The Entanglement Hamiltonian and others in one-dimensional critical and gapped systems

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Outline:

- Part I:
  Entanglement spectrum in 1+1d a conformal field theories (CFTs) and the sine-square deformation (SSD)
  [with Xueda Wen and Andreas Ludwig]

- Part II (optional):
  Entanglement spectrum in 1+1 d gapped (SPT) phases and boundary conformal field theories (BCFTs)
  [with Gil Cho, Ken Shiozaki, Andreas Ludwig]
Hamiltonians in CFT

- Let's start from a Hamiltonian of (1+1)d CFTs; On a lattice (chain), it would look like: \[ H = \sum_i h_{i,i+1} \]

- Deformed evolution operator: \[ H[f] = \sum_i f \left( \frac{x_i + x_{i+1}}{2} \right) h_{i,i+1} \]

- E.g. Entanglement Hamiltonian: \[ f(x) = \frac{R^2 - x^2}{2R} \]

- E.g. Sine-square deformation (SSD): \[ f(x) = \sin^2 \frac{\pi x}{L} \] [Gendiar-Krcmar-Nishino (2009) ...]

- Other applications: inhomogenous systems, quantum energy inequalities, etc.
What is the sine-square deformation (SSD)?

- What it does is to introduce an "optimal" or "infinitely smooth" cutoff.

- This may be of interest as a numerical technique. Reducing finite-size error, etc.

- Early numerical observations:
  - Correlation functions
  - \( \langle \Psi_{SSD} | \Psi_{PBC} \rangle \approx 1 \)
  - Entanglement scaling

[XXZ chain, \( \Delta = 0.5, M = 0 \)]

[Hikihara-Nishino (11)]
Key properties of SSD?

- The ground state of SSD = ground state of periodic chain
  Numerics: Hikihara-Nishino (11);
  Exactly solvable models: Katsura (11), Maruyama-Katsura-Hikihara (11),
  Okunishi-Katsura (15)
  Proof within CFT: Katsura (12)

- $1/L^2$ finite size scaling of energy levels
  was observed numerically.

[\text{Gendiar-Krcmar-Nishino (09), Hotta-Nishimoto-Shibata (13),
  LSM type analysis by Katsura ... } ]
- Grand canonical numerical analysis -- efficient extraction of physical quantities in the presence of an applied field.
  Shibata and Hotta (11)
  Hotta, Nishimoto and Shibata (13)

- SSD and string theory:

- Dipolar quantization:
  Ishibashi and Tada, arXiv:1504.00138[hep-th]

- Mobius quantization:
  Okunishi arXiv:1603.09543
Strategy

- We will discuss types of "deformations" $H[f] = \int dx \, f(x) \mathcal{H}$ generated by various conformal maps.

- Put differently, we are interested in deformations which we can "undo" by conformal maps.

- Will discuss spectral properties (finite size scaling) of $H[f]$
- CFT on the plane ↔ CFT on a cylinder (quantum lattice model on a circle)

- Radial evolution ↔ Hamiltonian (1/L scaling)
  (Dilatation)

- Angular evolution ↔ Hamiltonian with boundary
  ("Rindler" or "Modular"
  or "Entanglement" Hamiltonian)
Finite size scaling of CFT [Cardy]

- CFT on a cylinder of circumference L:

\[ \tilde{H} = \frac{1}{2\pi} \int_0^L dv \tilde{T}_{uu}(u_0, v) \]

\[ \tilde{H} = \frac{1}{2\pi} \oint_{C_w} dw \tilde{T}(w) + \text{(anti-hol)} \]

\[ \tilde{T}_{uu}(w) = \tilde{T}(w) + \tilde{T}(\bar{w}) \]

