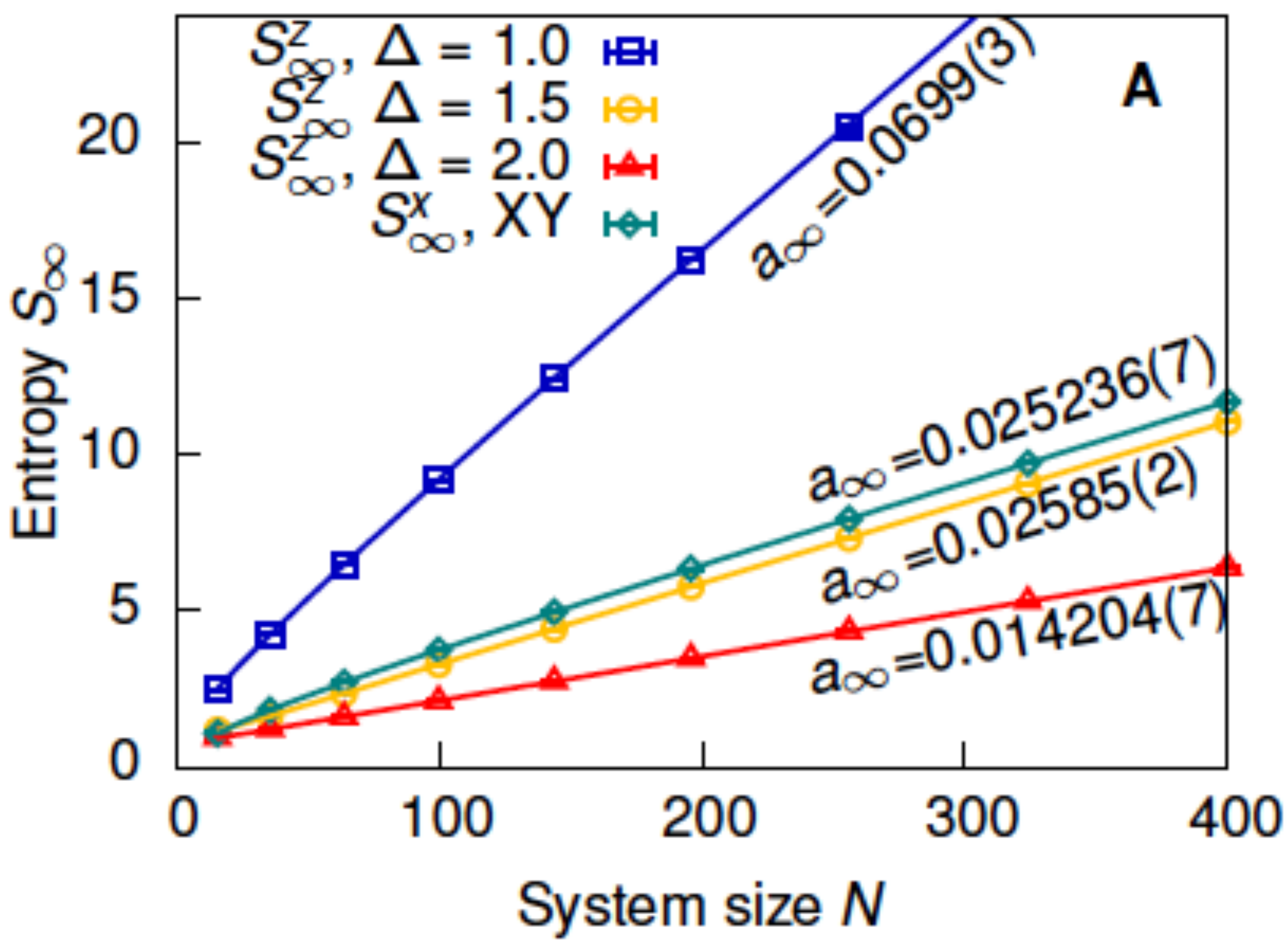
$\Delta=1$: SU(2) symmetry is spontaneously broken

$|\Delta|<1$: U(1) symmetry is spontaneously broken

These states have conventional order, but let us first study RSE in these well understood states

