Relative entropy in CFT

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Motivation

- Microstates of a black hole $|V\rangle \quad |W\rangle$

- They are "almost" same -> The foundation of the statistical mechanical interpretation of BH entropy.

- The difference between them is also important for information loss .

▪In addition to this, usally we employ a coarse grained view of the system,

$$\rho_W = tr_H |W\rangle\langle W| \qquad \rho_V = tr_H |V\rangle\langle V|$$

-> How can we quantify the difference between ρ_V and ρ_W ?

Relative entropy cf[Ooguri san, van Raamsdonk, Harlow, Casini's talk]

$$S(\rho||\sigma) = \mathrm{tr}\rho\log\rho - \mathrm{tr}\rho\log\sigma$$

It measures the distance between ρ and σ

Asymmetric under the exchange $\rho \leftrightarrow \sigma$

It can be written by the modular Hamiltonian $K_\sigma = -\log\sigma$

$$S(\rho||\sigma) = \Delta\langle K_\sigma\rangle - \Delta S \qquad \Delta\langle K_\sigma\rangle = \mathrm{tr}\rho K_\sigma - \mathrm{tr}\sigma K_\sigma$$

$$\Delta S = S(\rho) - S(\sigma)$$

Calculations of Relative entropy

A Calculation of relative entropy itself remains to be a difficult task.

Holographic computations: Available only for $S(\rho_V||\rho_0)$

RT formula + the vacuum modular Hamiltonian ex [Blanco Casini Hung Myers]

cf [Ooguri san, van Raamsdonk's talk]

CFT computations: 2d free scalar[Lashkari]

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Problem: Compute $S(\rho_V||\rho_W)$ for generic two excited states $|V\rangle$ $|W\rangle$ in an arbitrary CFT.

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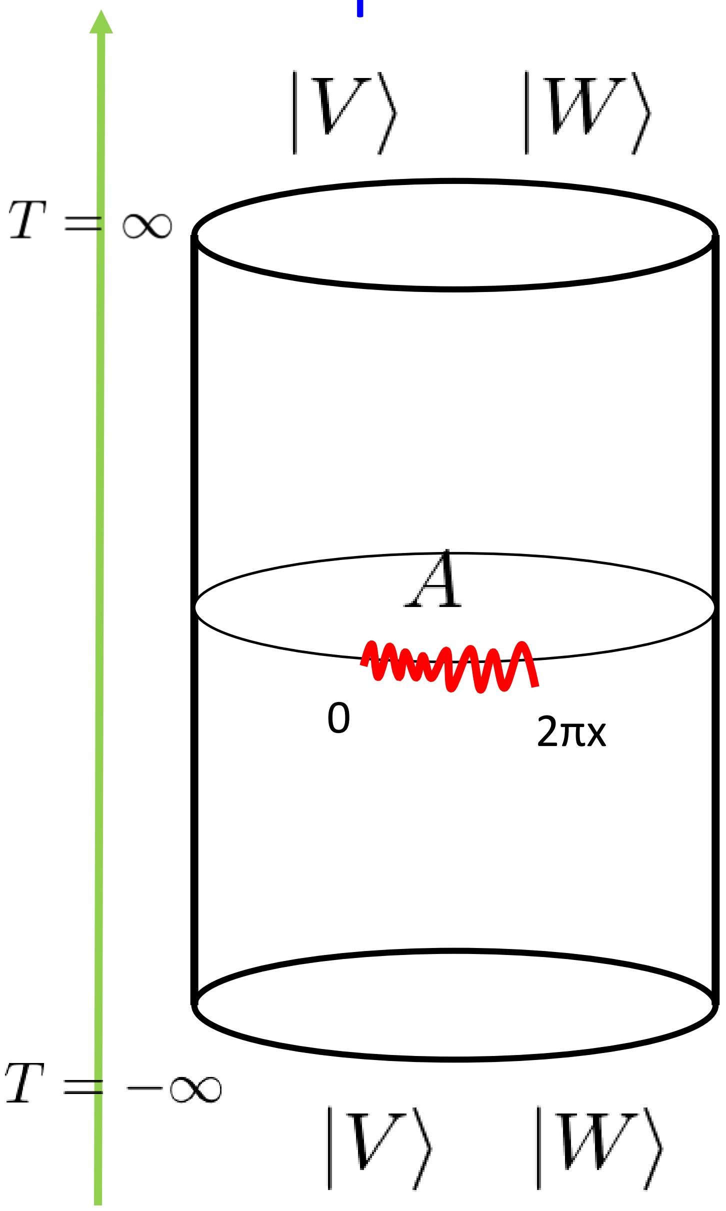
CFT computations: 2d Free scalar[Lashkari]

Problem: Compute $S(\rho_V||\rho_W)$ for generic two excited states $|V\rangle$ $|W\rangle$ in an arbitrary CFT.

