Thermodynamic Entropy as a Noether Invariant

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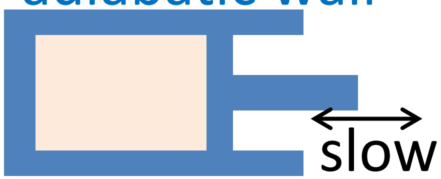
Thermodynamic Entropy as a Noether Invariant

- 2nd law

Adiabatic invariance

$$S_f \geq S_i$$



If a system is enclosed with adiabatic walls, the entropy never decreases for any process.

- Entropy (by Clausius):

$$S = S_0 + \int \frac{dQ}{T}$$

Thermodynamic Entropy as a **Noether Invariant**

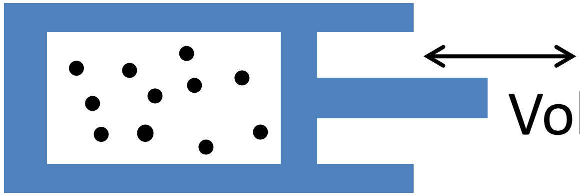
Noether theorem

Symmetry ⇔ Conservation law

question **What is the symmetry for adiabatic invariance $S_i = S_f$?**

Setup

- Consider N classical particles with short-range interaction in a box of volume V.



- The action

$$I[\hat{q}] = \int_{t_i}^{t_f} dt\, L\big(q(t), \dot{q}(t), V(t)\big)$$

$q(t) \in \mathbb{R}^{3N}$: a collection of coordinates of N particles
$V(t)$: external parameter⇒A protocol (functional form of $V(t)$) is fixed.

Example: $L = \sum_{i=1}^{N} \frac{1}{2} m \dot{\boldsymbol{r}}_i - \sum_{i<j} U_{int}\left(\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|\right) - \sum_{i=1}^{N} U_{Wall}(\boldsymbol{r}_i; V(t))$

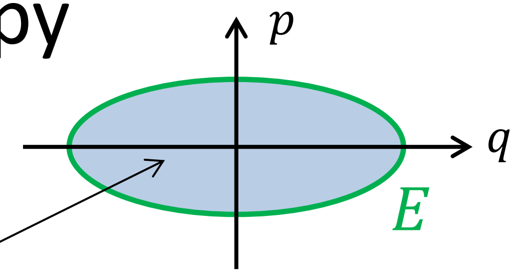
- The energy

$$E(q, \dot{q}, V) = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

Boltzmann Entropy

- The microscopic definition of entropy

$$S(E,V) = \log \frac{\Omega(E,V)}{N!}$$



$$\Omega(E,V) \equiv \int d\Gamma \theta\big(E - H(\Gamma,V)\big)$$

$$\Gamma = (q,p)$$

- The identity of $S(E,V)$:

The first law

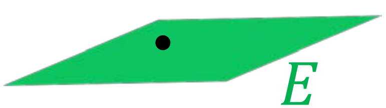
$$\boxed{dS = \beta dE + \beta P dV}$$

Inverse temperature

$$\beta(E,V) \equiv \frac{\Sigma(E,V)}{\Omega(E,V)}$$

micro-canonical ensemble

$$\rho = \frac{\delta}{\Sigma}$$



$$\langle A \rangle_{E,V}^{mc} \equiv \frac{1}{\Sigma(E,V)} \int d\Gamma A(\Gamma) \delta\big(E - H(\Gamma,V)\big)$$

