Loop optimization for tensor network renormalization

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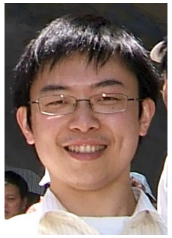
Perimeter Institute for Theoretical Physics, Waterloo, Canada



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(CUHK, PI)

Xiao-Gang Wen
(MIT, PI)

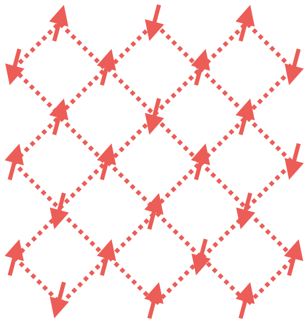
arXiv:1512.04938

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

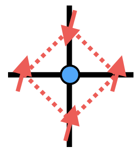
Overview

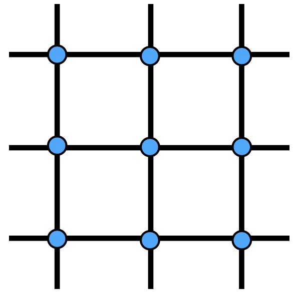
tensor network + renormalization group = tensor network renormalization

Partition function of classical statistical system

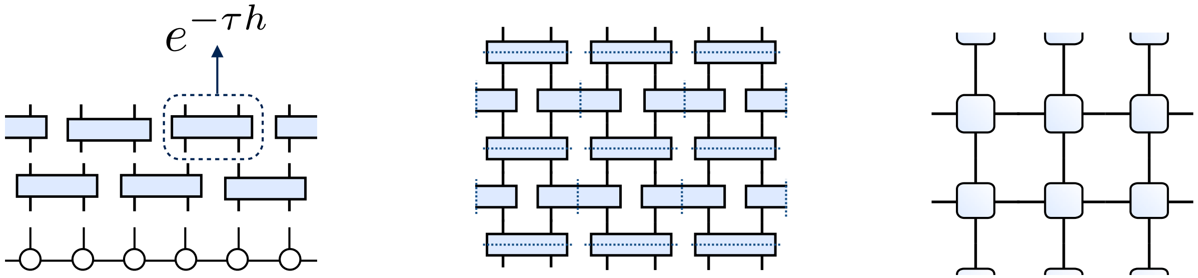


$$Z = \sum_{\{\sigma\}} \exp(\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j)$$

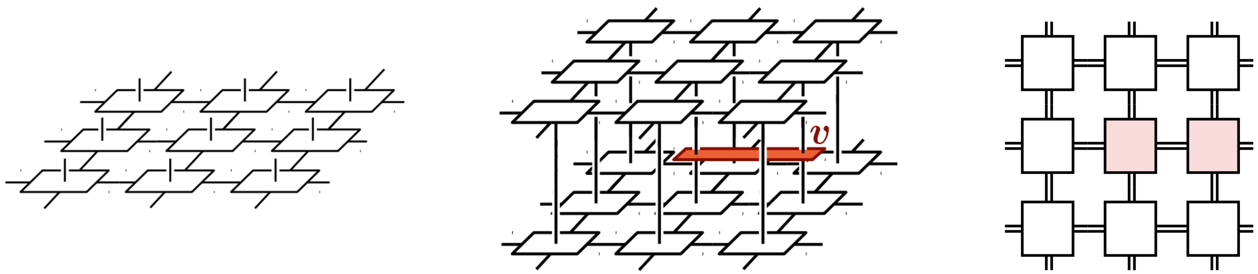




Euclidean path integral of 1D quantum system



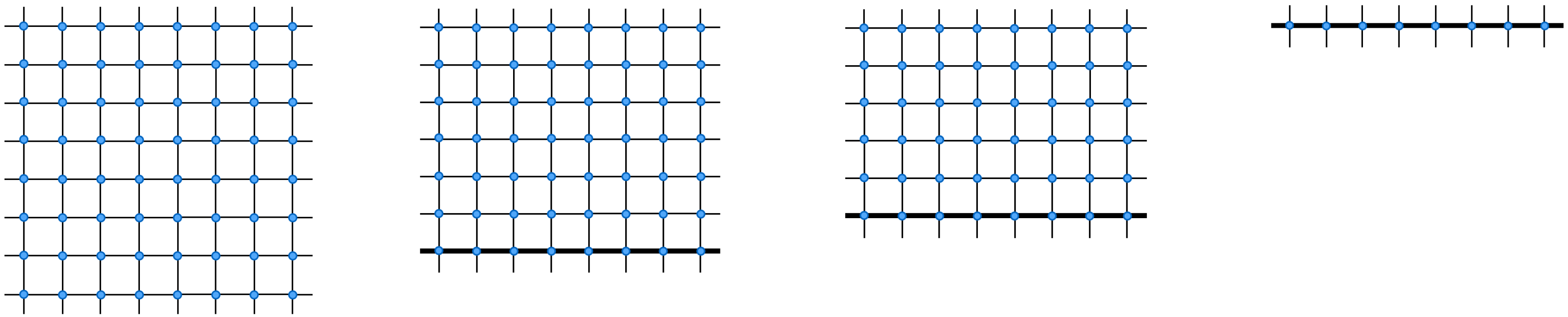
Physical properties of 2D quantum system



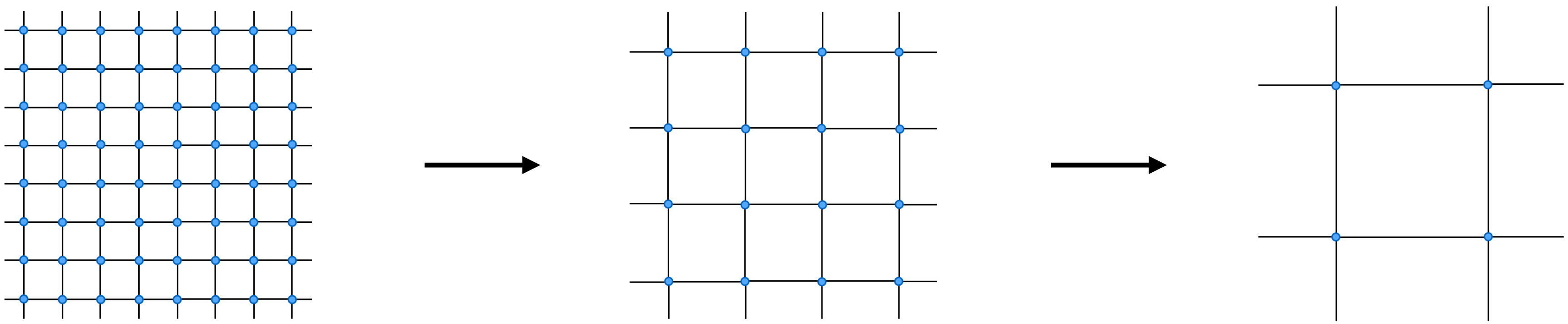
Overview

tensor network + renormalization group = tensor network renormalization

- boundary MPS



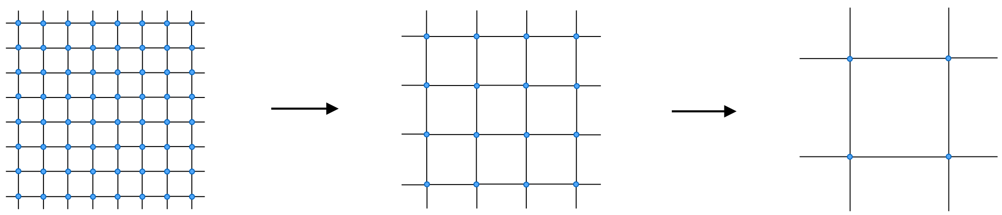
- fully isotropic coarse-graining



Overview

tensor network + renormalization group = tensor network renormalization

How the physics change with scale



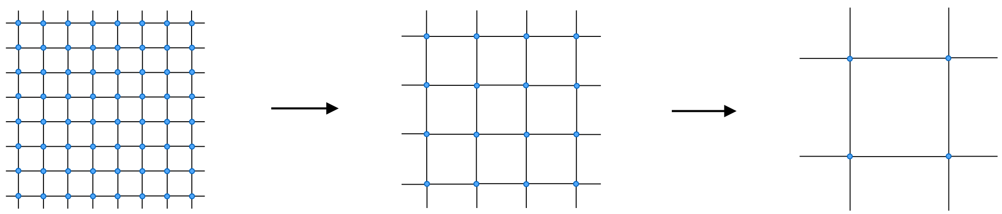
scale transformation

- Non-perturbative approach, suitable for strongly correlated systems
- Reproduce long-range physics

Overview

tensor network + renormalization group = tensor network renormalization

How the physics change with scale



scale transformation

- Non-perturbative approach, suitable for strongly correlated systems
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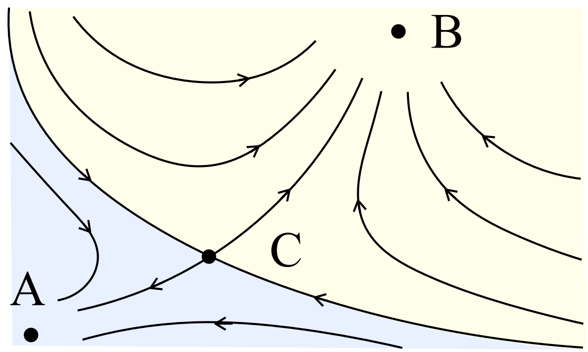
RG flow



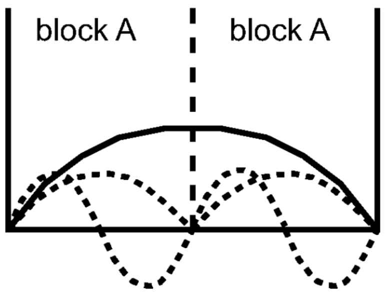
Aim

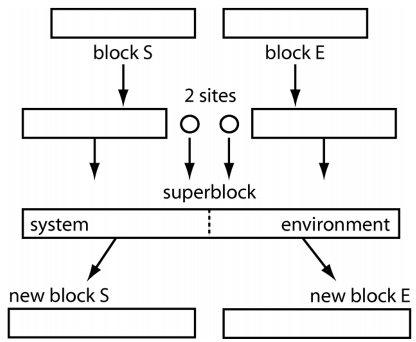
- Remove short-range entanglement / correlations
- Generate proper RG flow & correct fixed points
- Recover scale invariance at criticality

Real space renormalization group

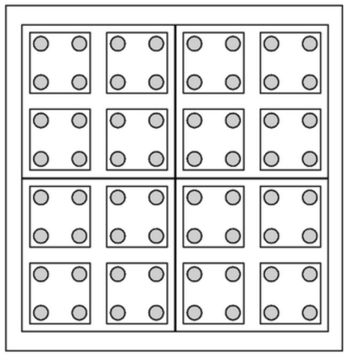
1D quantum: real space NRG —entanglement→ DMRG S. White (1992)

Real space analogue of NRG
Wilson (1975)
breakdown for "particle in a box" problem

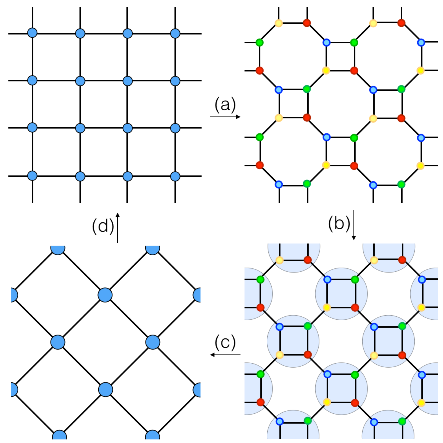




2D classical: block spin —entanglement→ TRG Levin & Nave (2007)

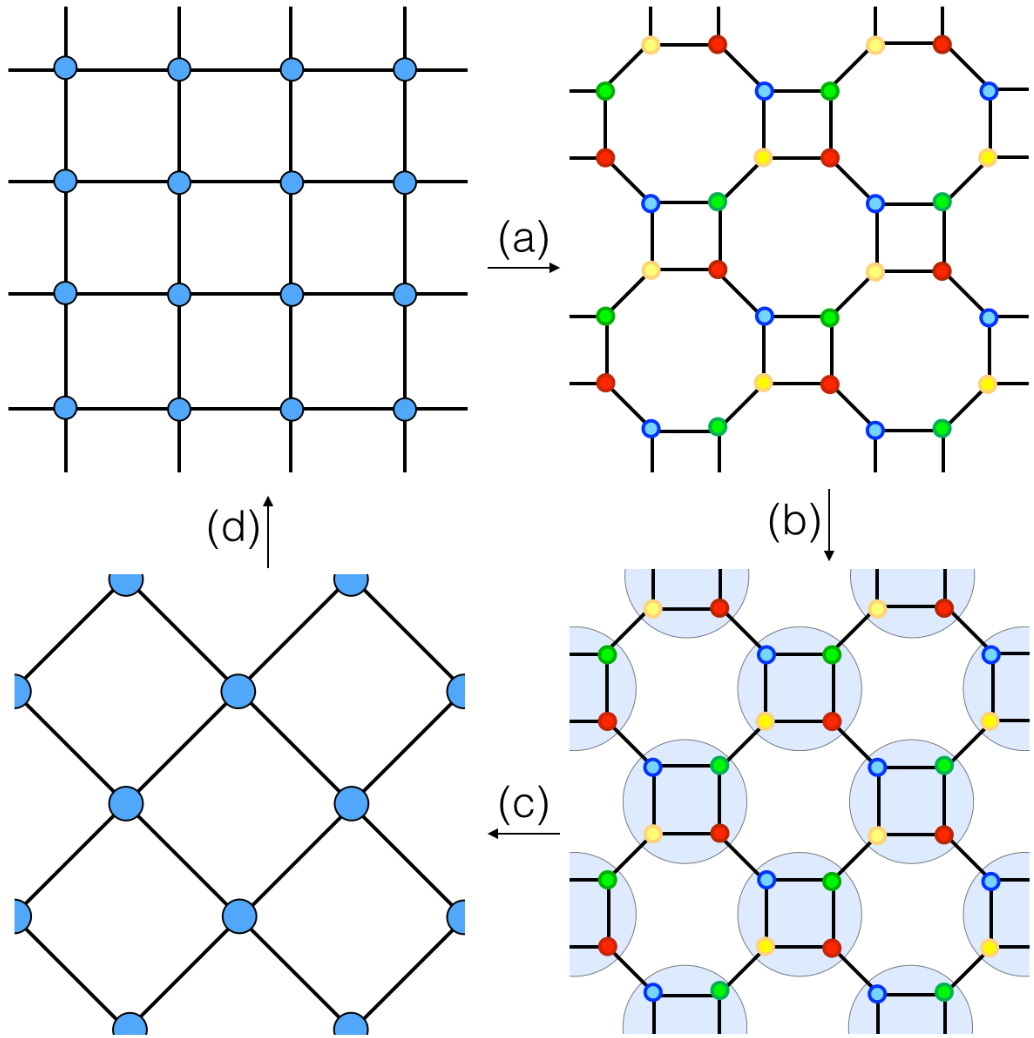


L.P. Kadanoff (1966)