Numerical Approach

Luitz-Alet-Laflorencie 2013

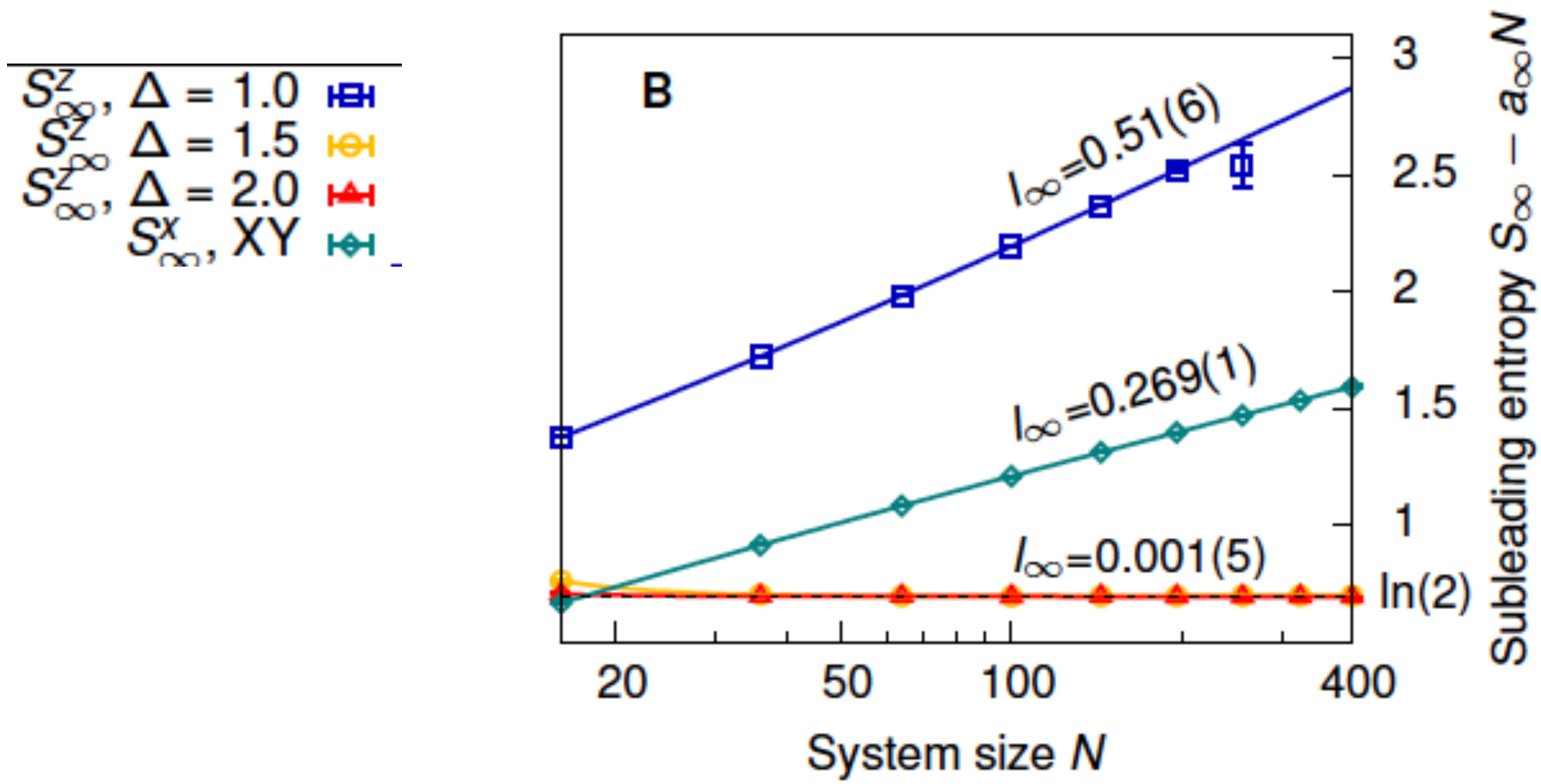


Plot: RSE in S^z- or S^x-basis

$$n \to \infty$$

Efficient Quantum Monte Carlo evaluation of the RSE: dominant "area law" contribution

Subleading term in RSE?