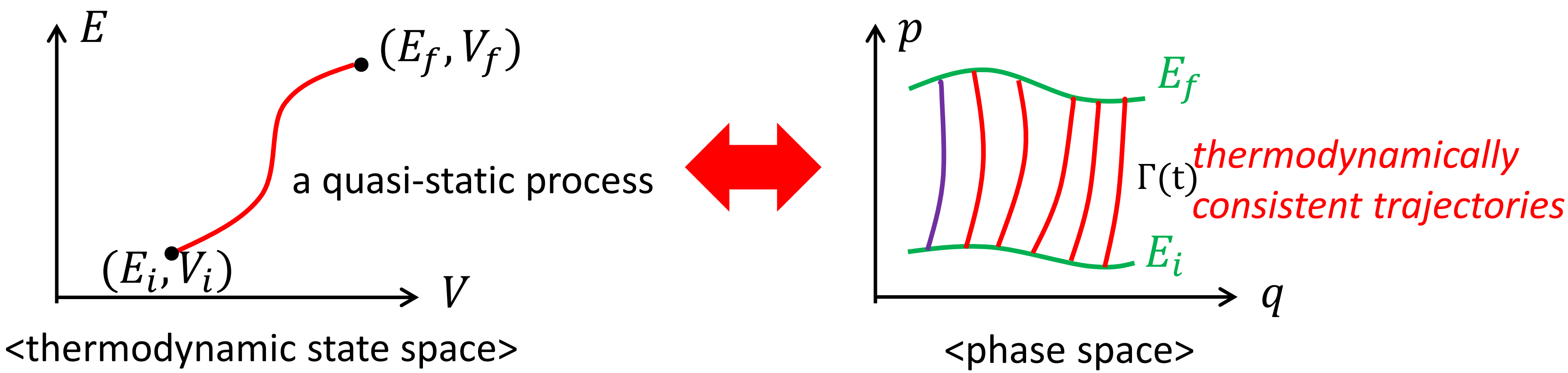
$$\Sigma(\mathrm{E},V) \equiv \int d\Gamma \delta\big(E - H(\Gamma,V)\big)$$

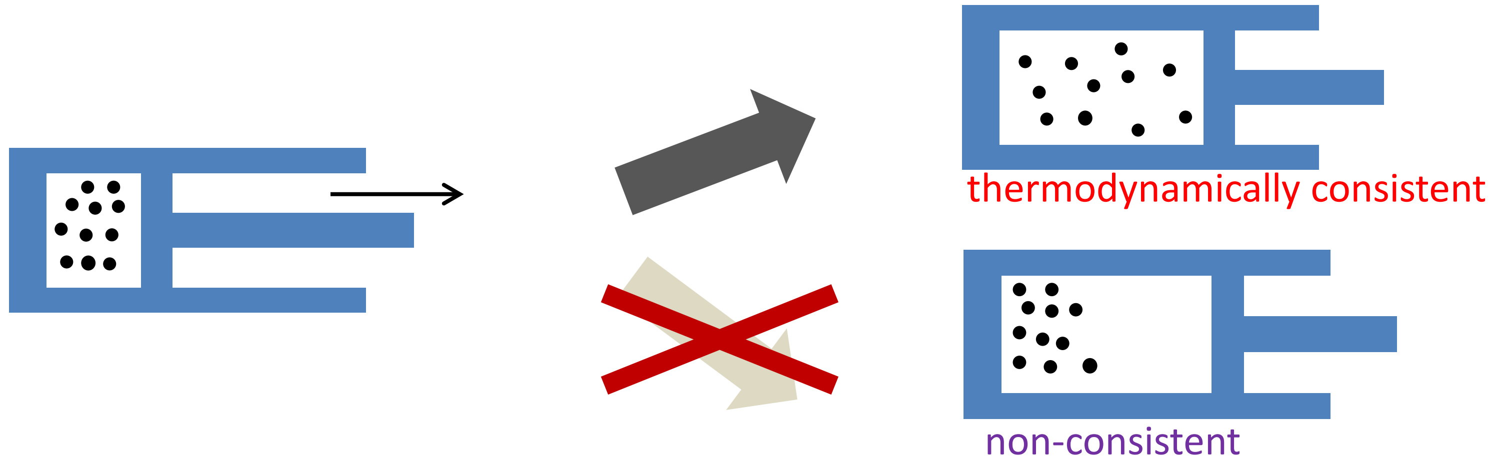
Thermodynamic pressure

$$P \equiv -\left\langle \frac{\partial H}{\partial V} \right\rangle_{E,V}^{mc}$$

Bridge between micro and macro

Restrict the domain of the action $I[\hat{q}]$ to a class of trajectories consistent with quasi-static processes in thermodynamics.