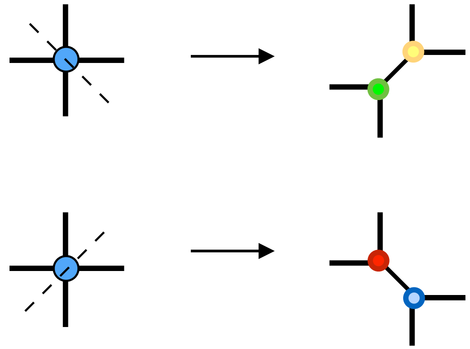
Tensor renormalization group

Levin & Nave (2007)   LN-TNR

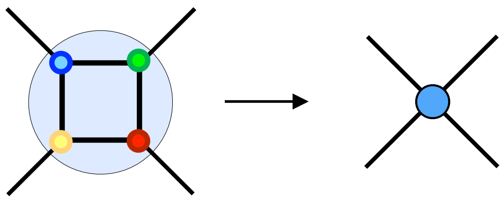


Three steps

1. Deform tensors, make a truncation

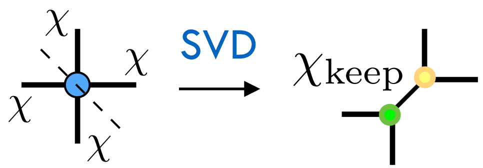


2. Coarse graining



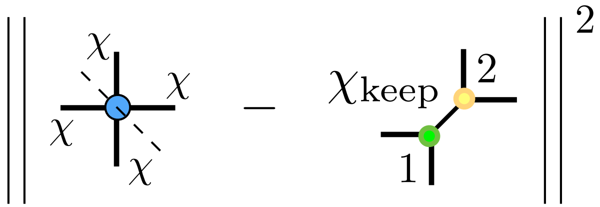
3. Renormalize tensors
(multiply tensor by a constant factor)

Tensor renormalization group



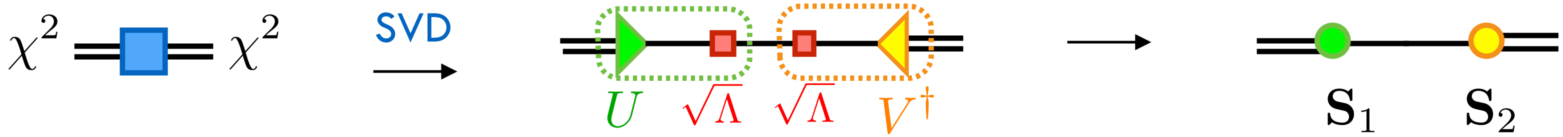
singular value decomposition (SVD)

only keep the largest χ_{keep} singular values



cost function

$||\mathbf{T} - \mathbf{S}_1 \cdot \mathbf{S}_2||^2$



original

$\chi_{\text{keep}} = \chi^2$

$\chi_{\text{keep}} = 2\chi$

$\chi_{\text{keep}} = \chi$

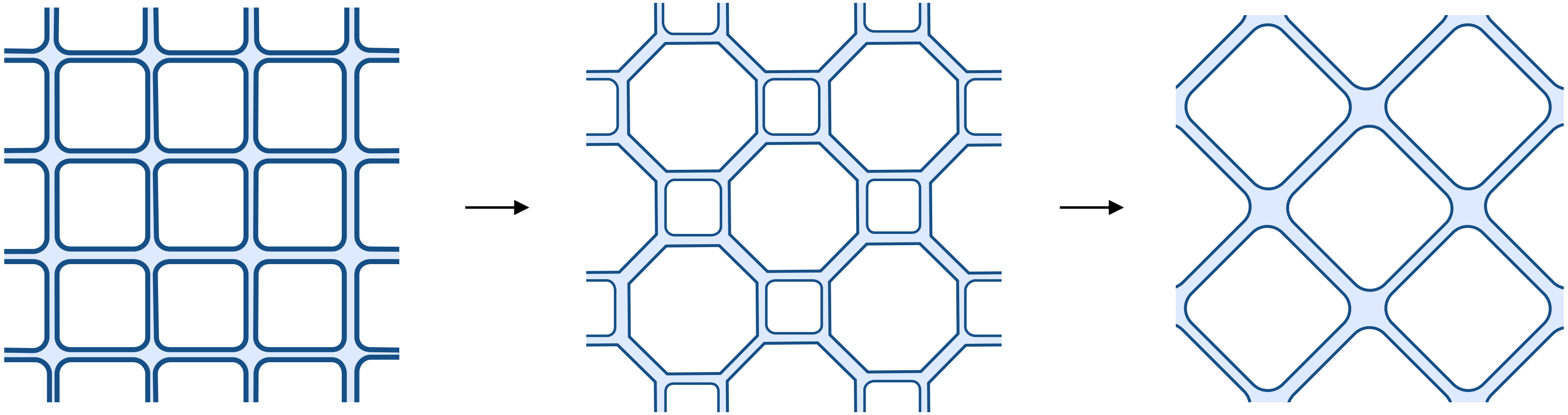
$\chi_{\text{keep}} = \chi/4$

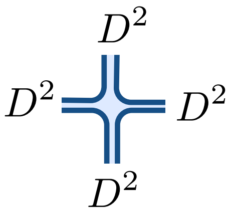
Drawbacks of LN-TNR

Off criticality

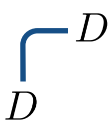
cannot remove conner-double-line (CDL) tensors

cannot give the correct structures of fixed points

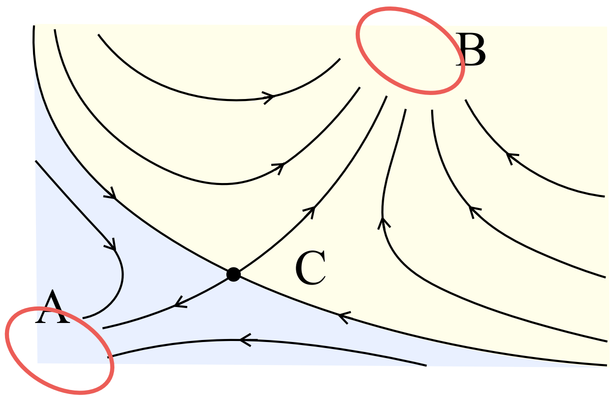




$$\chi = D^2$$



$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_D \end{pmatrix}$$



Drawbacks of LN-TNR

At criticality

cannot explicitly recover scale invariance

cannot completely remove short-range entanglement

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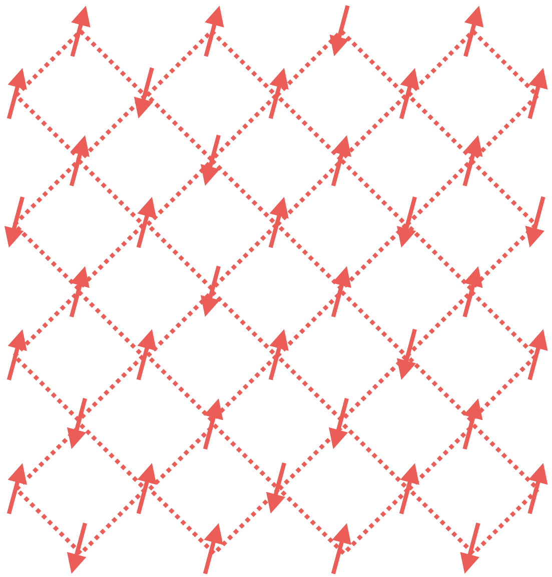
Example

classical Ising model



partition function $$Z = \sum_{\{\sigma\}} \exp(\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j)$$

Drawbacks of LN-TNR

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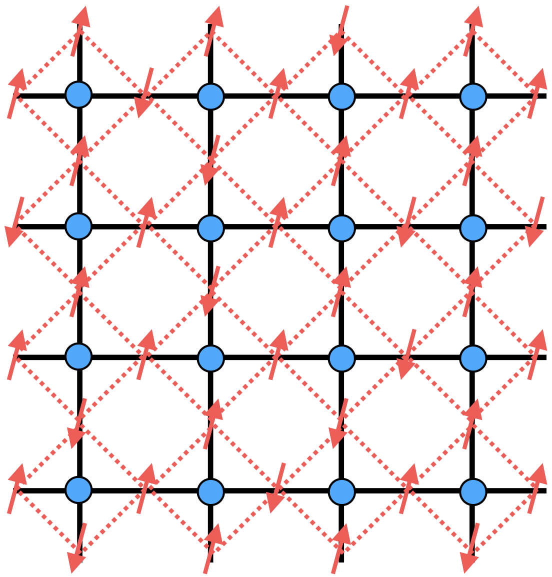
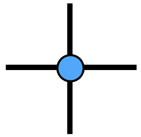


partition function $$Z = \sum_{\{\sigma\}} \exp(\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j)$$

local tensor



$$\mathbf{T} = T^{\text{Ising}}_{u,l,d,r}$$

$$T^{\text{Ising}}_{1,2,1,2} = e^{-4\beta},\ T^{\text{Ising}}_{2,1,2,1} = e^{-4\beta},$$
$$T^{\text{Ising}}_{1,1,1,1} = e^{4\beta},\ T^{\text{Ising}}_{2,2,2,2} = e^{4\beta},$$
$$\text{others} = 1.$$

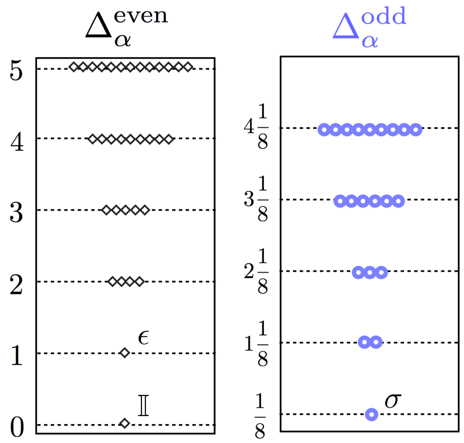
Drawbacks of LN-TNR

Ising CFT

central charge

$c = 1/2$

scaling dimensions →



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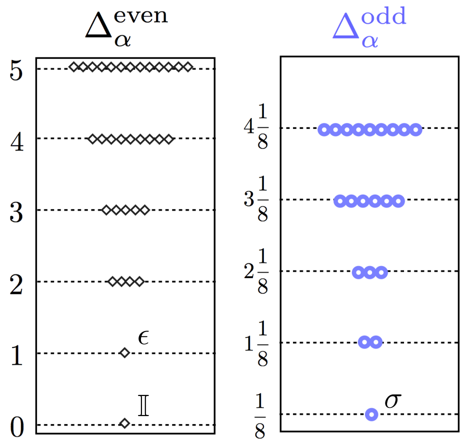
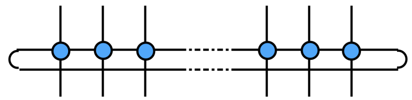
scaling dimensions $\longrightarrow$



Calculate scaling dimensions

transfer matrix



eigenvalues of the transfer matrix $\longrightarrow$ c, Δ_α

Zheng-Cheng Gu and Xiao-Gang Wen,
Phys. Rev. B **80**, 155131 (2009).

Drawbacks of LN-TNR

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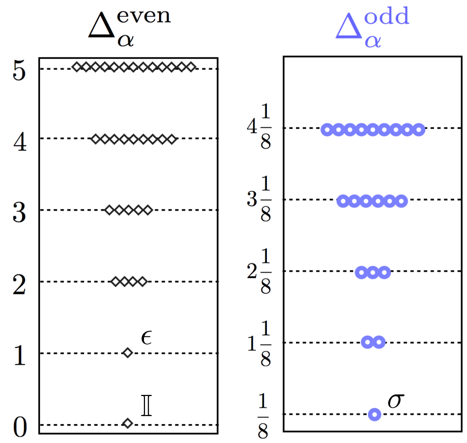

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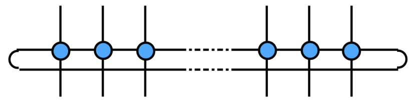

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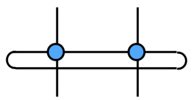
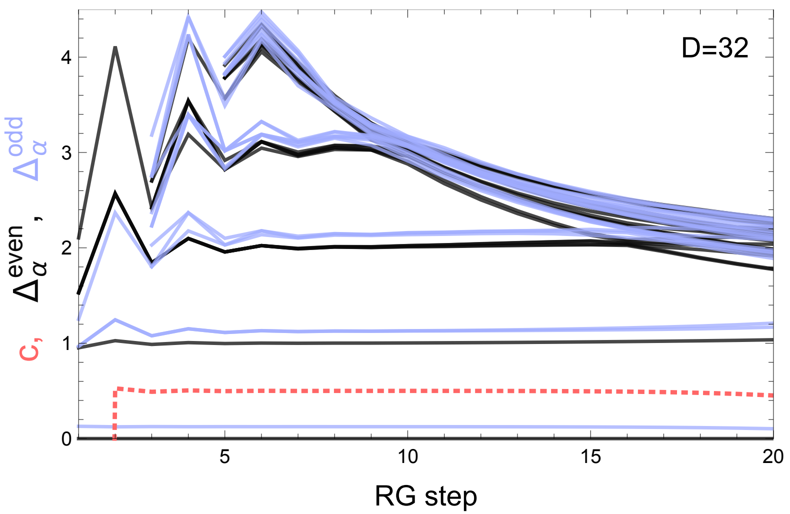
LN-TNR



$L = 2$



scaling dimensions change with RG step

cannot explicitly recover scale invariance

Drawbacks of LN-TNR

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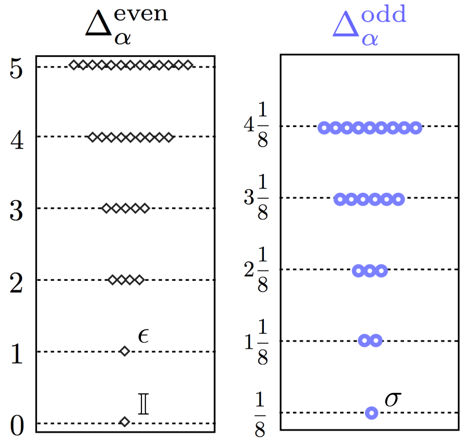
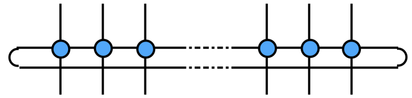
scaling dimensions →



Calculate scaling dimensions

transfer matrix



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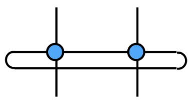
LN-TNR



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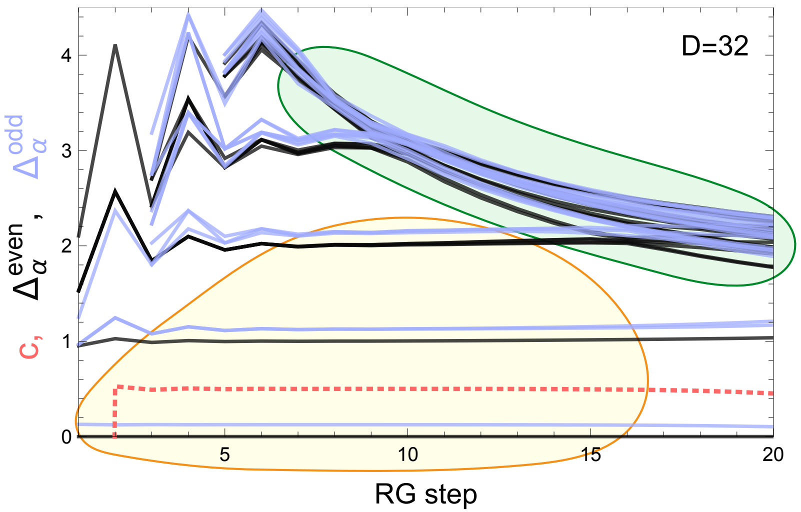


scaling dimensions change with RG step

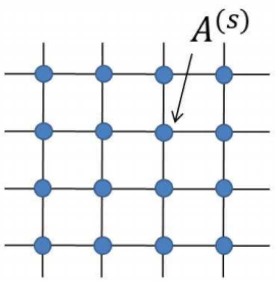
cannot explicitly recover scale invariance

