Renyi-Shannon Entropy and Boundary Field Theory

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Quantum Matter and Information?

Quantum Condensed Matter Physics

Realization of quantum computation/quantum information processing

New understanding/formulation in foundation
Efficient numerical algorithm

(Quantum) Information
How to distinguish phases?

Different orders (or their absence) characterize each phase — what is “order”?

**Ferromagnet: magnetic order**
- spontaneously breaks $Z_2$ symmetry (Ising model),
- $SU(2)$ symmetry (Heisenberg model)…..

**Superfluid: off-diagonal long-range order**
- spontaneously breaks $U(1)$ symmetry

“order” $\equiv$ Spontaneous Symmetry Breaking
Beyond Landau Paradigm

Many quantum phases, which are distinct but cannot be characterized by a (conventional) SSB have been found

“topological phases”

- quantum Hall states
- Haldane gap phase
- topological insulators/topological superconductors

……

How to characterize them? new tools will be useful!

→ “information theoretic” measures
  e.g. entanglement entropy
In this talk, I will discuss an “information theoretic” measure of a quantum state which is different from (but also related to) entanglement entropy.

Based on a collaboration with Grégoire Misguich and Vincent Pasquier, also on several earlier works.
Rényi-Shannon Entropy

\[ |\psi\rangle = \sum_i c_i |i\rangle \]

Define Rényi/von Neumann entropy from the prob. dist.

\[ p_i = |c_i|^2 \]

\[ S_n = \frac{1}{1 - n} \log \left( \sum_i p_i^n \right) \]

\[ S_{\text{vN}} = -\sum_i p_i \log p_i \]

\[ = \lim_{n \to 1} S_n \]

This DOES depend on the choice of the basis

Anyway, Rényi-Shannon Entropy can be defined for a given quantum state (and choice of basis) can be used as a new characterization??
XXZ chain in 1+1D

\[ \mathcal{H} = \sum_j S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \]

The ground state is critical (gapless) for \(-1 \leq \Delta < 1\)

Effective theory: Tomonaga-Luttinger liquid
(free boson field theory in 1+1D)

\[ \mathcal{L} = \frac{g}{4\pi} (\partial_\mu \phi)^2. \quad \phi \sim \phi + 2\pi R \]

Bethe Ansatz

exact solution \(\rightarrow\)

\[ \sqrt{gR} = \sqrt{\frac{1}{2\pi} \left( 1 - \frac{1}{\pi} \cos^{-1} \Delta \right)} \]
Rényi-Shannon Entropy

of the $S=1/2$ XXZ chain in the $S^z$ basis:

$$S^z_j \sim \frac{1}{2\pi R} \partial_x \phi + \text{const.}(-1)^j \cos \frac{\phi}{R}$$

$S^z$ basis $\Leftrightarrow$ $\Phi$-basis

$$p_i = \frac{1}{Z_F} \int D\phi e^{-S[\phi]} \bigg|_{\phi(\tau=0,x) = \phi^{(b)}(x)}$$

$$S_n = \frac{1}{1-n} \log \left[ \left( \frac{1}{Z_F} \right)^n \int D\phi^{(b)}(x) \int \prod_{j=1}^{n} D\phi_j e^{-\sum_j S[\phi_j]} \bigg|_{\phi_j(\tau=0,x) = \phi^{(b)}(x)} \right]$$
RSE of XXZ chain

Stephan-Furukawa-Misguich-Pasquier 2009, MO 2010

\[ S_n = \frac{1}{1 - n} \log \left[ \left( \frac{1}{z_F} \right)^n \int \mathcal{D}\phi(x) \int \prod_{j=1}^{n} \mathcal{D}\phi_j e^{-\sum_j S[\phi_j]} |\phi_j(\tau=0,x) = \phi(b)(x)\rangle \right] \]

\[ = \frac{1}{1 - n} \log \left[ \left( \frac{1}{z_F} \right)^n \int \prod_{j=1}^{n} \mathcal{D}\phi_j e^{-\sum_j S[\phi_j]} |\phi_1(\tau=0,x) = \phi_2(\tau=0,x) = \phi_3(\tau=0,x) = \ldots\rangle \right] \]

\[ = \text{const.} \times \text{area} + \log (\sqrt{2gR}) + \frac{\log n}{2(1 - n)} \]

universal

non-universal

“area (=system length) law”

Rényi-Shannon entropy does contain the universal characteristics of the quantum state (TL liquid)
Relation to EE

Entanglement Entropy of "Quantum Lifschitz" state in 2 spatial dimensions

RSE of free boson field theory in 1+1 dimensions
(=Tomonaga-Luttinger Liquid / S=1/2 XXZ chain)
Quantum Lifshitz Field Theory