- Conformal map: cylinder --> plane

\[ w = \frac{L}{2\pi} \log z. \]

\[ \tilde{T}(w) = \left( \frac{2\pi}{L} \right)^2 \left[ z^2 T(z) - \frac{c}{24} \right] \]

\[ \tilde{H} = \frac{2\pi}{L} \left( L_0 + \bar{L}_0 - \frac{c}{24} \right) \]

\[ \oint_{C_w} dw \tilde{T}(w) \]

\[ = \oint_{C_z} d\bar{z} \frac{dw}{d\bar{z}} \left( \frac{2\pi}{L} \right)^2 \left[ z^2 T(z) - \frac{c}{24} \right] \]

\[ = \oint_{C_z} dz \left( \frac{L}{2\pi} \right) \left[ z T(z) - \frac{c}{24} \frac{1}{z} \right]. \]
-- For a given tower of states, all levels are equally spaced (with degeneracy, which depends on details of the theory)

\[
\frac{(E - E_{GS})}{(E_1 - E_{GS})}
\]

\[
H = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)
\]

-- Level spacing scales as \(1/L\)
- Entanglement Hamiltonian on finite interval \([-R, R]\)  
  \(<--\) Hamiltonian with boundaries

- Transforming from strip to plane:

  \[ w(z) = \ln \left( \frac{z + R}{z - R} \right) \]

- Entanglement spec: \(1/\log R\) scaling.

[See, e.g: Casini-Huerta-Myers (11), Cardy @ 2015 KITP conference]
What is the evolution orthogonal to the evolution by Entanglement $H$?

- Let's take a circle as a Cauchy surf

$$
\left(x + \frac{\cosh u_0}{\sinh u_0} R\right)^2 + y^2 = \frac{R^2}{(\sinh u_0)^2}
$$

We have chosen:

$$r_0 := \frac{R}{\sinh u_0}$$

Circumference:

$$L = 2\pi r_0$$

- Evolution operator

$$H = \int_0^\pi dv T_{uu}(u_0, v) = r_0^2 \int_0^{2\pi} d\theta \frac{\cos \theta + \cosh u_0}{\sinh u_0} T_{rr}(r, \theta)$$

- "Regularized" version of the SSD:

$$H = \frac{L}{2\pi} \frac{1}{\sinh u_0} \int_0^L ds \left( \cos \frac{2\pi s}{L} + \cosh u_0 \right) T_{rr} \left( r = \frac{L}{2\pi}, \theta = \frac{2\pi s}{L} \right)$$
"Regularized" SSD

\[ H = \frac{L}{2\pi} \frac{1}{\sinh u_0} \int_0^L ds \left( \cos \frac{2\pi s}{L} + \cosh u_0 \right) T_{rr} \left( r = \frac{L}{2\pi}, \theta = \frac{2\pi s}{L} \right) \]

- By construction, this operator has the spectrum of CFT on a circle with level spacing of order one.

- Define:  \[ H_{rSSD} = \int_0^L ds \left( \cos \frac{2\pi s}{L} + \cosh u_0 \right) T_{rr} \left( \frac{L}{2\pi}, \frac{2\pi s}{L} \right) \]

- The envelope function:
  \[
  f(s) = \cos \left( \frac{2\pi s}{L} \right) + \cosh u_0 \\
  = \cos \left( \frac{2\pi s}{L} \right) + \sqrt{1 + \left( \frac{2\pi R}{L} \right)^2}.
  \]

- "Regularized" version of the SSD:
  \[ R, \ \text{the distance between vortices, is the regularization parameter.} \]

- Scaling:
  (i) fix \( u_0 \), change \( R \) --> 1/L scaling

  (ii) fix \( R \), change \( u_0 \) --> 1/L^2 scaling
The dipolar limit

- Can take the dipolar limit $R \to 0$ \text{rSSD} \to \text{SSD}:

- In the dipolar limit, the $w$-plane ($u$-$v$ plane) is an infinit plane
  --> Infinite system length limit, continuum spectrum \text{[Ishibashi-Tada (15,16)]}

- The prefactor $L^2$ is indicative of the $1/L^2$ scaling seen in numerics.

\[ w(z) = \frac{1}{z} \]