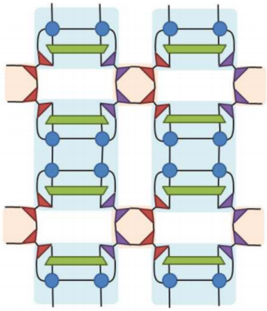
high-index parts will destroy low-index parts

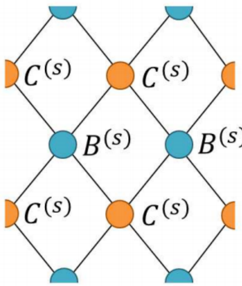
accuracy is fine, stability is bad

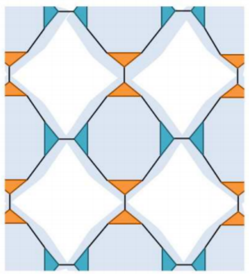
Improvements of TRG

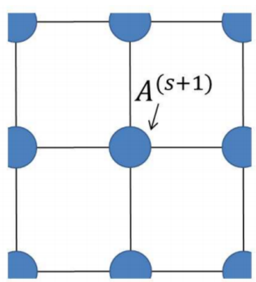
	off-critical	critical
TRG (Levin & Nave, 2006)		
SRG (Xie, Jiang, Weng, Xiang, 2008)		
TEFR (Gu & Wen, 2009)		
TNR (Evenbly & Vidal, 2014)		











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TNR (Evenbly & Vidal, 2014)		
Loop-TRG (current)		

Renormalization Group

Momentum space

- Tree level approximation — no loop

- Beyond tree level — one loop

Real space

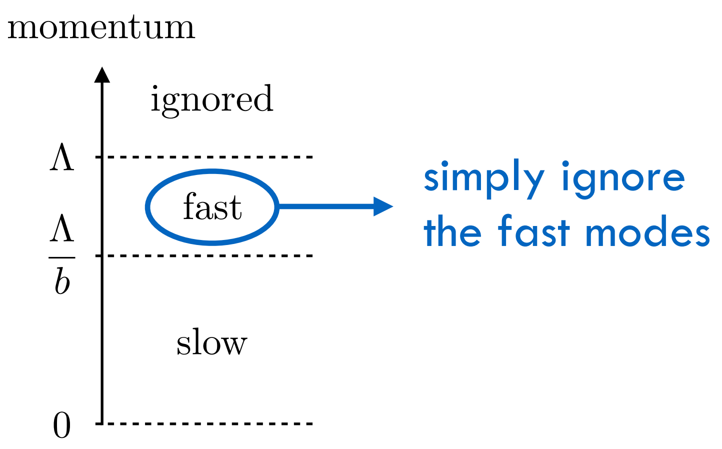
- Tree level approximation — no loop

- Beyond tree level — one loop

Renormalization Group

Momentum space

- Tree level approximation — no loop



- Beyond tree level — one loop

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Renormalization Group

Momentum space

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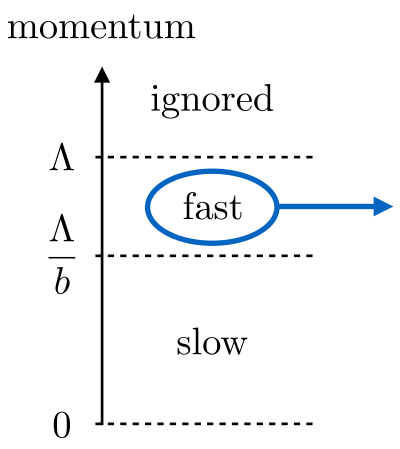
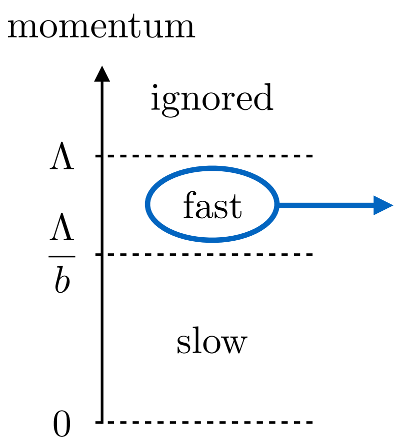
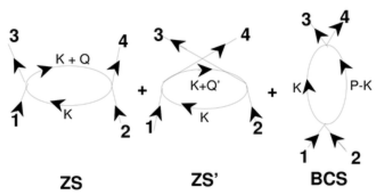


simply ignore
the fast modes

- Beyond tree level — one loop



integrate over
the fast modes



Real space

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Renormalization Group

Momentum space

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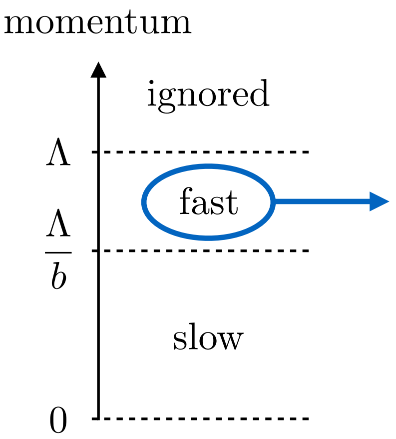

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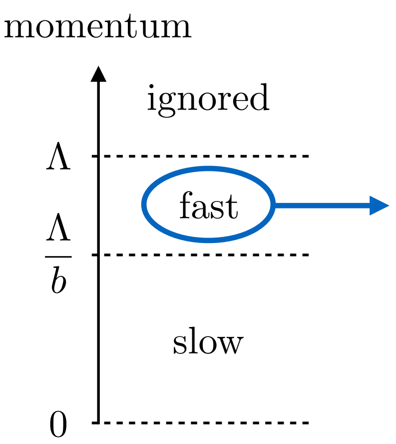

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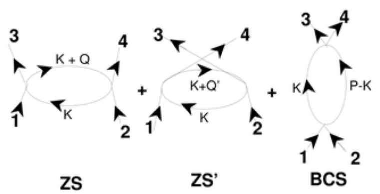
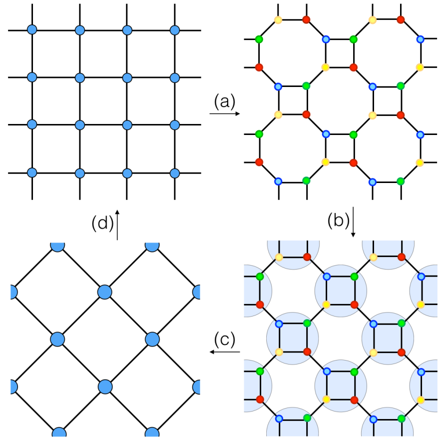


Real space

- Tree level approximation — no loop



LN-TNR

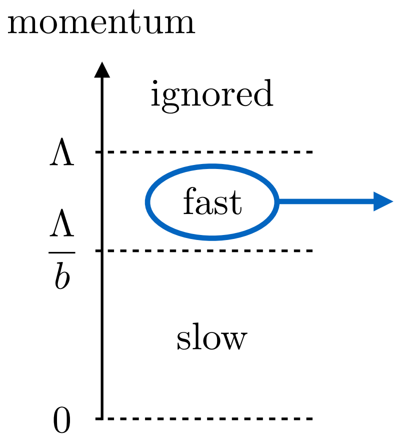
remove short-range
entanglement by a
local SVD

- Beyond tree level — one loop

Renormalization Group

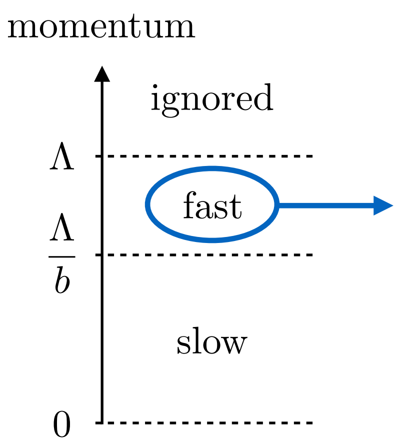
Momentum space

- Tree level approximation — no loop

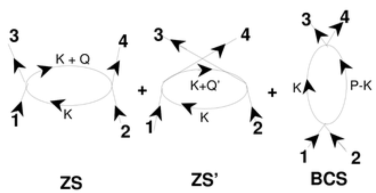


simply ignore the fast modes

- Beyond tree level — one loop

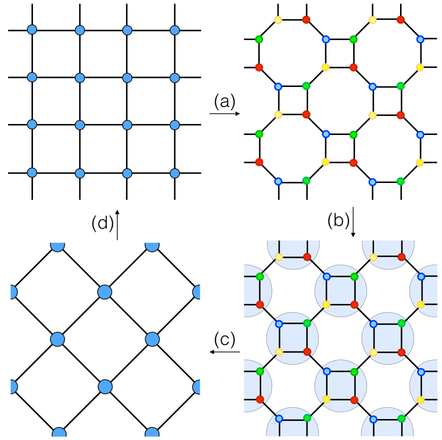


integrate over the fast modes



Real space

- Tree level approximation — no loop



LN-TNR

remove short-range entanglement by a local SVD

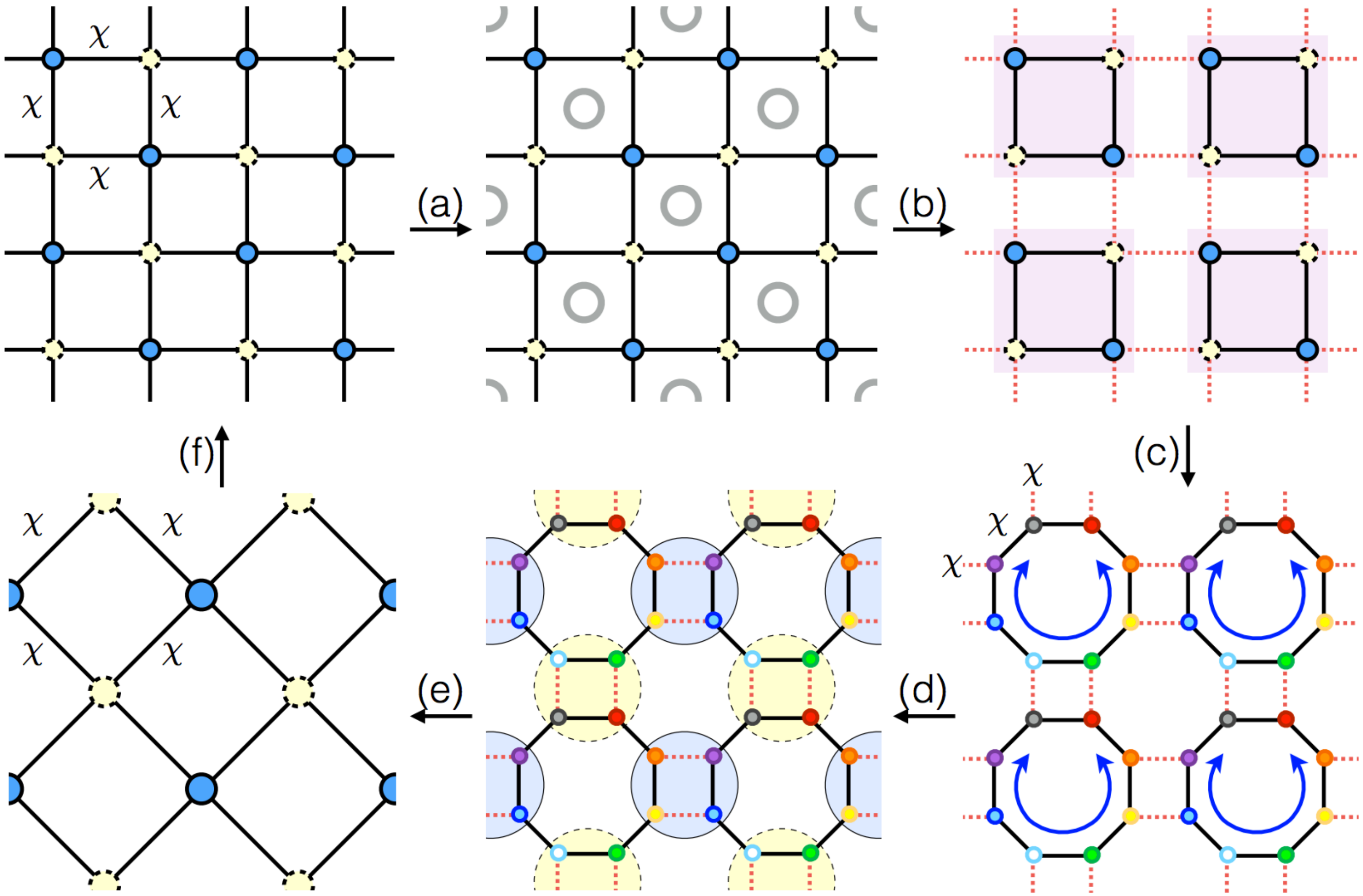
- Beyond tree level — one loop

LN-TNR + loop optimization

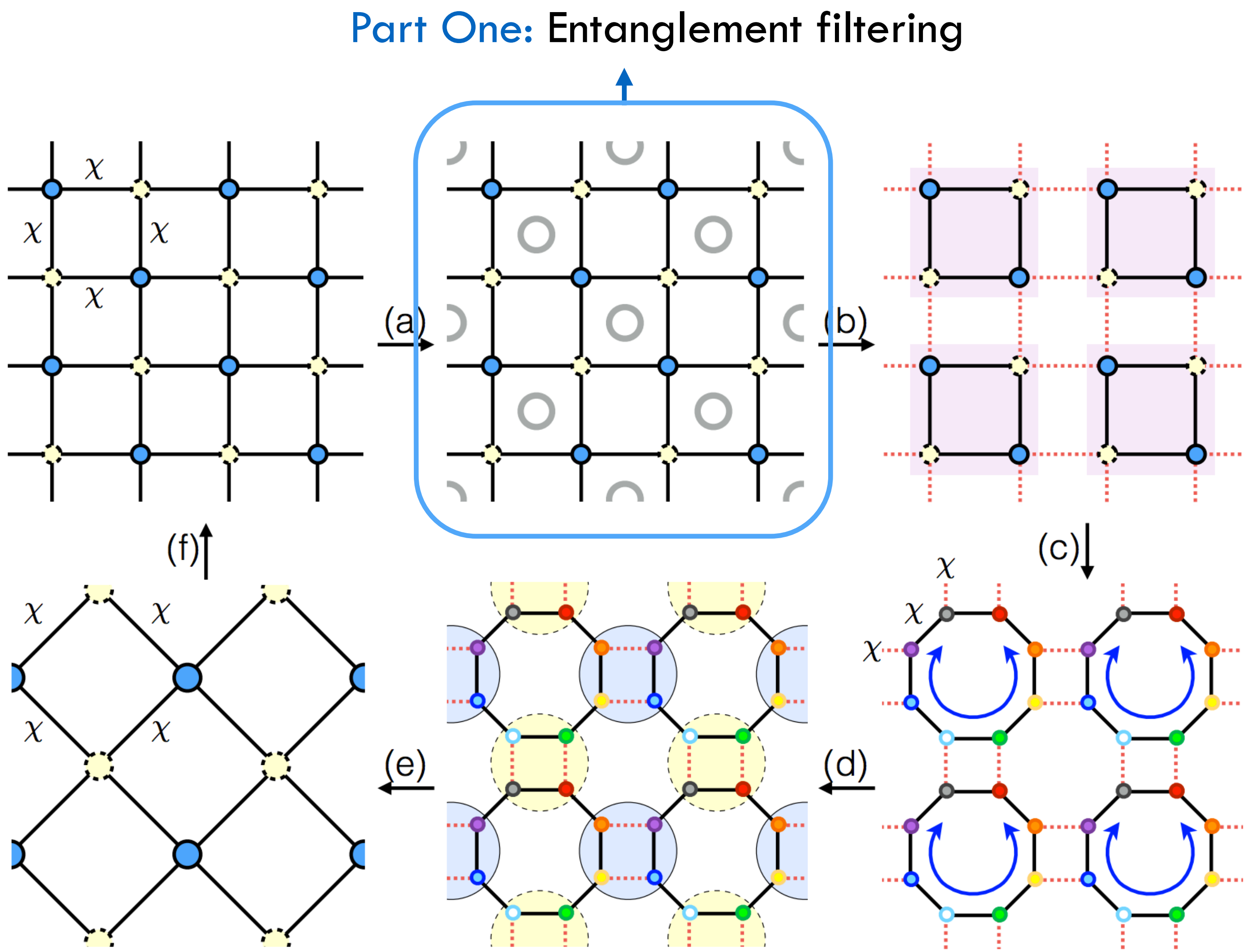
further remove short-range entanglement inside a loop

This work !