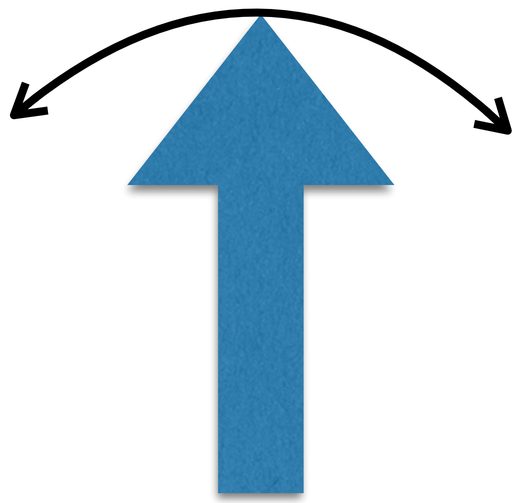
subleading log(N) contribution to RSE

only exist when a continuous (SU(2) or U(1)) symmetry is broken spontaneously;

universal?

Effective Field Theory

Nambu-Goldstone mode



order parameter

For the antiferromagnet, NG mode is described by free boson in 2+1D

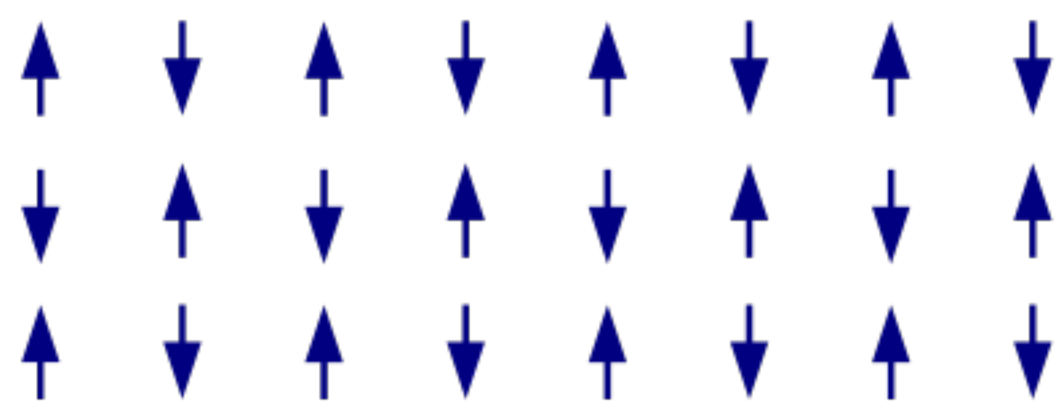
$$H = \frac{1}{2}\int d^2\mathbf{r}\left[\chi_\perp \Pi_\mathbf{r}^2 + \rho_s\left(\nabla\phi_\mathbf{r}\right)^2\right]$$

Simplest case: S_∞

$$S_n = \frac{1}{1-n} \log \left(\sum_i {p_i}^n \right) \qquad S_\infty \sim -\log p_{\max}$$