Thermodynamically consistent trajectory

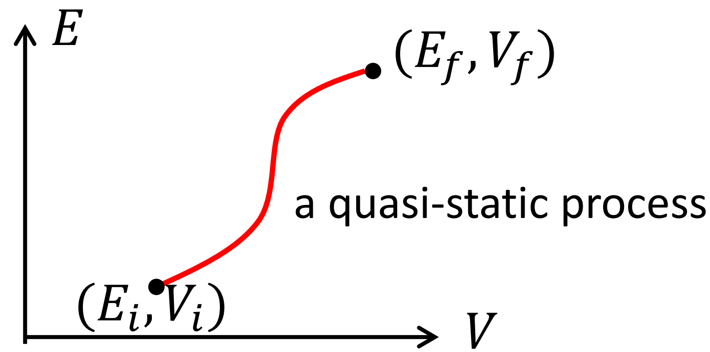
quasi-static limit

Consider a parameter $V(t) = \bar{V}(\epsilon t)$,

introduce $\tau = \epsilon t$,

and take $\epsilon \to 0$ with $\tau_i = \epsilon t_i, \tau_f = \epsilon t_f$ fixed.

$$\Rightarrow \frac{dV}{dt} = \epsilon \frac{d\bar{V}}{d\tau} = \epsilon \mathcal{O}(1) \ll 1$$



Thermodynamically consistent trajectory Γ(t) is defined by

$$(1) \quad E(t) = \underbrace{\bar{E}(\epsilon t)}_{\text{slow}} + \epsilon \underbrace{g(t)}_{\text{rapid}}$$

$$E(t) = H\big(\Gamma(t), V(t)\big)$$

$$(2) \quad \int_{\tau_i}^{\tau_f} d\tau \frac{d\bar{V}}{d\tau} \frac{\partial H}{\partial V} = \int_{\tau_i}^{\tau_f} d\tau \frac{d\bar{V}}{d\tau} \left\langle \frac{\partial H}{\partial V} \right\rangle^{mc}_{\bar{E}(\tau), \bar{V}(\tau)}$$

mechanical work = thermodynamic work

in the quasi-static limit $\epsilon \to 0$.

Note: We can check that these conditions hold for almost all solution trajectories and also for non-solution ones corresponding to isothermal processes.

A generalized Noether theorem

- The Noether theorem in text books

$$\delta_G I = 0 \text{ for any } \hat{q} \Leftrightarrow \frac{d}{dt} O_G \Big|_* = 0 \text{ for solutions } \hat{q}_*$$

under $\hat{q} \to \hat{q} + \delta_G \hat{q}$

generalized

- A generalized Noether theorem

[Trautman 1967, Sarlet-Cantrijin 1981]

$$\delta_G I = \int_{t_i}^{t_f} dt \frac{df(q,\dot{q},t)}{dt} \text{ for any } \hat{q}$$

under $\hat{q} \to \hat{q} + \delta_G \hat{q}$

$$\Leftrightarrow \frac{d}{dt} O_G \Big|_* = 0 \text{ for solutions } \hat{q}_*$$

Even if there remains total derivative of a function, the correspondence between symmetry and conservation holds.

Note: This type of symmetry belongs to a larger class of symmetry in the usual Noether theorem.

A non-uniform time translation

- Consider a non-uniform time translation

$$t \to t' = t + \eta\xi(q, \dot{q}, t)$$

η: infinitesimal parameter

Passive view

$$\begin{cases} q(t) \to q'(t') = q(t) \\ V(t) \to V'(t') = V(t') \end{cases}$$

(←just relabeling)

(←The protocol is fixed.)

$$\Rightarrow \delta_G I = \eta \int_{t_i}^{t_f} dt \left[-\mathcal{E}\dot{q} + \frac{d}{dt}(\xi E) \right]$$

$$\boxed{\mathcal{E} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}}$$

- Suppose that there exist $\xi(q, \dot{q}, t)$ and $\psi(q, \dot{q}, t)$ s.t.

←non-trivial!

Symmetry condition

$$\delta_G I = \eta \int_{t_i}^{t_f} dt \frac{d\psi}{dt}$$

$$\Longrightarrow - \int_{t_i}^{t_f} dt\, \mathcal{E}\, \dot{q}\, \xi = [\psi + \xi E]_{t_i}^{t_f}$$

$\Rightarrow \psi + \xi E$
is conserved on $q_*(t)$!