Algorithms of Loop-TNR

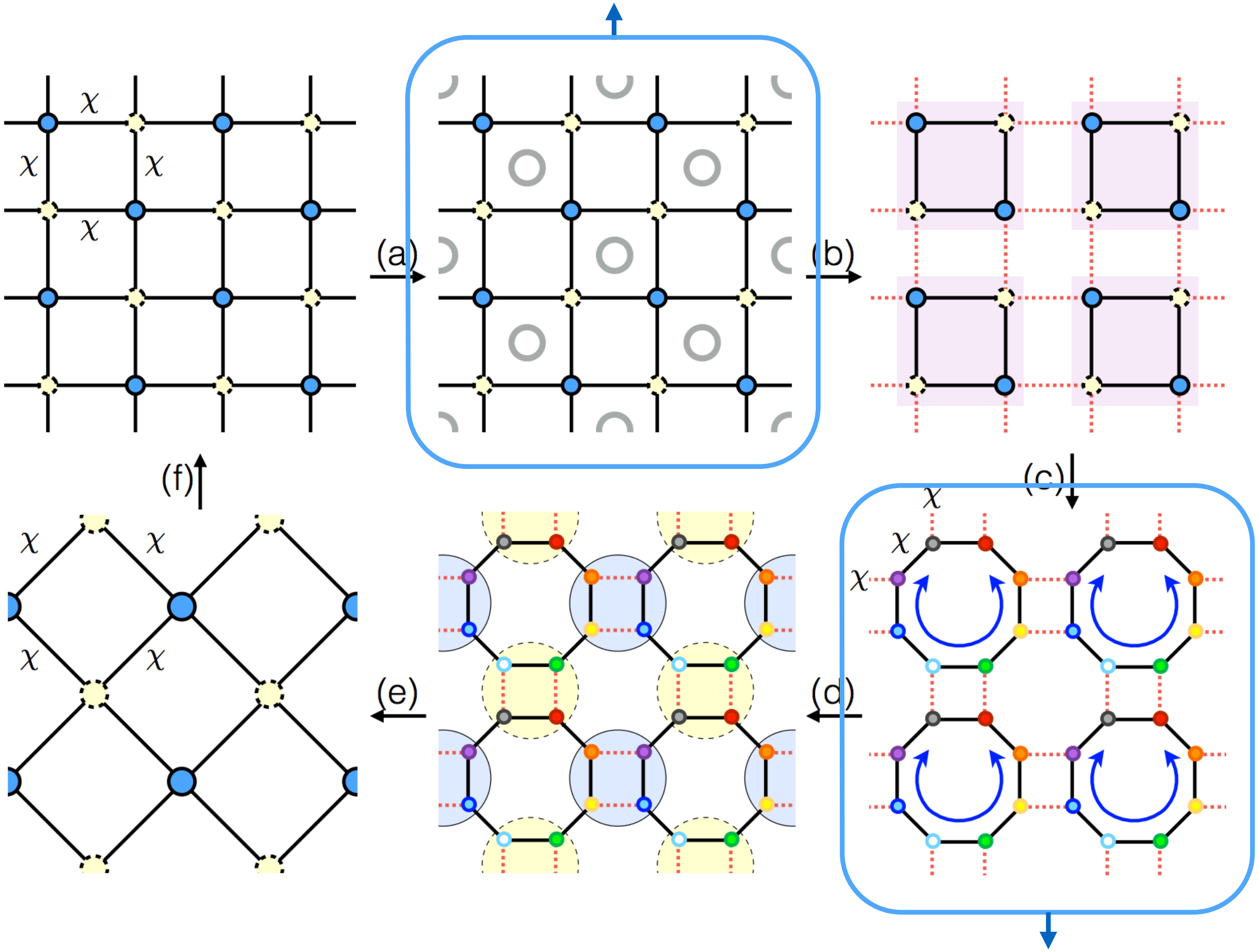


Algorithms of Loop-TNR



Algorithms of Loop-TNR

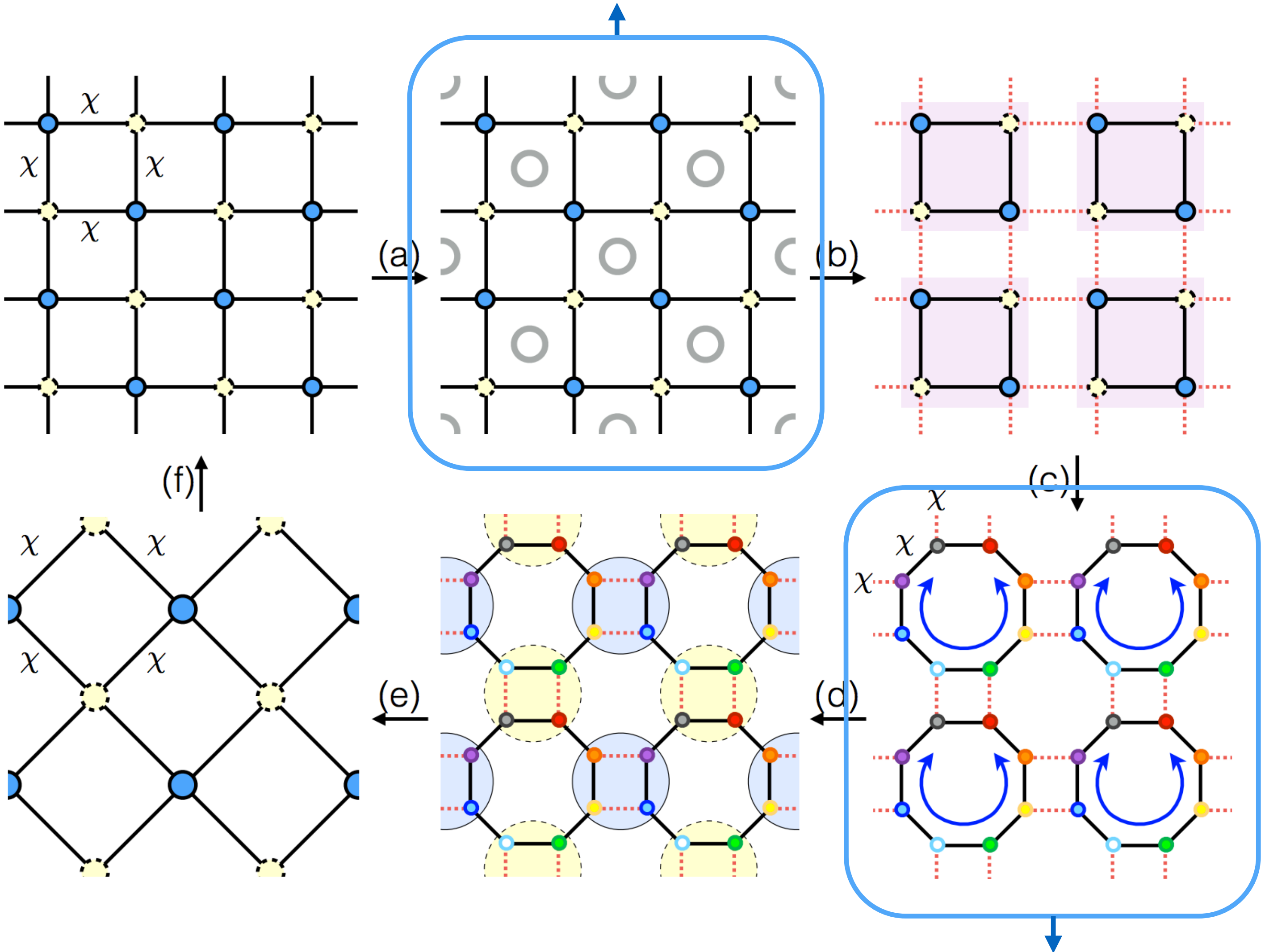
Part One: Entanglement filtering



Part Two: Optimizing tensors on a loop

Algorithms of Loop-TNR

Part One: Entanglement filtering



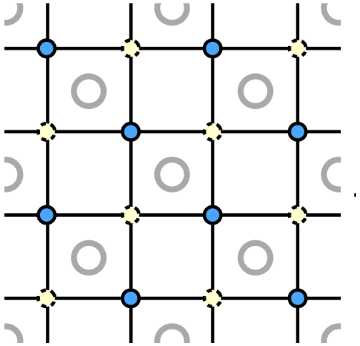
Together: Complete remove short-range entanglement

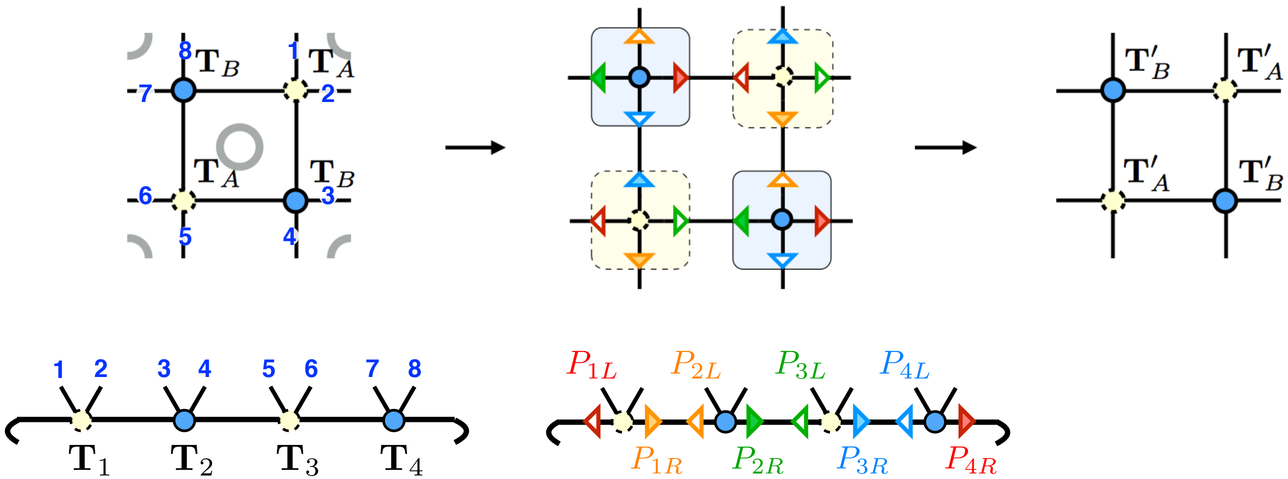
Part Two: Optimizing tensors on a loop

Part One — Entanglement filtering

How

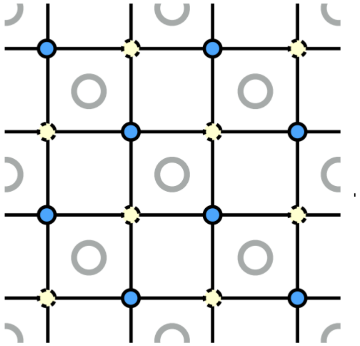
1. Find & insert projectors
2. Define new tensors





Part One — Entanglement filtering

How



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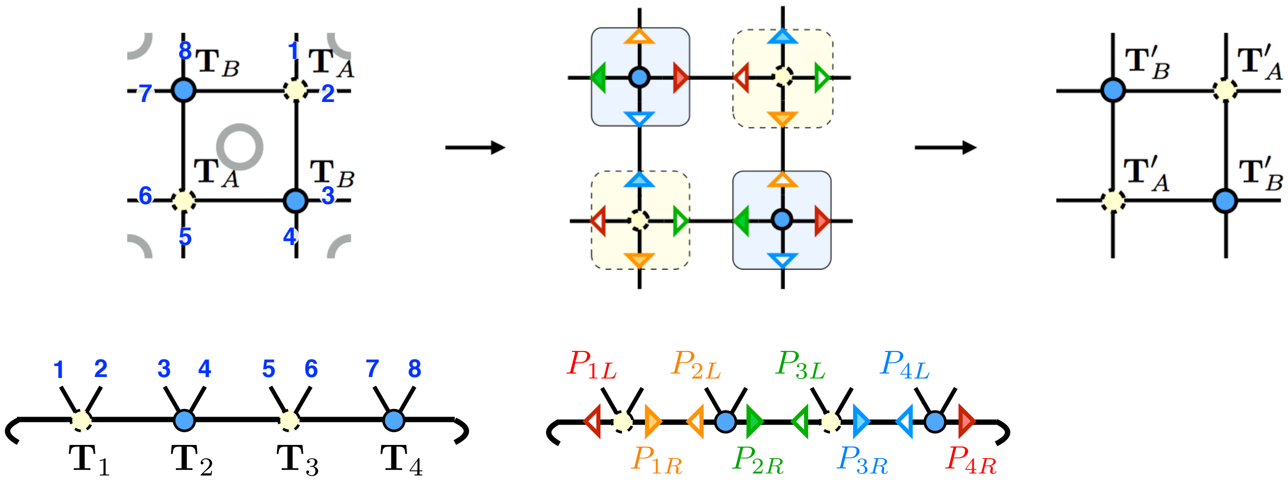
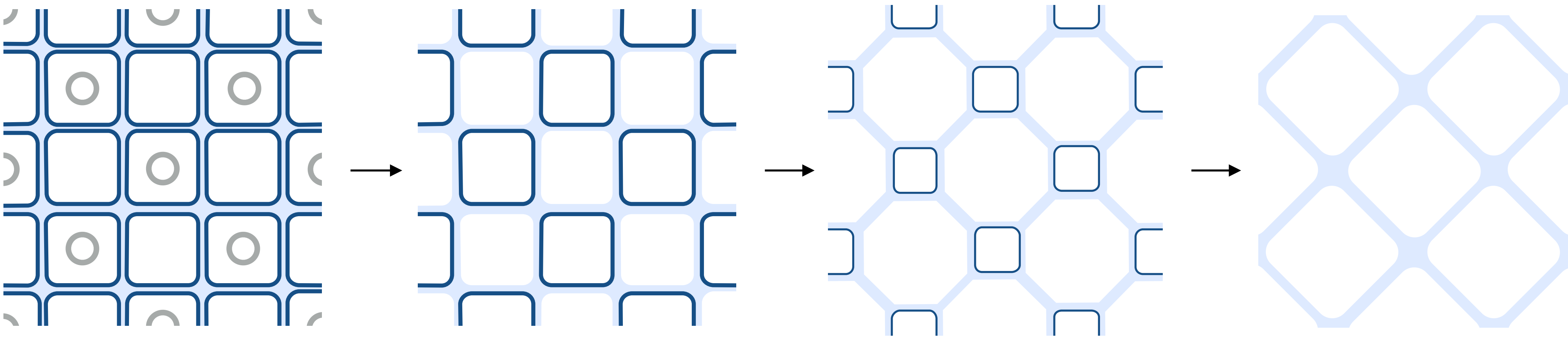