- In the dipolar limit, the $w$-plane ($u$-$v$ plane) is an infinit plane
  --> Infinite system length limit, continuum spectrum \text{[Ishibashi-Tada (15,16)]}

\[
H = \int_{-\infty}^{+\infty} dv \, T_{uu}(u_0, v) = 4r_0^3 \int_0^{2\pi} d\phi \sin^2(\phi/2) T_{rr}(r_0, \theta)
\]

\[
= \frac{L^2}{\pi^2} \int_0^L ds \sin^2 \left( \frac{\pi s}{L} \right) T_{rr} \left( \frac{L}{2\pi}, \frac{2\pi s}{L} \right)
\]

- The prefactor $L^2$ is indicative of the $1/L^2$ scaling seen in numerics.
Numerics (rSSD) $H = \sum_i (S^x_i S^{x}_{i+1} + S^y_i S^{y}_{i+1})$
SSD spectrum does not much physical spectrum, 
$1/L^2$ scaling
Other examples

- Engineering conformal map and evolution operator
- "square root deformation" $\tilde{H} = \int_{-R+\epsilon}^{+R-\epsilon} dx \sqrt{R^2 - x^2} T_{yy}$
- Known in the context of "perfect state transfer" (Thanks: Hosho Katsura)
"Square root" deformation

Physical spectrum (OBC)
Summary (Part I)

- Setup a general discussion of "deformed" Hamiltonians in CFTs
- Proposed a "regularized" version of SSD (rSSD).
- Original SSD can be viewed as a "singular" limit of rSSD
- Spectrum of rSSD is easy to understand. Shed light on 1/L^2 scaling of SSD.

- Issues:

Exciations?
Relation to the classification of conformal vacua [Candelas-Dowker (1979)]
(Part II) Going away from CFTs

- Add a relevant deformation --> go into a massive phase

\[ S_{z, \bar{z}} = S_* + g \int d^2 z \phi(z, \bar{z}) \]

- Consider the entanglement Hamiltonian for the half space; The entanglement spectrum?

- Repeat the conformal map analysis

\[ S_* + g \int_{u_1}^{\infty} du \int_{0}^{2\pi} dv \ e^{yu} \Phi(w, \bar{w}) \]

- Massive perturbation creates an exponentially growing potential

\[ z = (x + iy) = \exp(w) = \exp(u + iv) \]
- Massive perturbation creates an artificial boundary in the entanglement Hamiltonian.

- To a good approximation, the entanglement Hamiltonian is the Hamiltonian of a CFT with boundaries; Boundary CFT.

- Comment: this argument shows the low-energy part of the entanglement spec. of a massive theory is given by BCFT. There are integrable massive models, whose corner transfer matrices are given exactly by Virasoro characters (BCFT).

\[
\rho_A \propto \exp(-H_e) \\
H_e = \text{const.} \frac{L_0}{\log(\xi/a_0)}
\]
ES for gapped phases is given by nearby boundary conformal field theory

\[ \rho_A \propto \exp(-H_e) \quad H_e = \text{const.} \frac{L_0}{\log(\xi/a_0)} \]

Q: Which boundary condition? \(<----->\) Which gapped phase?

Let's focus on the case when the massive phase is a SPT phase

i.e.: (i) unique ground state
   (ii) topologically distinct in the presence of some symmetry

Result:
For a given symmetry \( G \), and a given boundary state \( |B\rangle \),
found a method to compute the topological invariant
of the corresponding SPT phase.

\[ \hat{g} |B\rangle_h = \varepsilon(g|h)|B\rangle_h \]

Related to symmetry-protected degeneracy of ES

Relation to physics of fractional branes
Symmetry-protected degeneracy

- E.g. 1d lattice fermion model ("SSH" model)

\[ H = t \sum_i (a_i^\dagger b_i + h.c.) + t' \sum_i (b_i^\dagger a_{i+1} + h.c.) \]

Symmetry: \( a_i \rightarrow a_i^\dagger \quad b_i \rightarrow -b_i^\dagger \)

Phase diagram:

- Symmetry-protected contribution to EE [SR-Hatsugai (06)]

\[ S_A \sim (1/6) \log(\xi/a_0) + \ln 2 \]
Symmetry-protected degeneracy

- (1+1)d SPT phase:
  E.g. the Haldane phase, the Kitaev chain

- Symmetry-protected degeneracy in ES: \[\text{[Pollmann-Berg-Turner-Oshikawa (10)]}\]

\[
H = \sum_j S_j \cdot S_{j+1} + U_{zz} \sum_i (S_j^z)^2
\]
- Symmetry-protected degeneracy $\implies$ Vanishing of partition function:

$$Z_{AB}^h = \text{Tr}_{\mathcal{H}_{AB}} \left[ \hat{h} e^{-\beta \hat{H}_{\text{open}}^{\text{closed}}} \right] = 0,$$

- Exchange time and space:

$$Z_{AB}^h = h \langle A | e^{-\frac{\ell}{2} \hat{H}_{\text{closed}}} | B \rangle_h = h \langle A | \tilde{q}^{\frac{1}{2}}(\hat{H}_L + \hat{H}_R) | B \rangle_h,$$

$$h \langle A | e^{-\frac{\ell}{2} \hat{H}_{\text{closed}}} | B \rangle_h = 0.$$

- Act with a symmetry on $|B\rangle$

$$\hat{g} | B \rangle_h = \varepsilon_B (g | h) | B \rangle_h, \quad \text{when} \quad g \in N_h.$$

- Symmetry-enforced vanishing of partition function

$$h \langle A | \tilde{g} \tilde{q}^{\frac{1}{2}}(\hat{H}_L + \hat{H}_R) | B \rangle_h = h \langle A | \tilde{q}^{\frac{1}{2}}(\hat{H}_L + \hat{H}_R) \hat{g} | B \rangle_h,$$

$$\varepsilon_A (g | h) | A \rangle \langle A | \tilde{q}^{\frac{1}{2}}(\hat{H}_L + \hat{H}_R) | B \rangle_h = \varepsilon_B (g | h) \langle A | \tilde{q}^{\frac{\ell}{2}}(\hat{H}_L + \hat{H}_R) | B \rangle_h.$$
Anomalous boundary states

- Ideal lead obeys B.C. set by SPT
  \[ \Phi(\sigma_2) - U \cdot \Phi(\sigma_2) = 0 \]
  \[ [\Phi(\sigma_2) - U \cdot \Phi(\sigma_2)]|B\rangle = 0 \]

- Symmetry G acts on fundamental fields
  \[ G \cdot \Phi(\sigma_2) \cdot G^{-1} = U_G \cdot \Phi(\sigma_2) \]

- B.C. is invariant under G:
  \[ G \left[ \Phi - U \cdot \Phi \right] G^{-1} = U_G \cdot \Phi - U_G \cdot U \cdot \Phi \]

- But boundary state may not be:
  \[ G \cdot |B\rangle \neq |B\rangle \]

- \( Z_8 \) classification of TRS Kitaev chain, Haldane phase
Analysis and result: Fidkowski-Kitaev problem

- Ideal lead

\[ H = \sum_{a=1}^{N_f} \int_0^\ell dx \left[ \psi_L^a(-vi\partial_x)\psi_L^a + \psi_R^a(+vi\partial_x)\psi_R^a \right] \]

- Symmetry group:  \( \{ T, G_f, T \times G_f \} \)

- Boundary states

\[
\begin{align*}
[\psi_L(\sigma_2) - i\eta_1 \psi_R(\sigma_2)] |B(\eta_1, \eta_2)\rangle &= 0 \\
[\psi_R(\sigma_2) + i\eta_2 \psi_L(\sigma_2)] |B(\eta_1, \eta_2)\rangle &= 0
\end{align*}
\]

- Symmetry action on fermion number parity:

\[
\begin{align*}
G_f |B(\eta_1 = -\eta_2)\rangle &= |B(\eta_1 = -\eta_2)\rangle \\
G_f |B(\eta_1 = \eta_2)\rangle &= (-1)^{N_f} |B(\eta_1 = \eta_2)\rangle
\end{align*}
\]

Anomalous relative sign goes away for 2N copies --> \( \mathbb{Z}_2 \)

- Time reversal:

\[ T |B(\eta_1 = \eta_2)\rangle = e^{i\pi N_f/4} |B(\eta_1 = \eta_2)\rangle \]