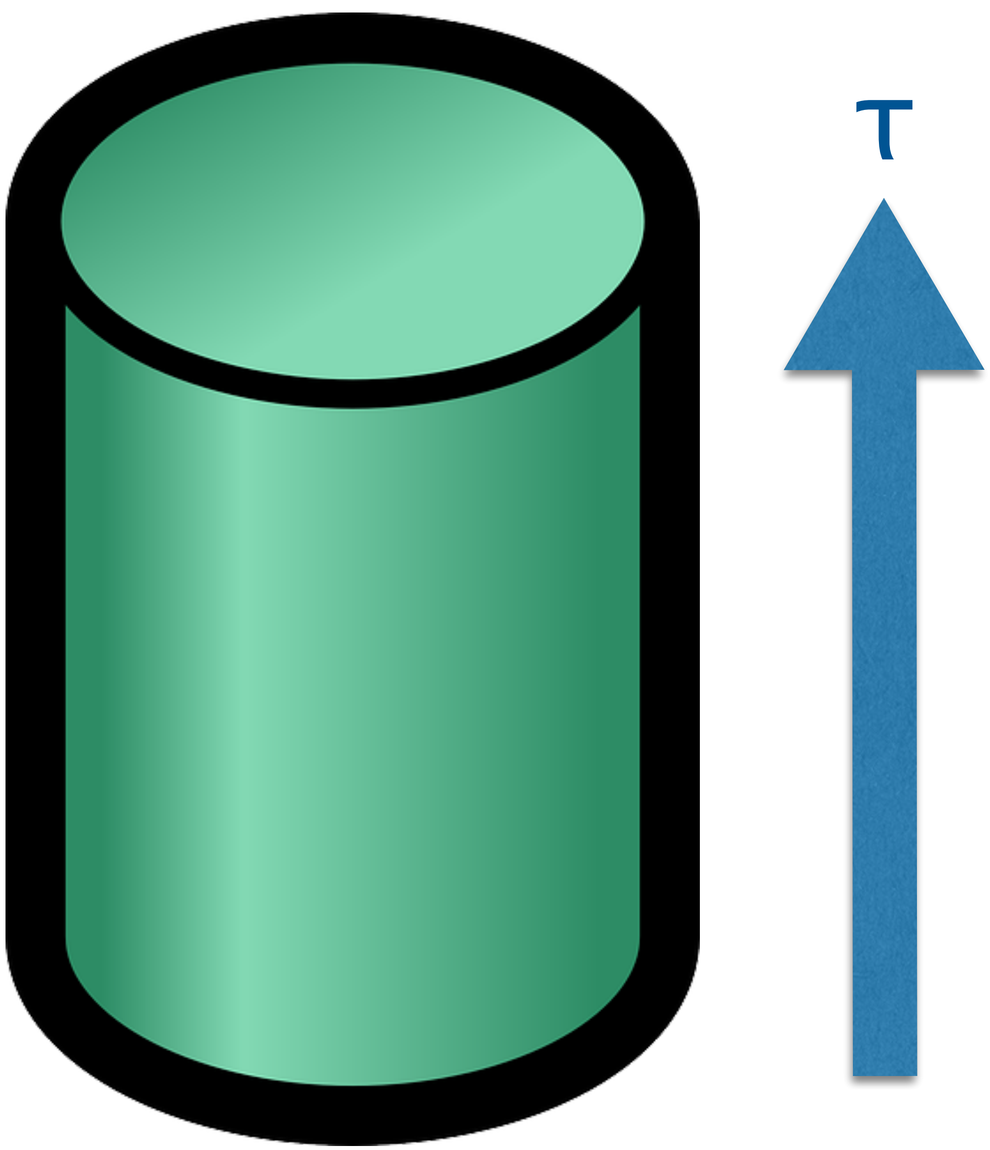
Q: Which state (in the Sz-basis) has the maximum amplitude (probability)?

A: Néel state!



Boundary Formulation

$$p_{\max} \sim \left|\langle \text{Néel} | \Psi \rangle\right|^2$$



Boundary condition at τ=0:
no fluctuation of NG mode

$$\phi(\vec{r}, \tau = 0) = 0$$

Dirichlet boundary condition

What is p_{max}?

cf.) single harmonic oscillator $H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$

$$\psi(x) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2}x^2\right).$$

$$p_{\max} = |\psi(0)|^2 = \left(\frac{m\omega}{\pi}\right)^{1/2}$$

NG modes: ∞ collection of harmonic oscillators labelled by the wave number $\mathbf{k}$ $\omega_{\mathbf{k}} = c|\mathbf{k}|$

$$p_{\max}^{\rm osc} = \prod_{\mathbf{k}\neq 0} p_{\max}(\mathbf{k}) = \prod_{\mathbf{k}\neq 0} \left(\frac{\rho_s|\mathbf{k}|}{\pi c}\right)^{1/2}$$