Aim

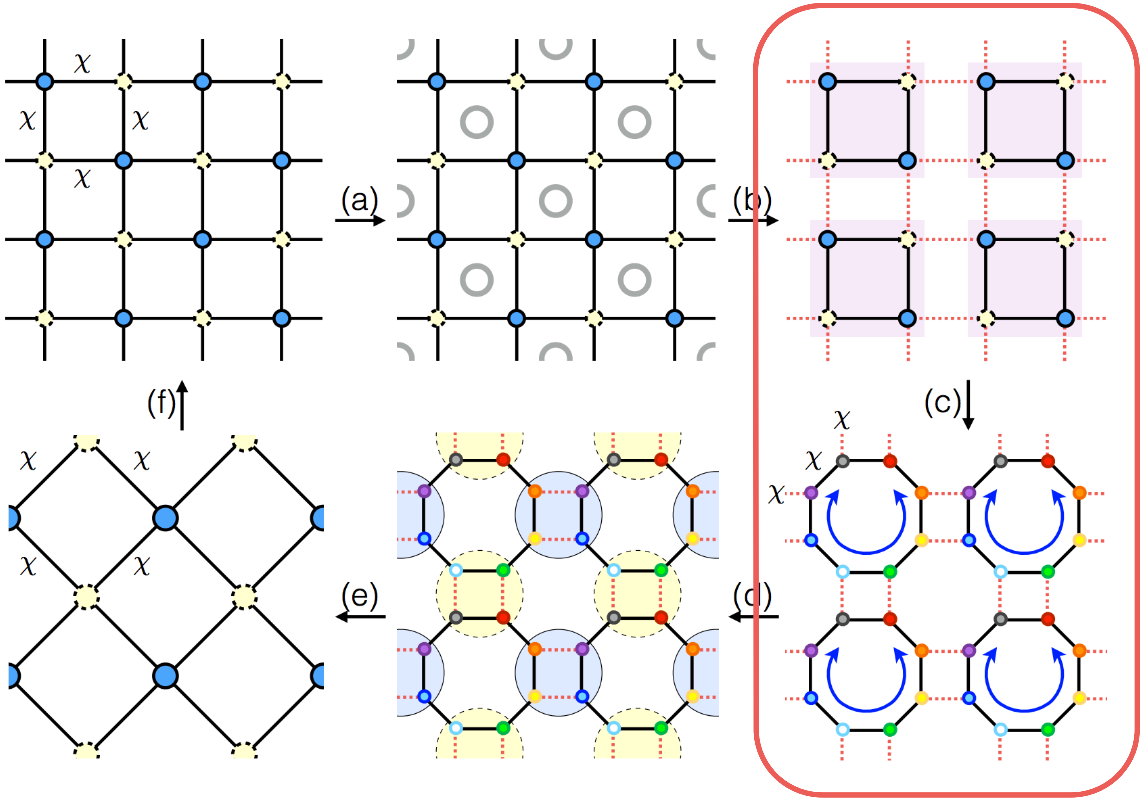
Remove conner double line (CDL) tensors

Generate local canonical gauge

Zheng-Cheng Gu and Xiao-Gang Wen, Phys. Rev. B **80**, 155131 (2009).



Part Two — Optimizing tensors on a loop



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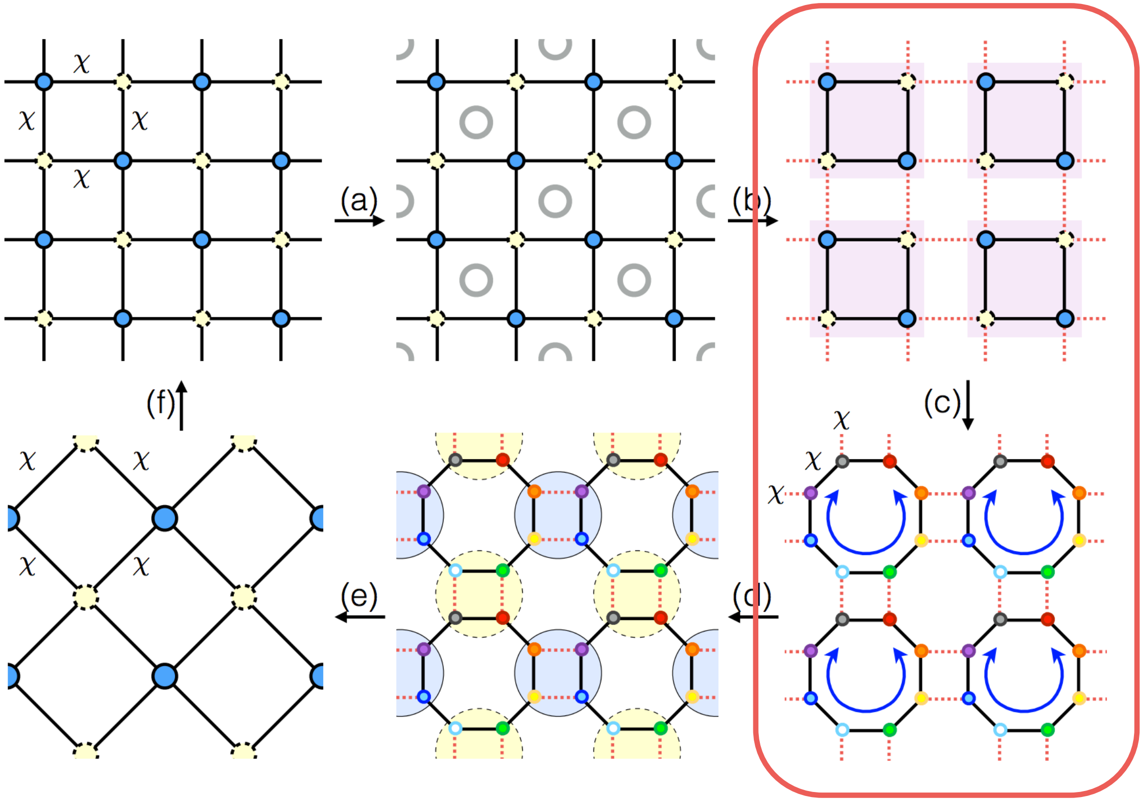
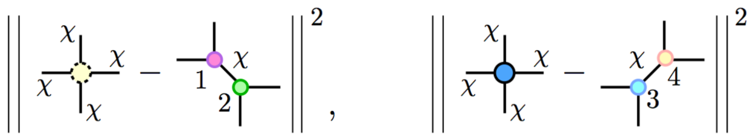
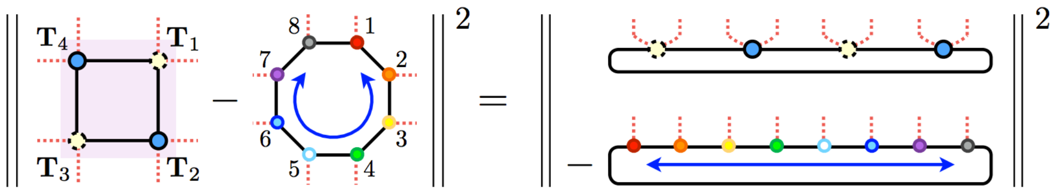


cost function

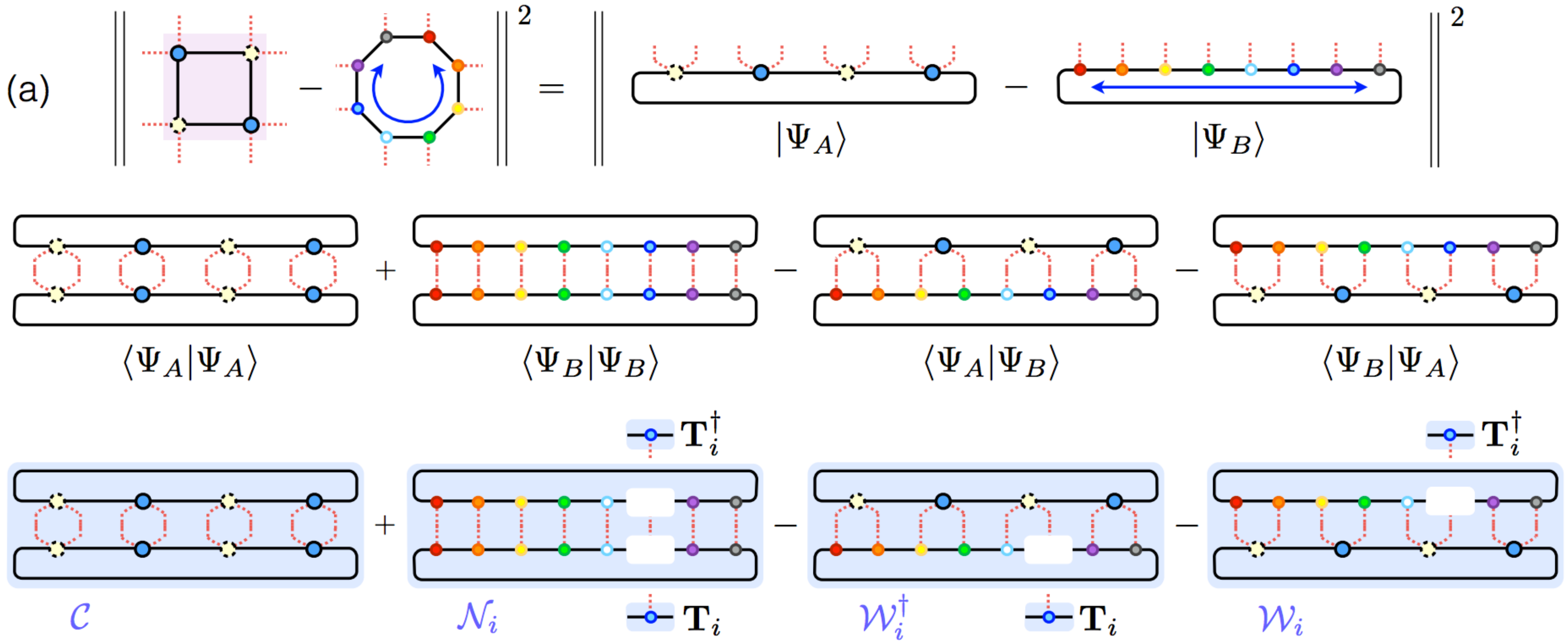
- LN-TNR



- Loop-TNR



Part Two — Optimizing tensors on a loop

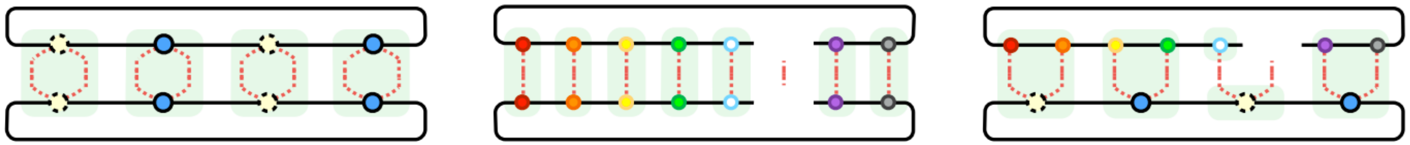


$$f(\mathbf{T}_i) = \||\Psi_A\rangle - |\Psi_B\rangle\| = \langle\Psi_A|\Psi_A\rangle + \langle\Psi_B|\Psi_B\rangle - \langle\Psi_A|\Psi_B\rangle - \langle\Psi_B|\Psi_A\rangle$$
$$= \mathcal{C} + \mathbf{T}_i^\dagger \mathcal{N}_i \mathbf{T}_i - \mathcal{W}_i^\dagger \mathbf{T}_i - \mathbf{T}_i^\dagger \mathcal{W}_i,$$

solve the linear equation $\qquad \mathcal{N}_i \mathbf{T}_i = \mathcal{W}_i.$

Part Two — Optimizing tensors on a loop

(b) contraction order



	$\mathcal{C}$	$\mathcal{N}_i$	$\mathcal{W}_i$
computational cost	$\mathcal{O}(7\chi^6)$	$\mathcal{O}(6\chi^6 + 7\chi^5)$	$\mathcal{O}(6\chi^6 + 4\chi^5)$
memory cost	$\mathcal{O}(\chi^4)$	$\mathcal{O}(\chi^4)$	$\mathcal{O}(\chi^4)$

Part Two — Optimizing tensors on a loop

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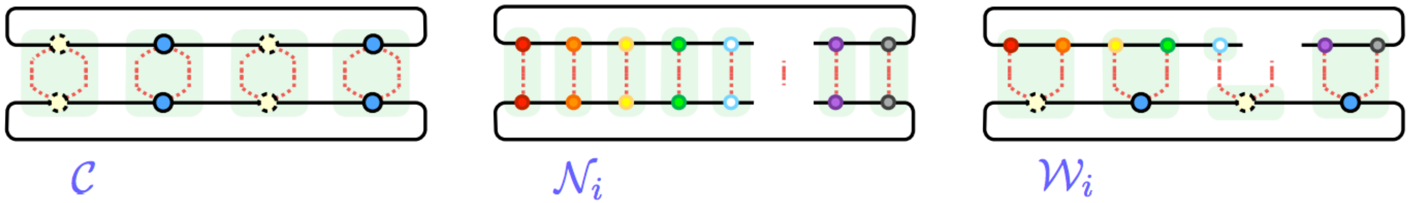

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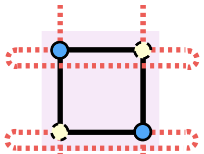
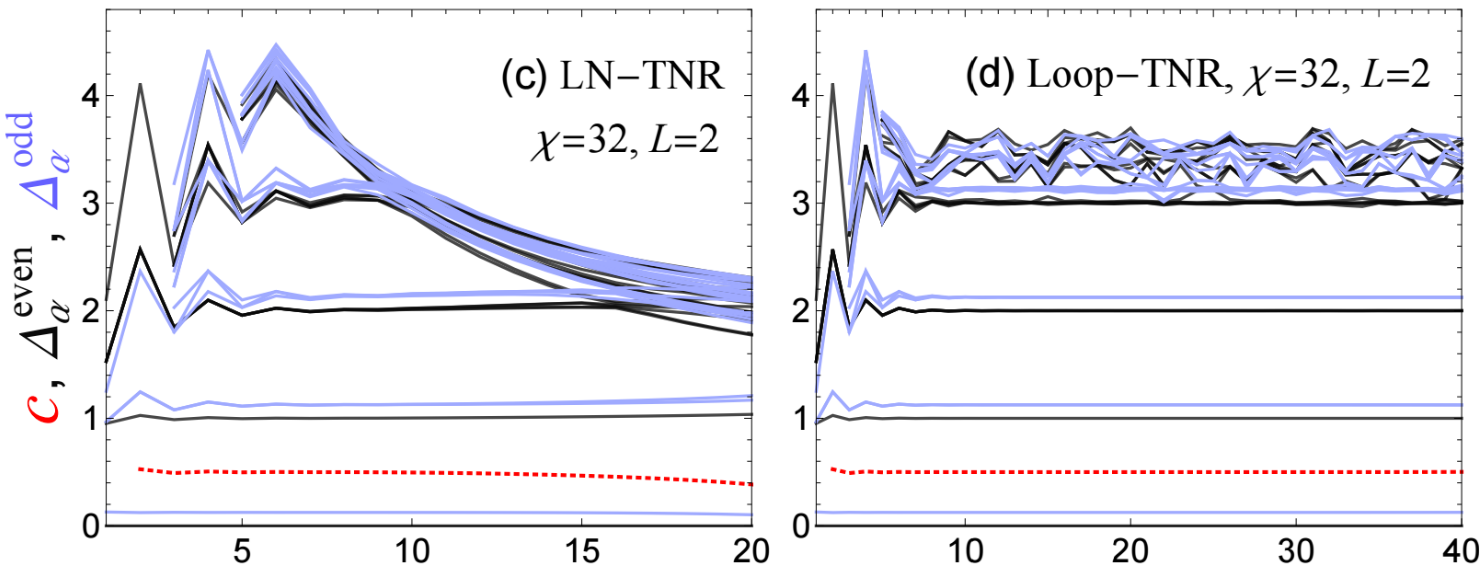
Loop-TNR Results