Ardonne-Fendley-Fradkin 2004

e.g. critical point of quantum dimer model on a square lattice

\[ S = \int d^3x \left[ \frac{1}{2} (\partial_t \varphi)^2 - \frac{\kappa^2}{2} (\nabla^2 \varphi)^2 \right] \]

\[ H = \int d^2x \; Q^\dagger(\vec{x})Q(\vec{x}) \]

\[ Q(x) = \frac{1}{\sqrt{2}} \left( \frac{\delta}{\delta \varphi} + \kappa \nabla^2 \varphi \right) \]

Groundstate wavefunction

\[ \Psi_0[\varphi] = \frac{1}{\sqrt{Z}} e^{-\frac{\kappa}{2} \int d^2x \left( \nabla \varphi(x) \right)^2} \]

EE of this state in replica formalism

= same formula as RSE of XXZ chain (TL liquid)

Phase Transition in RSE

Stephan-Misguich-Pasquier 2011

numerical

theory
(discussed earlier)

discrepancy for $n>n_c$?
Boundary Perturbations

The effective field theory, in general, contains all the possible perturbations which are not forbidden by the symmetries.

Bulk: all the perturbations are irrelevant in the gapless regime, irrespective of the Renyi parameter $n$.

Boundary: $n$ replica fields are coupled, and we need to consider possible perturbations w.r.t. “center of mass” field

$$\Phi_0 = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi_j$$
Boundary Perturbations

compactification of individual fields: \( \phi_j \sim \phi_j + 2\pi R \)

replica condition at the boundary: \( \phi_1 = \phi_2 = \ldots = \phi_n \)

effective compactification of the c.o.m. field

\[
\Phi_0 = \frac{1}{\sqrt{n}} \sum_j \phi_j \sim \Phi_0 + 2\pi \sqrt{n} R
\]

Possible boundary perturbations:

\[
\cos \frac{m \Phi_0}{\sqrt{n} R} \quad \sin \frac{m \Phi_0}{\sqrt{n} R}
\]

Scaling dim. \( x = \frac{m^2}{ngR^2} \) relevant if \( < 1 \)
Boundary Phase Transition

$m=1$: forbidden by the translation symmetry

\[ S^z_j \sim \frac{1}{2\pi R} \partial_x \phi + \text{const.} (-1)^j \cos \frac{\phi}{R} \]

$m=2$: most relevant perturbation in generic case

relevant if \( n > n_c = \frac{2}{gR^2} \)

SU(2) AF Heisenberg (\( \Delta=1 \)): \( gR^2 = 1 \quad n_c = 2 \)

XY (\( \Delta=0 \)): \( gR^2 = \frac{1}{2} \quad n_c = 4 \)
RSE above $n_c$

relevant $\Rightarrow$ boundary condition is

$\Phi=\text{const. i.e. "Néel state"}$

$$S_{n>n_c} = \frac{1}{1-n} \log [2(p_{max})^n]$$

$$p_{max} = |\langle + - + - + - \ldots |\Psi_0\rangle|^2$$

$$= \frac{Z_D}{Z} = \frac{1}{\sqrt{2gR}}$$
Phase Transition in RSE

Stephan-Misguich-Pasquier 2011

Theory

$n > n_c$

Theory

$n < n_c$
RSE of XXZ model in 2+1D

\[ \mathcal{H} = \sum_{\langle j,k \rangle} S_j^x S_k^x + S_j^y S_k^y + \Delta S_j^z S_k^z \]

On a bipartite lattice (square etc.) there is no geometric frustration; the ground state has a long-range (Néel) order

\(|\Delta|>1: \mathbb{Z}_2 \text{ symmetry is spontaneously broken} \\
\Delta=1: \text{SU}(2) \text{ symmetry is spontaneously broken} \\
|\Delta|<1: \text{U}(1) \text{ symmetry is spontaneously broken} \]

These states have conventional order, but let us first study RSE in these well understood states
Numerical Approach

Efficient Quantum Monte Carlo evaluation of the RSE: dominant “area law” contribution

Plot: RSE in $S^z$- or $S^x$-basis

$n \to \infty$
Subleading term in RSE?

Subleading $\log(N)$ contribution to RSE only exist when a continuous (SU(2) or U(1)) symmetry is broken spontaneously; universal?
Effective Field Theory

Nambu-Goldstone mode

For the antiferromagnet, NG mode is described by free boson in 2+1D

\[ H = \frac{1}{2} \int d^2r \left[ \chi_{\perp} \Pi^2_r + \rho_s (\nabla \phi_r)^2 \right] \]
Simplest case: $S_\infty$

$$S_n = \frac{1}{1 - n} \log \left( \sum_i p_i^n \right)$$

$S_\infty \sim - \log p_{\text{max}}$

Q: Which state (in the Sz-basis) has the maximum amplitude (probability)?
A: Néel state!
Boundary Formulation

\[ \rho_{\text{max}} \sim |\langle \text{Néel} | \Psi \rangle|^2 \]