Determinant of Laplacian

$$-\log\left(p_{\max}^{\mathrm{osc}}\right) = -\frac{1}{2}\sum_{\mathbf{k}\neq 0}\log\left(\frac{\rho_s}{\pi c}\right) - \frac{1}{4}\sum_{\mathbf{k}\neq 0}\log \mathbf{k}^2.$$

$$-\sum_{\mathbf{k}\neq 0}\log \mathbf{k}^2 = \log \det{}'\Delta$$

Determinant of Laplacian

M. Kac 1966 etc.

$$\log \det{}'\Delta \simeq \mathcal{O}(L^2) + \left(1 - \frac{\chi}{6}\right)\log\left(\mathrm{L}^2\right)$$

χ: Euler characteristics of the spatial manifold

$$-\log\left(p_{\max}^{\mathrm{osc}}\right) = \frac{1}{4}\log L^2$$ for torus (χ=0)

“GS degeneracy” factor

Finite-size ground state: generally symmetric
(even in the SSB phase!)

$$|\text{symmetry-broken ground state}\rangle \sim \sum |\begin{matrix}\text{nearly-degenerate} \\ \text{finite-size ground states}\end{matrix}\rangle$$

SSB ⇔ existence of nearly degenerate ground states

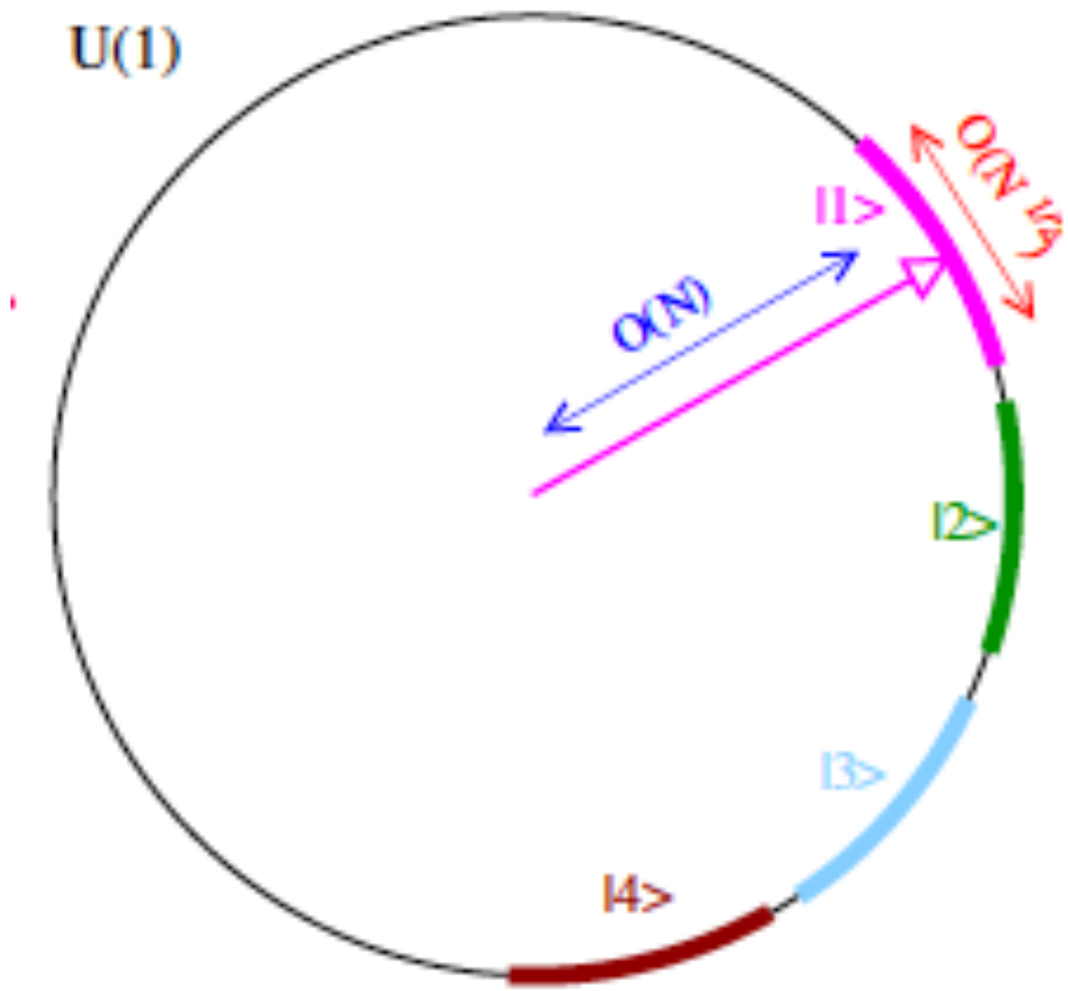
in finite size (“Anderson tower of states”)