$L = 2$



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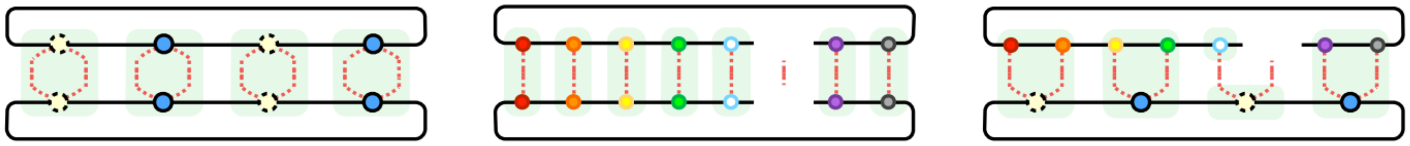

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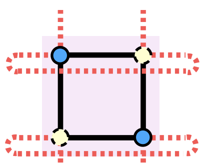
Loop-TNR Results

- scaling dimensions does not change with scale ~ scale invariance
- a clear gap between high-level parts and low-level parts



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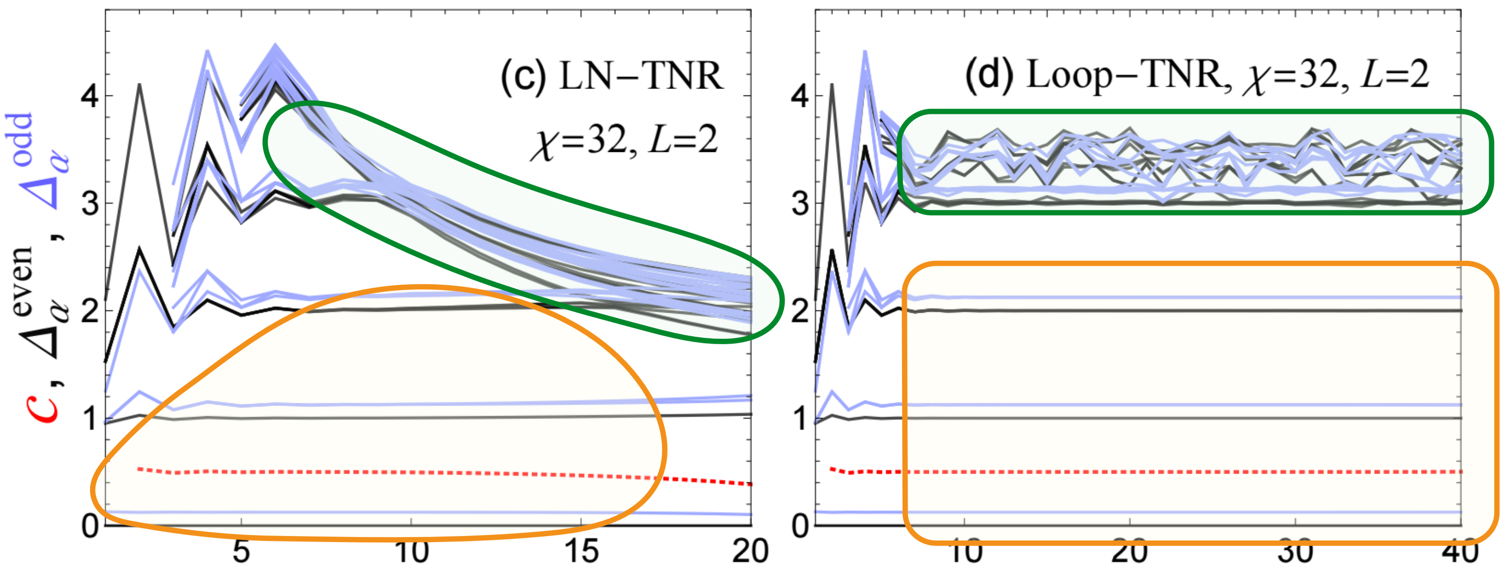
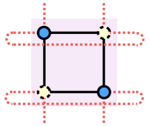
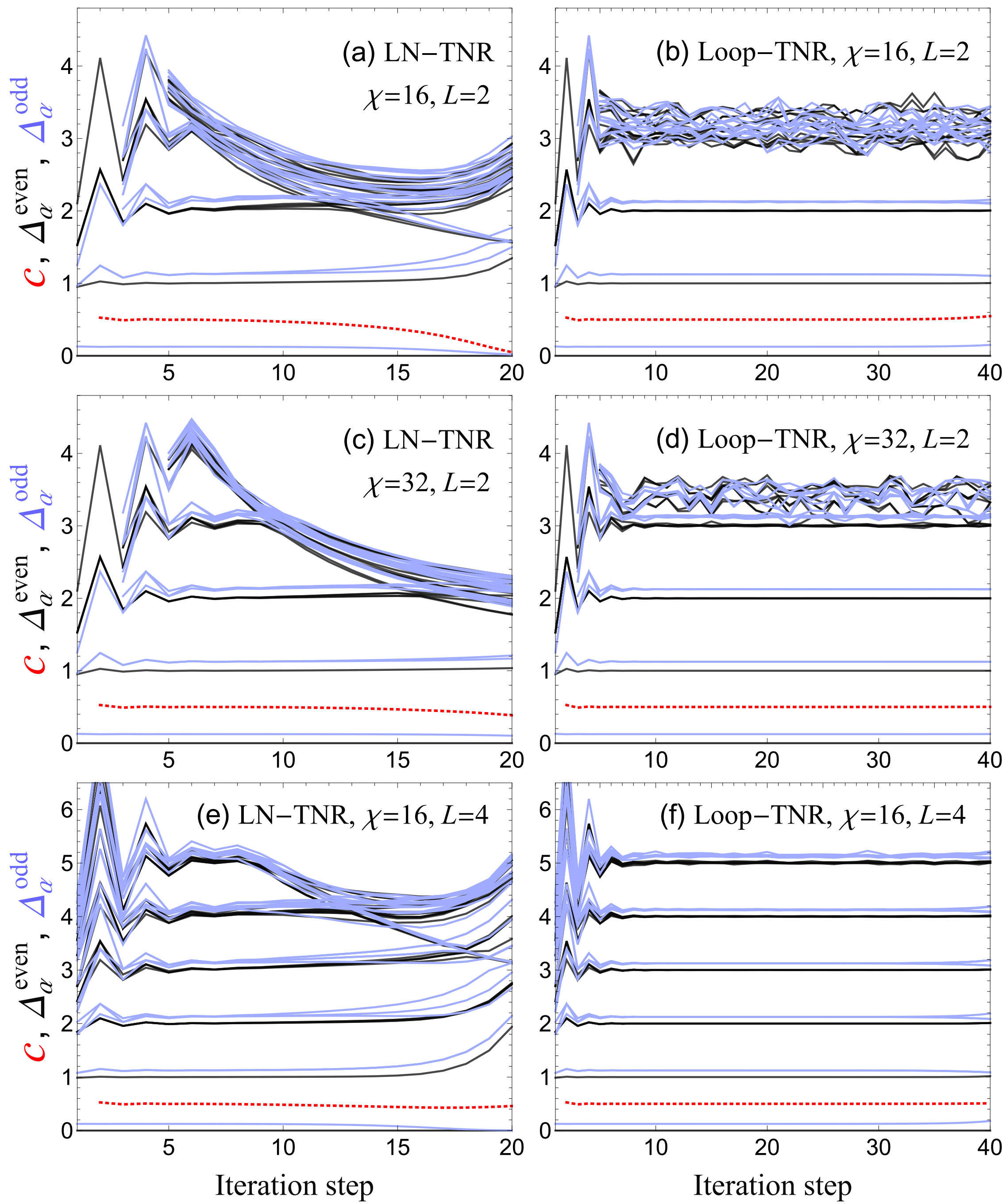
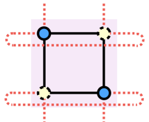


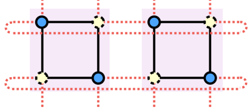
Stability





(a) LN−TNR χ=16, L=2
(b) Loop−TNR, χ=16, L=2
(c) LN−TNR χ=32, L=2
(d) Loop−TNR, χ=32, L=2
(e) LN−TNR, χ=16, L=4
(f) Loop−TNR, χ=16, L=4
c, $\Delta_\alpha^{\mathrm{even}}$, $\Delta_\alpha^{\mathrm{odd}}$
Iteration step
Iteration step





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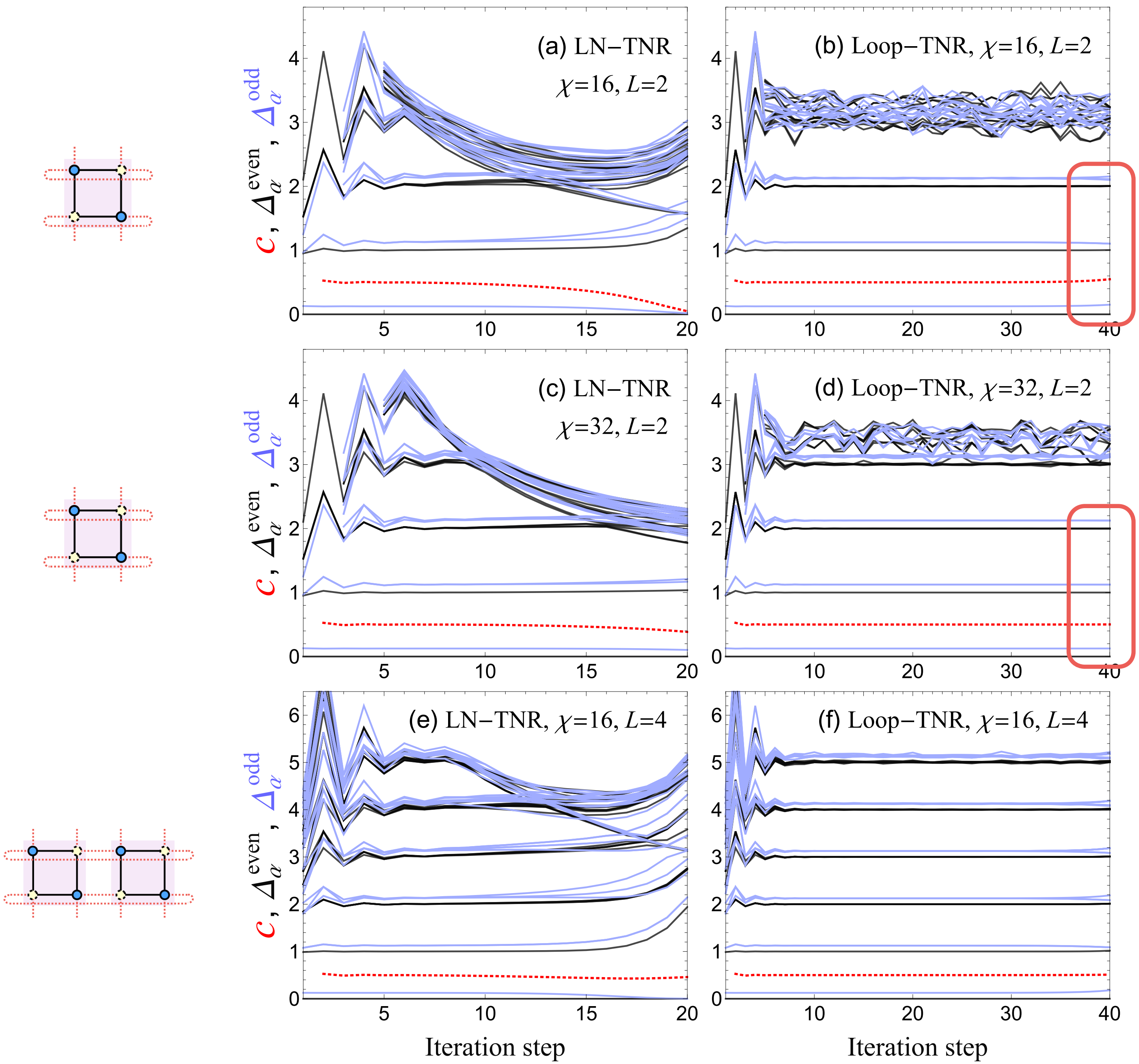