Main claim: We found a general formula for it, in the small subsystem size limit.

Relative entropy in 2d CFT

Set up



- A 2d CFT on a Cylinder $\mathbb{R}\times S^1$
- A subsystem $A:[0,\ 2\pi x]$ $0<x<1$
- Excited states $|V\rangle$ $|W\rangle$ at $T=\pm\infty$
- Reduced density matrices $\rho_V=\mathrm{tr}_{A_c}|V\rangle\langle V|$

$$\rho_W=\mathrm{tr}_{A_c}|W\rangle\langle W|$$

Main result

In the small interval limit x->0, the leading behavior of relative entropy is given by

$$\boxed{S(\rho_V||\rho_W) = \frac{\Gamma(\frac{3}{2})\Gamma(\Delta+1)}{\Gamma(\Delta+\frac{3}{2})}\sum_{\alpha}(C_{\mathcal{O}_\alpha VV} - C_{\mathcal{O}_\alpha WW})^2(\pi x)^{2\Delta}}$$

$\{\mathcal{O}_\alpha\}$ Set of lightest primary operators with $C_{\mathcal{O}_\alpha VV} - C_{\mathcal{O}_\alpha WW} \neq 0$

$$C_{\mathcal{O}_\alpha VV} = \langle V|\mathcal{O}_\alpha|V\rangle \qquad \Delta = \dim\,\mathcal{O}_\alpha$$

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At the leading order, the relative entropy is captured by the difference of the one point function of these excited states   V and W.

Main result(2)

When the lightest operator is the stress tensor,

$$S(\rho_V||\rho_W) = \frac{4}{15c}(\Delta_V - \Delta_W)^2(\pi x)^4$$

Sketch of the derivation

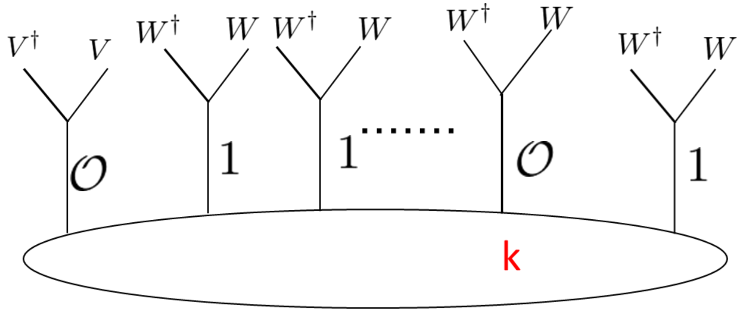
- Step1: Introduce the replica trick[Lashkari]

$$S(\rho_V||\rho_W) = \lim_{n\to 1}\frac{1}{n-1}\left(\mathrm{tr}\rho_V^n - \mathrm{tr}\rho_V\,\rho_W^{n-1}\right)$$

- Step2 : Write each term by a correlation function on an n-sheeted plane Σ_n :

$$\mathrm{tr}\rho_V\,\rho_W^{n-1} = \frac{\langle V^\dagger(\infty_0)V(0_0)\prod_{k=1}^{n-1}W^\dagger(\infty_k)W(0_k)\rangle_{\Sigma_n}}{\langle V(\infty)V(0)\rangle_{\Sigma_1}\langle W(\infty)W(0)\rangle_{\Sigma_1}^{n-1}}$$

- Step 3: expand these correlation functions in the x->0 limit, using OPEs



[Alcaraz,Berganza,Sierra], [Carabrese Cardy Tonni], [Headrick]+ a lot of papers

Example(1):Relative entropy in free boson theory

- Lashkari computed the relative entropy between two chiral vertex operators $\mathcal{V}_\alpha = e^{i\alpha X(z)}$, $\mathcal{V}_\beta = e^{i\beta X(z)}$ in free boson theory,

$$S(\rho_{\mathcal{V}_\beta}||\rho_{\mathcal{V}_\alpha}) = (\alpha-\beta)^2(1-\pi x\cot\pi x) = \frac{1}{3}(\alpha-\beta)^2(\pi x)^2 + o(x^4)$$

・By using our formula, this leading behavior is explained by the exchange of the U(1) current operator, $J_z = i\partial X(z)$ with the conformal dimension $\Delta_J = 1$ $C_{J\mathcal{V}_\alpha\mathcal{V}_{-\alpha}=\alpha}$
and the gamma function factor=1/3.