$$|\Psi\rangle = \frac{1}{\sqrt{Q}} \left(|1\rangle + |2\rangle + \cdots + |Q\rangle \right)$$

Finite-size GS

(almost) linearly independent
symmetry-broken GSs

How many ground states?

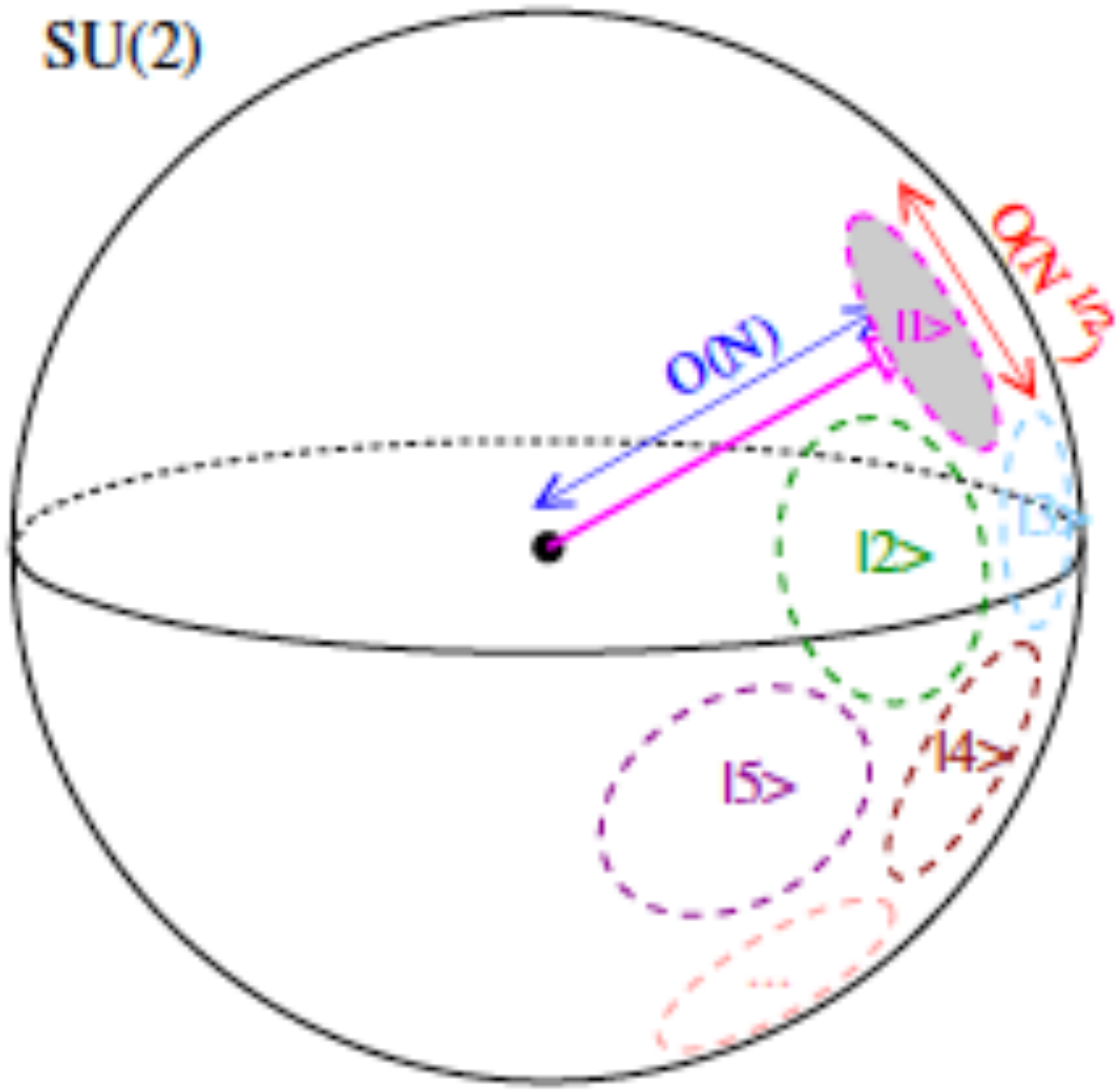


U(1) SSB phase:

$$Q \sim O(\frac{N}{\sqrt{N}}) \sim O(\sqrt{N})$$

$$N = L^2$$

How many ground states?