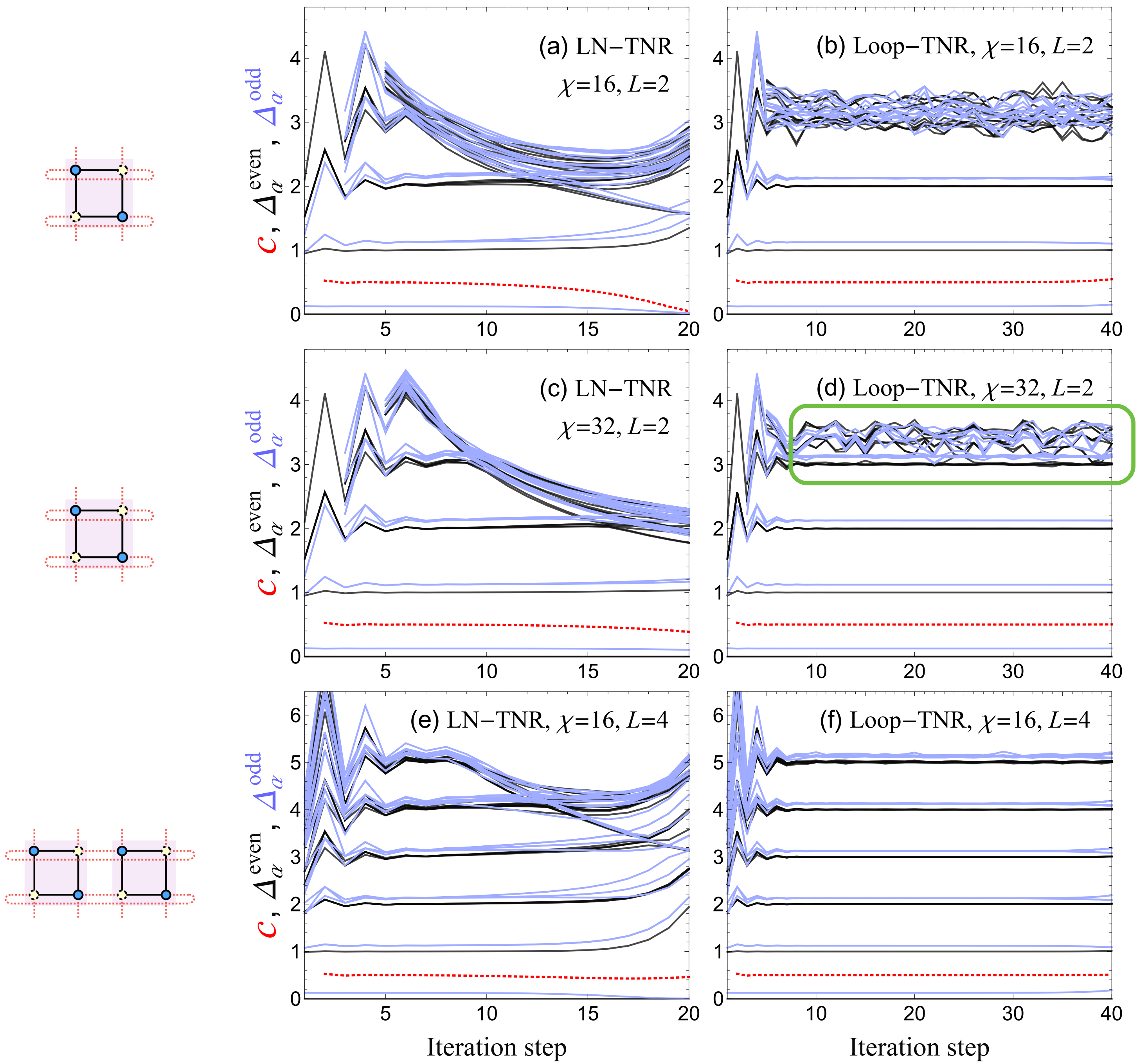
$\chi = 16$

remain accurate up to 40 iteration steps

even longer for $\chi = 32$

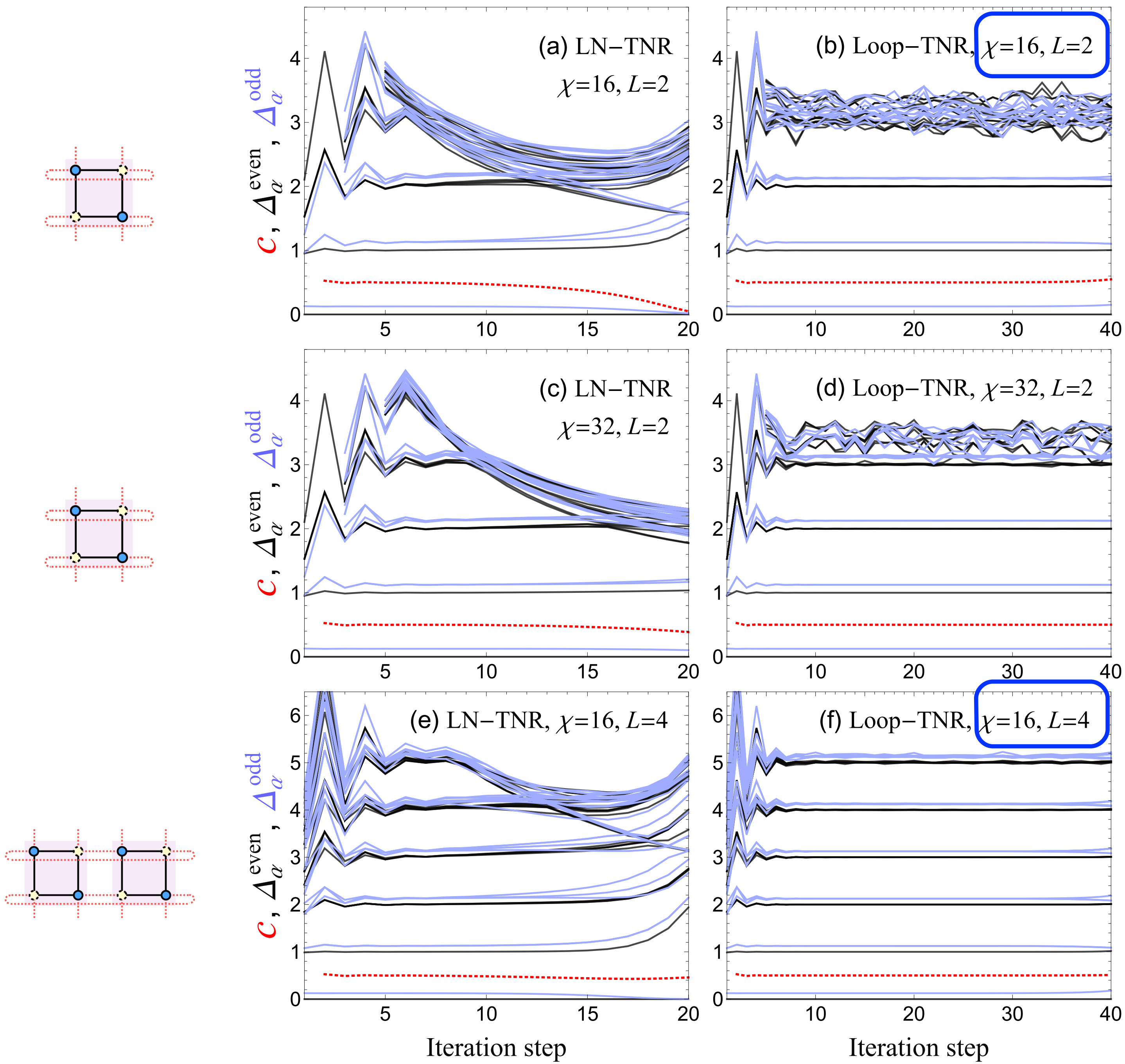
the proper RG flow last longer for larger χ

Stability



increasing χ , more scaling dimensions can be resolved

Stability



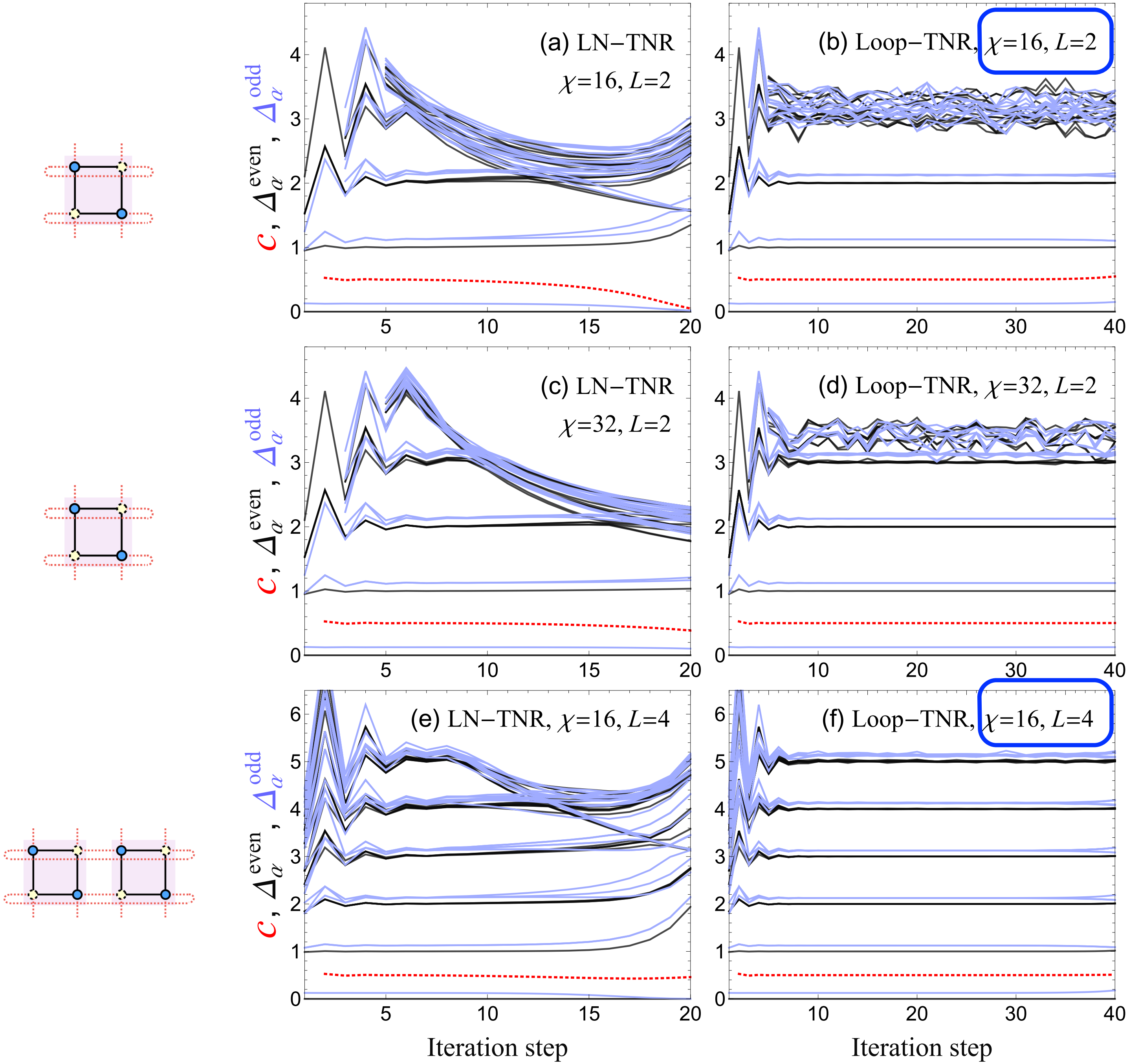
effectively, $\chi = 16$

effectively, $\chi = 16^2 = 256$

much more scaling dimensions can be read off

accuracy is higher

Stability



effectively, $\chi = 16$

effectively, $\chi = 16^2 = 256$

much more scaling dimensions can be read off

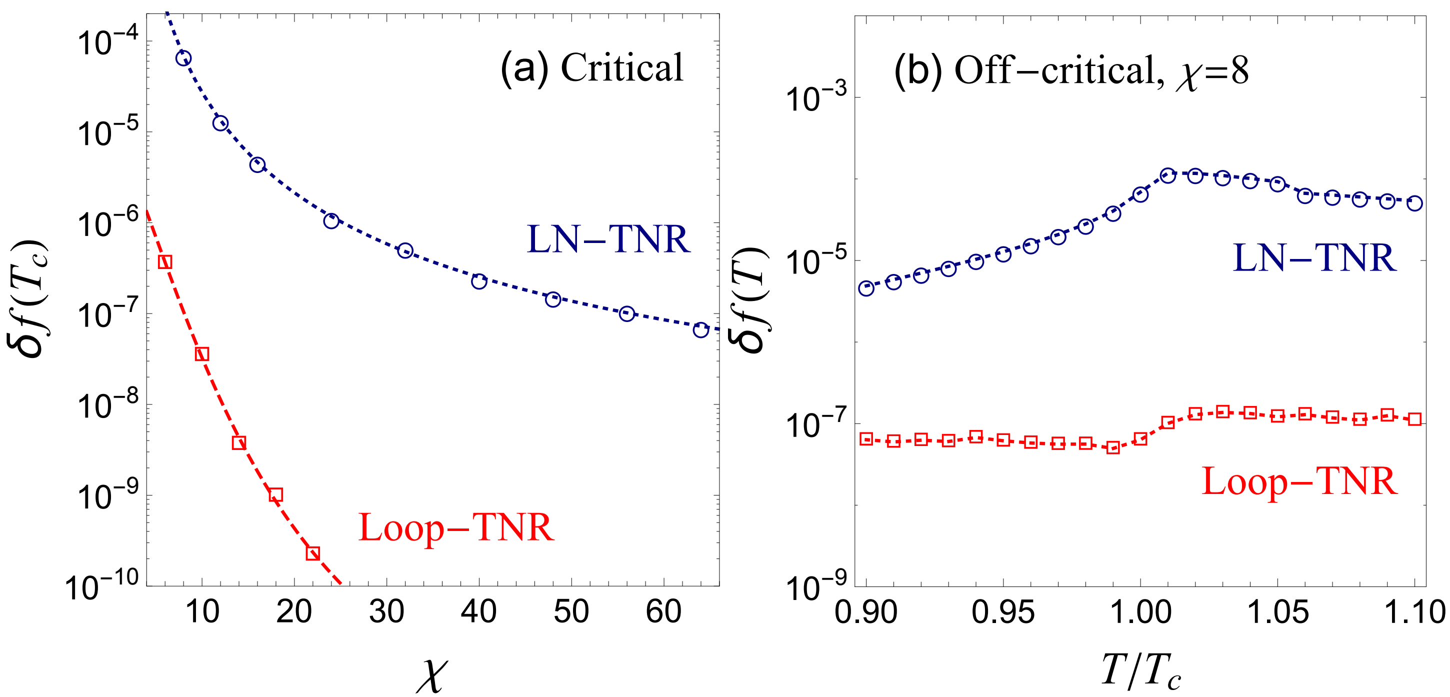
accuracy is higher

infinite χ ~ infinite dimensional fixed point tensor described by Ising CFT

Accuracy

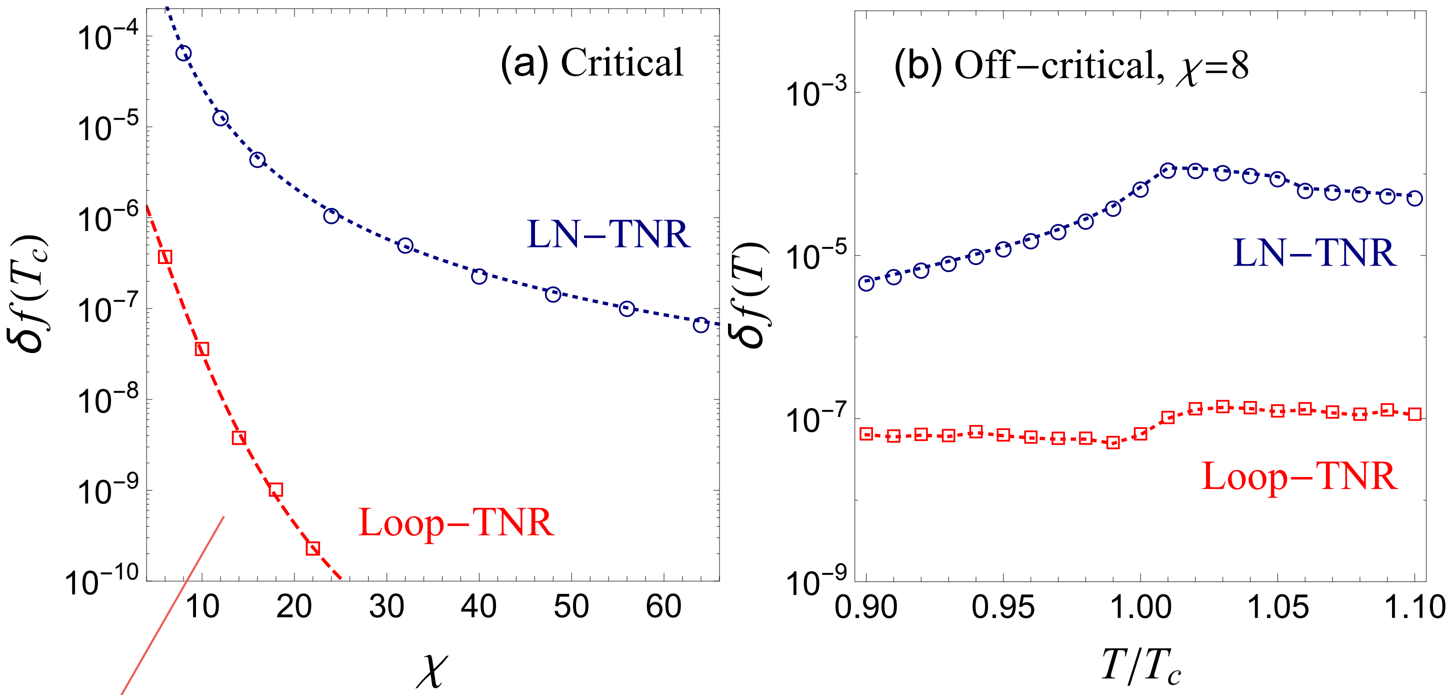
	Exact	LN-TNR $\chi = 64$ $L = 1$ 2^{11} spins	LN-TNR $\chi = 64$ $L = 2$ 2^{11} spins	Loop-TNR $\chi = 16$ $L = 2$ 2^{18} spins	Loop-TNR $\chi = 24$ $L = 2$ 2^{18} spins	Loop-TNR $\chi = 16$ $L = 4$ 2^{18} spins	Loop-TNR $\chi = 24$ $L = 4$ 2^{18} spins	EV-TNR [50] $\chi = 24$ $L = 2$ 2^{18} spins
c	0.5	0.49946958	0.49970058	0.50001491	0.50000165	0.50009255	0.50008794	0.50001
σ	0.125	0.12504027	0.12500837	0.12500528	0.12500011	0.12501117	0.12499789	0.1250004
ϵ	1	1.00028269	0.99996784	1.00000566	1.00000601	0.99999403	1.00000507	1.00009
	1.125	1.12368834	1.12444247	1.12495187	1.12499400	1.12498755	1.12500559	1.12492
	1.125	1.12394625	1.12450246	1.12510600	1.12500464	1.12498755	1.12500559	1.12510
	2	1.92334948	1.99811859	2.00000743	1.99970911	1.99999517	2.00000985	1.99922
	2	1.96264143	1.99815644	2.00066117	2.00016629	1.99999517	2.00000985	1.99986
	2	1.97496787	1.99868822	2.00066117	2.00031103	2.00002744	2.00001690	2.00006
	2	2.00274974	1.99948966	2.00586886	2.00131384	2.00006203	2.00002745	2.00168