Ex(2):Relative entropy from the modular Hamiltonian

- The relative entropy between an excited states V and the ground state has been calculated in the somewhat different way by using the modular Hamiltonian, ex [Blanco Casini Hung Myers]..

$$S(\rho_V||\rho_0) = \Delta\langle K\rangle - \Delta S$$

$$K = -\log \rho_0 = 2\pi \int_0^{2\pi x} d\phi \left[\frac{\cos(\phi - \pi x) - \cos(\pi x)}{\sin(\pi x)}\right] T_{00} \qquad \Delta\langle K\rangle = \langle V|K|V\rangle - \langle 0|K|0\rangle$$

ΔS is the difference of entanglement entropy. When there are no primary operator lighter than the stress tensor ,we can compute ΔS in the small x limit from the vacuum conformal block. Then

$$S(\rho_V||\rho_0) = \frac{4}{15c}\Delta_V^2(\pi x)^4$$

This is a special case of our formula, $S(\rho_V||\rho_W) = \frac{4}{15c}(\Delta_V - \Delta_W)^2(\pi x)^4$

Ex(3):Relative entropy between two generalized free fields

Generalized free fields =The CFT dual of bulk free scalar fields.

The correlation functions of the generalized free fields are calculated as if they are free fields (ie Wick contractions).

This property allows us to compute the relative entropy between two generalized free fields without using the general formula,

$$S(\rho_V||\rho_W) = \frac{\Gamma(\frac{3}{2})\Gamma(2\Delta+1)}{\Gamma(2\Delta+\frac{3}{2})}\left[1-|\langle V|W\rangle|^2\right](\pi x)^{4\Delta}$$

We checked this result can be reproduced from the general formula.

Generalizations to higher dimensions

Relative entropy in d dimensional CFT

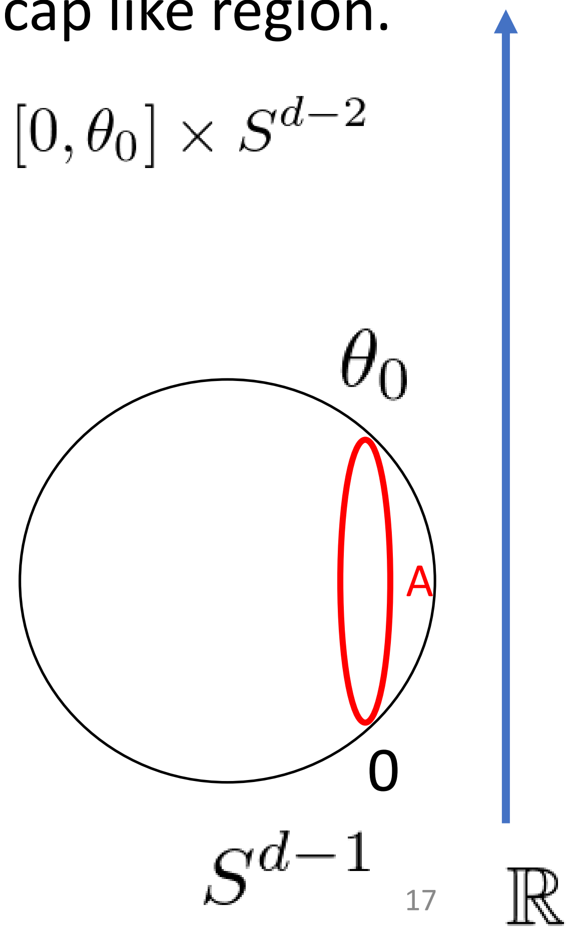
- Higher dimensional generalizations of this calculation are straightforward.
- We start from a CFT on $\mathbb{R} \times S^{d-1}$, and take the subsystem A : a cap like region.
- By using the Casini Huerta Myers conformal map,

  $\mathrm{tr}\rho_V^n$ and $\mathrm{tr}\rho_V \rho_W^{n-1}$ are now computed by

  the correlation functions on $S^1 \times H^{d-1}$ with the periodicity $2\pi n$

- Technically, the most difficult part is the analytic continuation of the sum of two point functions $\sigma_n = \sum_{k=1}^{n} G_n(2\pi k)$ on $S^1 \times H^{d-1}$.

This has been done in [Agon, Faulkner] in last year.



The result

- When the lightest operator is the stress tensor, then,

$$S(\rho_V||\rho_W) = \frac{1}{4C_T}\frac{d}{d-1}\frac{\Gamma(\frac{1}{2})\Gamma(d+1)}{\Gamma(d+\frac{3}{2})}(\varepsilon_V - \varepsilon_W)^2(\theta_0)^{2d}$$

CT: the coefficient of the two point function of the stress tensor. $\langle TT\rangle \sim \frac{C_T}{x^{2d}}$

$\varepsilon_V = \frac{\Delta_V}{\mathrm{Vol}(S^{d-1})}$ energy density of the excited state V.