Boundary condition at \( \tau = 0 \):
no fluctuation of NG mode

\[ \phi(\vec{r}, \tau = 0) = 0 \]

Dirichlet boundary condition
What is $p_{\text{max}}$?

cf.) single harmonic oscillator

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$$

$$\psi(x) = \left(\frac{m\omega}{\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2}x^2\right).$$

$$p_{\text{max}} = |\psi(0)|^2 = \left(\frac{m\omega}{\pi}\right)^{1/2}$$

NG modes: $\infty$ collection of harmonic oscillators labelled by the wave number $k$

$$\omega_k = c|k|$$

$$p_{\text{max}}^{\text{osc}} = \prod_{k \neq 0} p_{\text{max}}(k) = \prod_{k \neq 0} \left(\frac{\rho_s|k|}{\pi c}\right)^{1/2}$$
Determinant of Laplacian

\[ -\log(p_{\text{max}}^{\text{osc}}) = -\frac{1}{2} \sum_{k \neq 0} \log \left( \frac{\rho_s}{\pi c} \right) - \frac{1}{4} \sum_{k \neq 0} \log k^2. \]

\[ -\sum_{k \neq 0} \log k^2 = \log \det' \Delta \]

\[ \log \det' \Delta \simeq O(L^2) + \left( 1 - \frac{\chi}{6} \right) \log (L^2) \]

\( \chi \): Euler characteristics of the spatial manifold

\[ -\log(p_{\text{max}}^{\text{osc}}) = \frac{1}{4} \log L^2 \quad \text{for torus (} \chi = 0 \text{)} \]
“GS degeneracy” factor

Finite-size ground state: generally symmetric
(even in the SSB phase!)

\[ |\text{symmetry-broken ground state}\rangle \sim \sum |\text{nearly-degenerate finite-size ground states}\rangle \]

SSB ⇔ existence of nearly degenerate ground states
in finite size (“Anderson tower of states”)

\[ |\Psi\rangle = \frac{1}{\sqrt{Q}} (|1\rangle + |2\rangle + \cdots + |Q\rangle) \]

Finite-size GS

(almost) linearly independent symmetry-broken GSs
How many ground states?

\[ Q \sim O\left(\frac{N}{\sqrt{N}}\right) \sim O(\sqrt{N}) \]

\[ N = L^2 \]
How many ground states?

SU(2) SSB phase

\[ Q \sim O\left(\frac{N^2}{\sqrt{N}}\right) \sim O(N) \]

\[ N = L^2 \]
Universal term in $S_\infty$

$$- \log (p_{\text{max}}) \sim - \log (p_{\text{osc}}) - \log \left( \frac{1}{Q} \right)$$

$$\sim - \frac{N_{\text{NG}}}{4} \log N + \frac{N_{\text{NG}}}{2} \log N$$

$$\sim + \frac{N_{\text{NG}}}{4} \log N$$

$N_{\text{NG}}$: number of Nambu-Goldstone mode

(= number of broken symmetry generators)

1 for XY / XXZ (|\Delta| < 1)
2 for Heisenberg AF (XXX) (\Delta=1)

※ here we consider “relativistic” case (type-A NG modes) only
Comparison with Numerics

Our theory:

- $\Delta = 1$: $N_{NG} = 2$, $l_\infty = 2/4 = 0.5$
- $\Delta = 0$: $N_{NG} = 1$, $l_\infty = 1/4 = 0.25$
- $\Delta > 1$: $N_{NG} = 0$, $l_\infty = 0$
RSE for general $n$

Boundary phase transition?
- recall in 1+1D, boundary phase transition at $n=n_c$
- what about 2+1D?

Nambu-Goldstone mode in a SSB phase:
  - “small fluctuation” around the ordered state

Leading boundary perturbation: $\Phi_0^2$
  - “boundary mass”: always relevant!!

“fixed phase” at least for $n>1$
\[ S_{n>1} \sim \frac{1}{1 - n} \log (p_{\text{max}})^n \]
\[ \sim \frac{n}{n - 1} S_\infty \sim \frac{nN_{NG}}{4(n - 1)} \log N \]

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<th>Model</th>
<th>( n )</th>
<th>( \log(N) ) coef.</th>
<th>Ref.</th>
<th>( \frac{N_{NG}}{4} )</th>
<th>( \frac{n}{n-1} )</th>
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Conclusions

- Basis-dependent Rényi-Shannon Entropy (RSE) exhibits universal behaviors (shown analytically & numerically)
- could be useful in characterizing ground states of quantum many-body systems
- some similarity (and relation) to entanglement entropy, in some respects “simpler” than EE (good for practical applications??)
- So far elucidated only for “conventional” phases (Tomonaga-Luttinger liquid in 1+1D, SSB phase in 2+1D) — can we apply to more exotic phases?
- Numerical approach: Exact Diagonalization / Quantum Monte Carlo… tensor networks??