SU(2) SSB phase

$$Q \sim O(\frac{N^2}{\sqrt{N}^2}) \sim O(N)$$

$$N = L^2$$

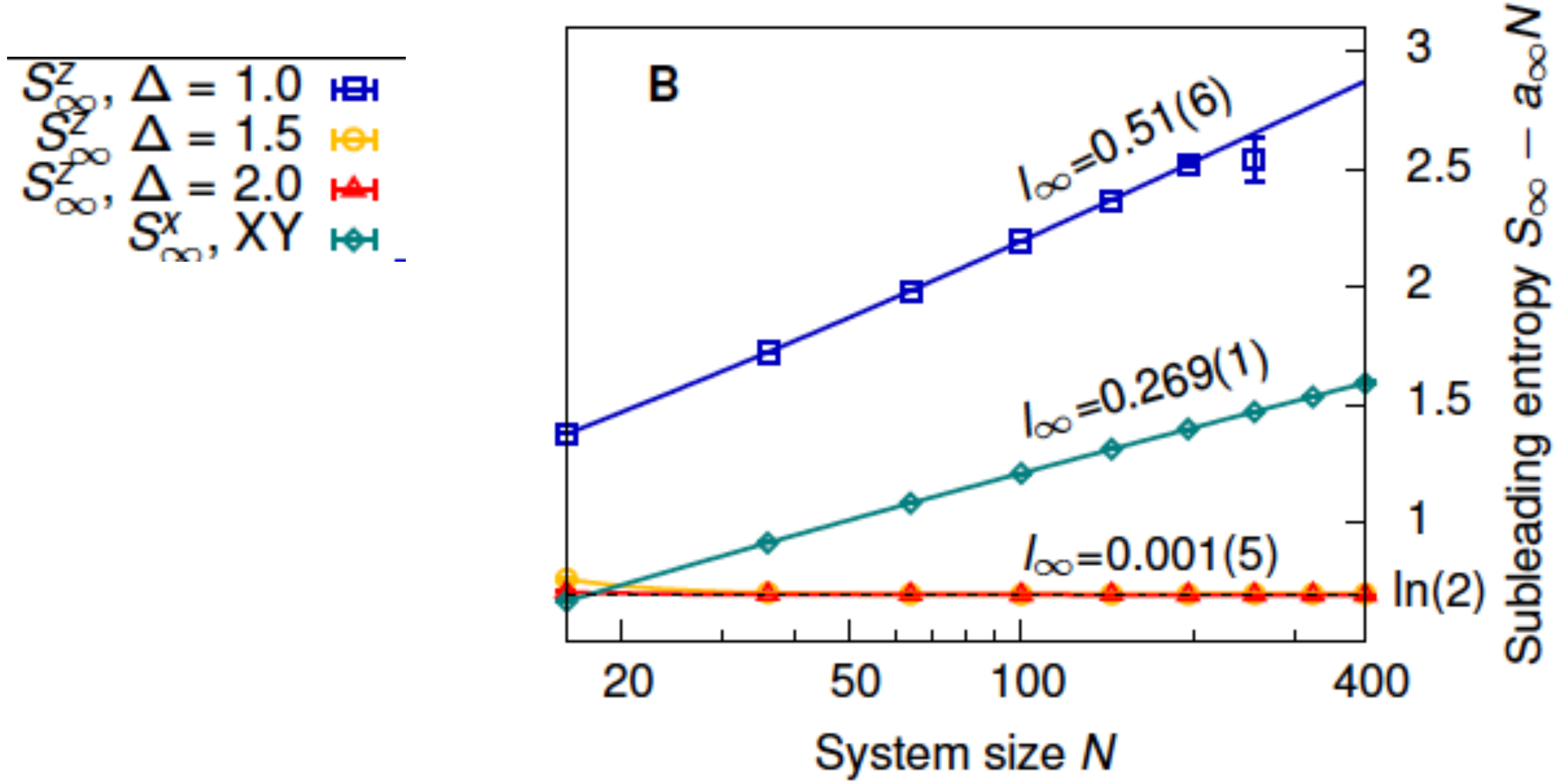
Universal term in S_∞

$$-\log(p_{\max}) \sim -\log(p_{\max}^{\mathrm{osc}}) - \log(\frac{1}{Q})$$
$$\sim -\frac{N_{NG}}{4}\log N + \frac{N_{NG}}{2}\log N$$
$$\sim +\frac{N_{NG}}{4}\log N$$

N_{NG} :number of Nambu-Goldstone mode
(= number of broken symmetry generators)
1 for XY / XXZ ($|\Delta| < 1$)
2 for Heisenberg AF (XXX) ($\Delta=1$)

※ here we consider "relativistic" case (type-A NG modes) only

Comparison with Numerics



Our theory:

Δ=1: N_{NG} = 2 $\quad l_\infty$ = 2/4 = 0.5

Δ=0: N_{NG} = 1 $\quad l_\infty$ = 1/4 = 0.25

Δ>1: N_{NG} = 0 $\quad l_\infty$ = 0

RSE for general *n*

Boundary phase transition?

recall in 1+1D, boundary phase transition at $n=n_c$

what about 2+1D?

Nambu-Goldstone mode in a SSB phase:

"small fluctuation" around the ordered state

Leading boundary perturbation: ${\Phi_0}^2$

"boundary mass": always relevant!!

"fixed phase" at least for $n>1$

$$S_{n>1} \sim \frac{1}{1-n} \log\left(p_{\max}\right)^n$$

$$\sim \frac{n}{n-1} S_\infty \qquad \sim \frac{n N_{NG}}{4(n-1)} \log N$$