When $\varepsilon_W = 0$ the result agrees with the holographic result : the RT surface area calculation on the AdS black brane with ε_V . [Blanco, Casini Hung Myers]

Relative entropy of two disjoint intervals

A trick

[Jafferis Maldacena Lewkowitz Suh] [Dong Harlow Wall] [Leichnauer: private communication]

We start from the entanglement entropy of ρ_W : $S_A(\rho_W) = -c_A(l)\mathrm{tr}[\rho_W \mathcal{O}]^2$

Slightly deform the density matrix : $\rho_W \to \rho_W + \delta\rho$

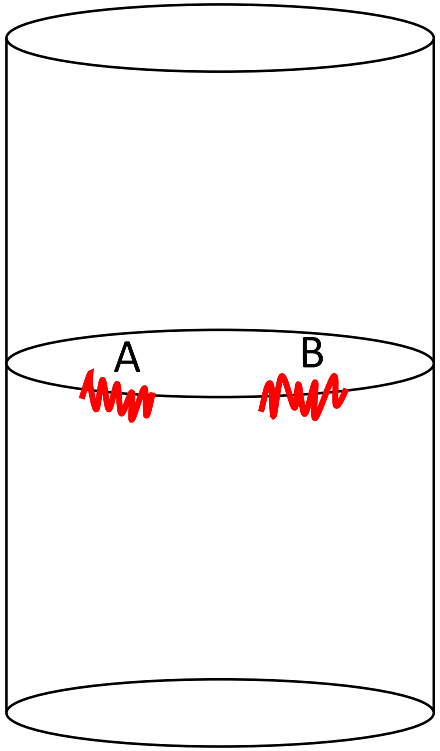
$$\delta S_A(\rho_W) = -c_A(l)\mathrm{tr}[\delta\rho\mathcal{O}]\langle\mathcal{O}\rangle_W = \mathrm{tr}[K_A^W \delta\rho]$$

Since this is true for any deformation $\delta\rho$, we can read off the modular Hamiltonian from it

$$K_A^W = -c_A(l)\langle\mathcal{O}\rangle_W \, \mathcal{O}_A$$

$$S(\rho_V||\rho_W) = \Delta\langle K_W\rangle - \Delta S = c_A(l)(\langle\mathcal{O}\rangle_V - \langle\mathcal{O}\rangle_W)^2$$

Relative entropy of two disjoint intervals



$$S_{A\cup B}(\rho_W) = S_A(\rho_W) + S_B(\rho_W) - I_W(A,B)$$

We find that in the small subsystem limit, $l_A, l_B \to 0$
the mutual information of the excited state is given by

$$I_W(A,B) = (l_A)^{2\Delta}(l_B)^{2\Delta}\frac{\Gamma(\frac{3}{2})\Gamma(2\Delta+1)}{\Gamma(2\Delta+\frac{3}{2})}\Big(\langle W|\mathcal{O}_A\mathcal{O}_B|W\rangle - \langle W|\mathcal{O}_A|W\rangle\langle W|\mathcal{O}_B|W\rangle\Big)^2$$

$$\equiv C_{AB}(l_A,l_B)(M^W_{AB})^2$$

Cf. [Wolf Verstraete Hastings Cirac]

Relative entropy of disjoint intervals

$$K_{A\cup B} = K_A + K_B$$

$$-2c_{AB}M^{W}_{AB}\left[\mathcal{O}_A\mathcal{O}_B - \langle\mathcal{O}_A\rangle_W\mathcal{O}_B - \langle\mathcal{O}_B\rangle_W\mathcal{O}_A\right]$$

Relative entropy of disjoint intervals

$$S_{A\cup B}(\rho_V||\rho_W) = S_A(\rho_V||\rho_W) + S_B(\rho_V||\rho_W)$$

$$+c_{AB}\left[(M^V_{AB} - M^W_{AB})^2 - 2M^W_{AB}(\langle\mathcal{O}\rangle_V - \langle\mathcal{O}\rangle_W)^2\right]$$

Relative entropy of disjoint intervals

$$S_{A\cup B}(\rho_V||\rho_W) = S_A(\rho_V||\rho_W) + S_B(\rho_V||\rho_W)$$

$$+c_{AB}\left[(M_{AB}^V - M_{AB}^W)^2 - \underline{2M_{AB}^W(\langle\mathcal{O}\rangle_V - \langle\mathcal{O}\rangle_W)^2}\right]$$

Asymmetric under the exchange $V \leftrightarrow W$

Although the relative entropy for a single interval starts from the symmetric term, the part involving both A and B contains the asymmetric part from the beginning.

Conclusions

Relative entropy in CFT.

Time dependent Process?

Bulk reconstruction? Radon transformation?

Another quantum information theoretic quantities?