Relative error of the free energy per site



Relative error of the free energy per site

- At critical point, the error of Loop-TNR decays much faster than the error of LN-TNR



almost decays exponentially with bond dimension

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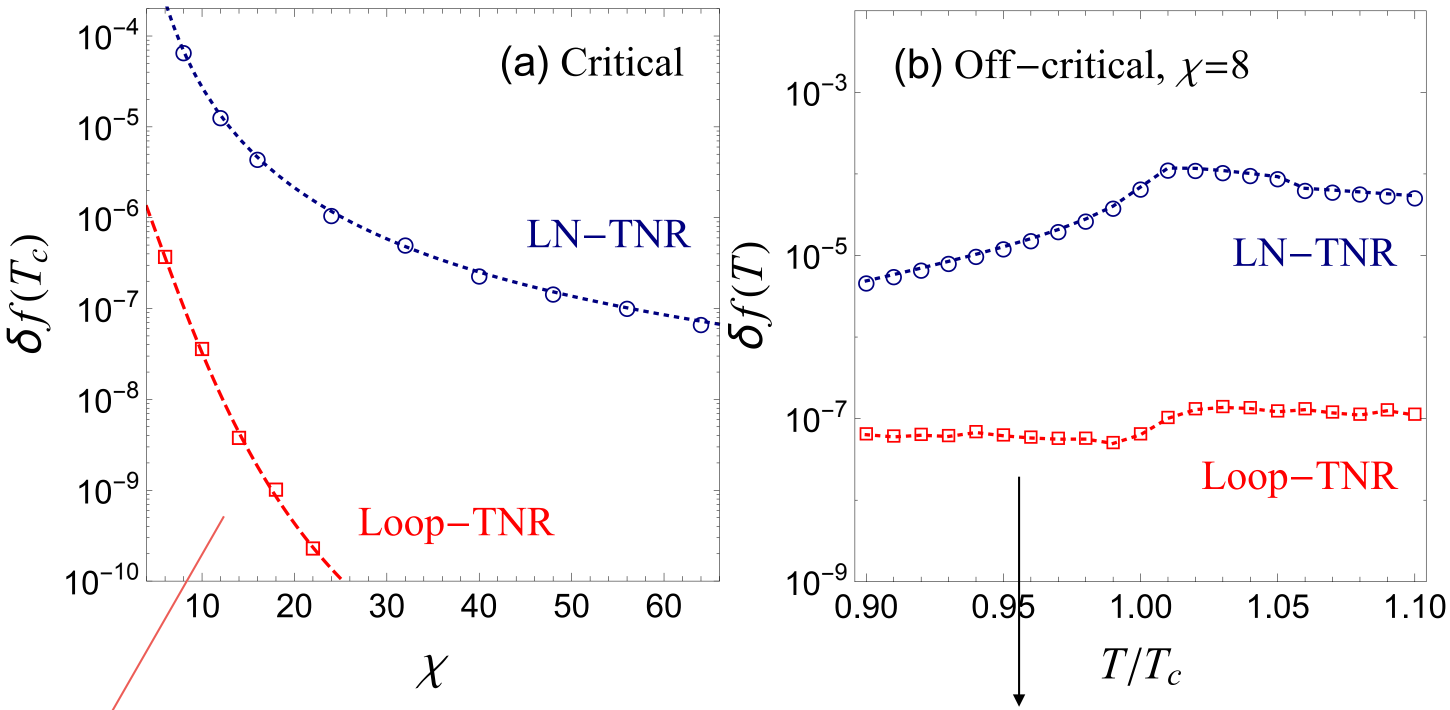
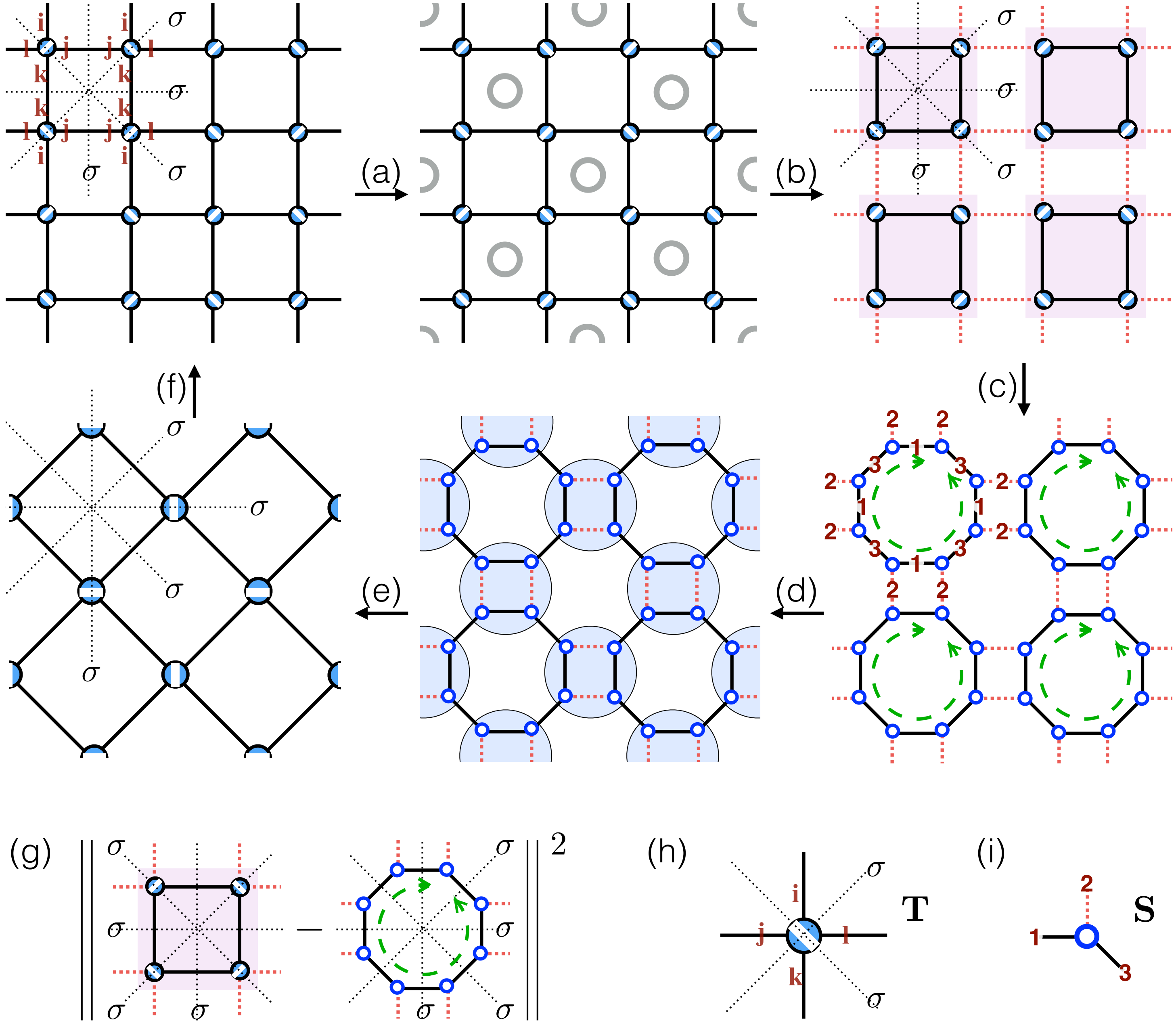


almost decays exponentially with bond dimension

- At off-critical points, the errors almost remains a constant for all temperatures near the critical point

Loop TRG with reflection symmetry



4 axis of symmetry

global C4 symmetry

fix the gauge

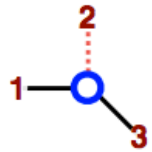
ideal case:

explicitly

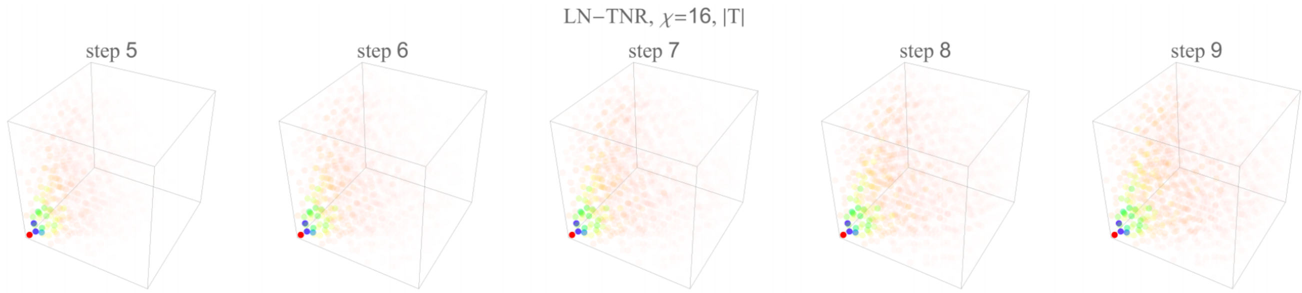
invariant fixed

point tensor S

Fixed point tensors

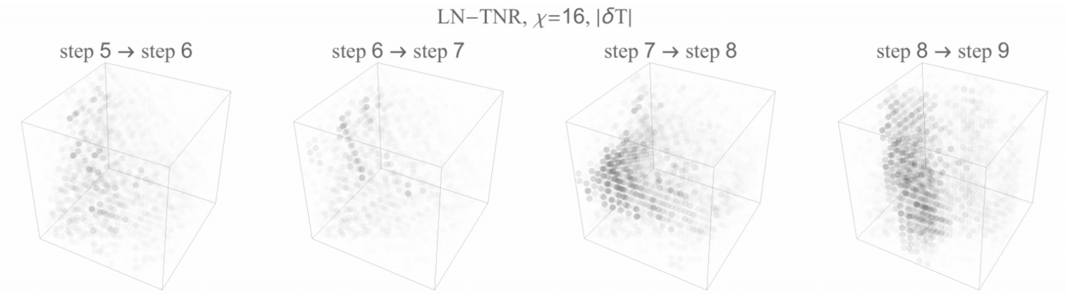


LN-TNR
absolute value



(a)

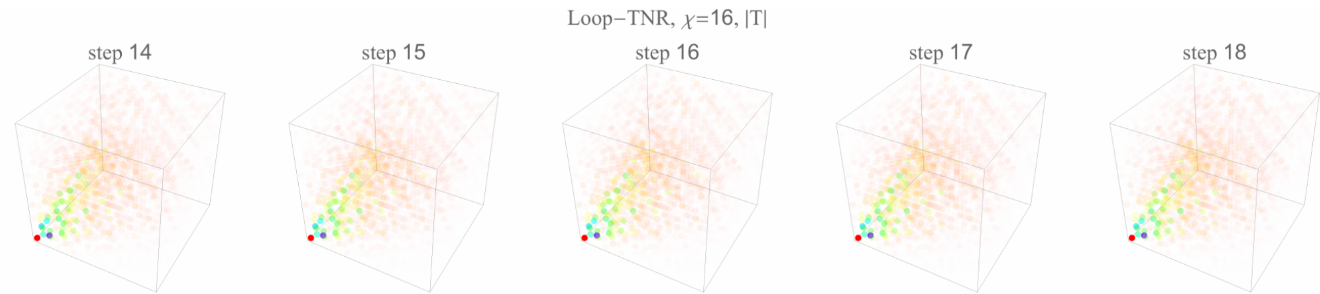
LN-TNR
absolute difference



(b)

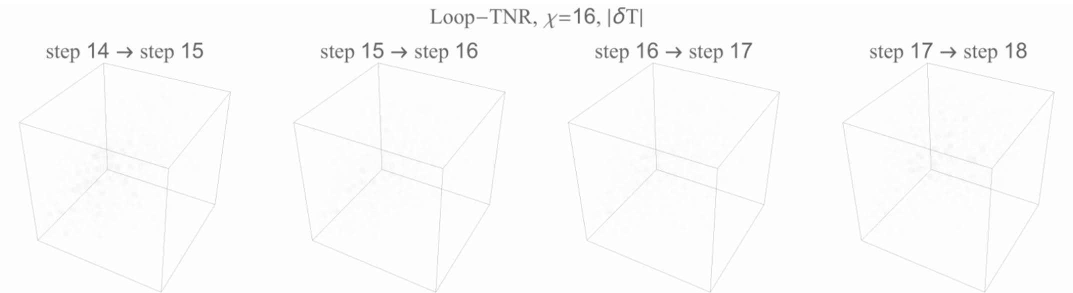
low-index parts
are approximately
invariant

Loop-TNR
absolute value



(c)

Loop-TNR
absolute difference



(d)

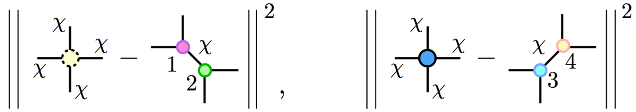
all tensor elements
are nearly invariant
~ scale invariance

Summary

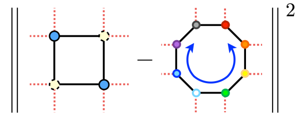
Summary

Real space TNR

- Tree level: LN-TNR (Levin & Nave, 2007)



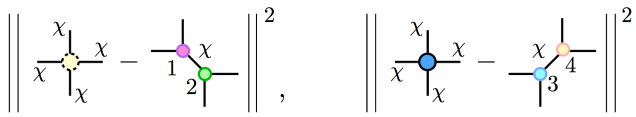
- One loop: GW-TNR, EV-TNR, Loop-TNR



Summary

Real space TNR

- Tree level: LN-TNR (Levin & Nave, 2007)



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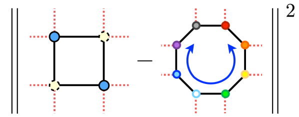
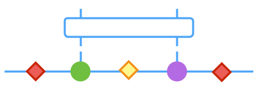
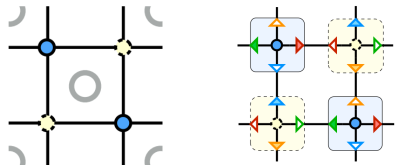
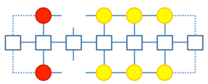
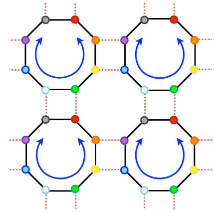
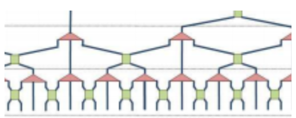
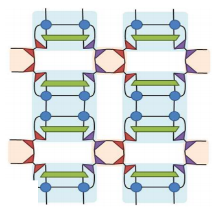


	1D algorithm	LN-TNR + 1D algorithm	
	iTEBD	GW-TNR (Gu & Wen, 2009) Loop-TNR Part One	
	DMRG Varational MPS	Loop-TNR Part Two	
	MERA	EV-TNR (Evenbly & Vidal, 2014)	

Thank you!