Model	n	$\log(N)$ coef. Ref. 18	$\frac{N_{\mathrm{NG}}}{4}\frac{n}{n-1}$
XY			
$J_2 = 0$	∞	0.281(8)	0.25
$J_2 = -1$	∞	0.282(3)	0.25
$J_2 = 0$	2	0.585(6)	0.5
$J_2 = -1$	2	0.598(4)	0.5
$J_2 = 0$	3	0.44(2)	0.375
$J_2 = -1$	3	0.432(7)	0.375
$J_2 = 0$	4	0.35(8)	0.333
$J_2 = -1$	4	0.38(2)	0.333

Model	n	$\log(N)$ coef. Ref. 18	$\frac{N_{\mathrm{NG}}}{4}\frac{n}{n-1}$
Heisenberg			
$J_2 = 0$	∞	0.460(5)	0.5
$J_2 = -5$	∞	0.58(2)	0.5
$J_2 = 0$	2	1.0(2)	1
$J_2 = -5$	2	1.25(4)	1
$J_2 = -5$	3	1.06(3)	0.75
$J_2 = -5$	4	1.0(1)	0.666

Conclusions

- Basis-dependent Rényi-Shannon Entropy (RSE) exhibits universal behaviors (shown analytically & numerically)
- could be useful in characterizing ground states of quantum many-body systems
- some similarity (and relation) to entanglement entropy, in some respects "simpler" than EE (good for practical applications??)
- So far elucidated only for "conventional" phases (Tomonaga-Luttinger liquid in 1+1D, SSB phase in 2+1D) — can we apply to more exotic phases?
- Numerical approach: Exact Diagonalization / Quantum Monte Carlo… tensor networks??