Derivation of the symmetry

1st step

- Consider

$$\xi = \Xi(\mathrm{E}(\mathrm{q}, \dot{\mathrm{q}}, V), V), \qquad \psi = \Psi(\mathrm{E}(\mathrm{q}, \dot{\mathrm{q}}, V), V).$$

for a thermodynamically consistent trajectory in the quasi-static limit.

2nd step

⇒The symmetry condition can be transformed as

$$-\int_{t_i}^{t_f} dt\, \mathcal{E}\dot{q}\xi = \int_{t_i}^{t_f} dt \frac{d}{dt}(\psi + \xi E)$$

$$\Leftrightarrow \Xi\beta^{-1} = \mathcal{F}(S)$$

arbitrary function of S

$$dS = \beta dE + \beta P dV$$

$$(1) \quad E(t) = \bar{E}(\epsilon t) + \epsilon\, g(t)$$

$$(2) \quad \int_{\tau_i}^{\tau_f} d\tau \frac{d\bar{V}}{d\tau}\left[\frac{\partial H}{\partial V} - \left\langle\frac{\partial H}{\partial V}\right\rangle_{\bar{E}(\tau), \bar{V}(\tau)}^{mc}\right] = 0$$

summary so far

⇒The symmetry

$$t \to t + \eta\, \beta(t)\, \mathcal{F}(S(t))$$

emerges, and the Noether invariant is

$$\Psi + E\Xi = \int^{S} dS'\, \mathcal{F}(S').$$

- Here we can express

$\Xi\beta^{-1} = \mathcal{F}(S)$

intensive

$$\mathcal{F}(S) = \bar{\mathcal{F}}(s; M).$$

$s \equiv \frac{S}{N}$

material dependence

3rd step

- Consider a composite system as our Lagrangian.

$$\beta_A = \beta_B$$
$$\Xi_A = \Xi_B$$

matter M_A | matter M_B

4th step

$\Xi\beta^{-1} = \mathcal{F}(S)$

$$\boxed{\bar{\mathcal{F}}(s_A; M_A) = \bar{\mathcal{F}}(s_B; M_B)}$$

- If $M_A = M_B = M$

$\Rightarrow \bar{\mathcal{F}}(s_A; M) = \bar{\mathcal{F}}(s_B; M)$ for any s_A, s_B

$\Rightarrow \bar{\mathcal{F}}(s; M) = c(M)$

- If $M_A \neq M_B$

$\Rightarrow c(M_A) = c(M_B)$ for any M_A, M_B

$\Rightarrow c(M) = c_*$

Final step

- Thus, we obtain

$$\underbrace{\Xi\beta^{-1}}_{\substack{[\text{time}]\times[\text{energy}]\\=[\text{action}]}} = \mathcal{F} = \underbrace{c_*}_{\substack{\text{independent of state and material}\\=\text{universal constant}}}$$

$$\Rightarrow \boxed{c_* \propto \hbar}$$

⇒Our framework based on classical theory has led to the existence of a Planck-constant-like universal constant. However, this discussion cannot determine the magnitude.

arbitrary proportionality constant

- Therefore, we have

$$\Xi = a\hbar\beta$$

$$\Psi + E\Xi = \int^{S} dS' \mathcal{F}(S') = a\hbar(S + const.)$$

Main result

- In the macroscopic system, the symmetry

$$\boxed{t \to t + \eta\hbar\beta\big(E(t), V(t)\big)}$$

emerges if we restrict the domain of the action $I[\hat{q}]$ to thermodynamically consistent trajectories in the quasi-static limit.

- Then, the Noether invariant is

$$\Psi + E\Xi = \hbar(S + bN).$$

Relation with BH Entropy?

- BH entropy⇔Noether charge for diff-invariance

$$S_{Wald} = \frac{1}{\hbar}\int_{Area} dA\, Q_{\bar{\xi}}.$$

A normalized horizon Killing vector

$$\bar{\xi} = \frac{2\pi}{\kappa}\xi = \hbar\beta_H \frac{\partial}{\partial v}$$

v: Killing parameter
κ: surface gravity

⇒ state-dependent diffeo:

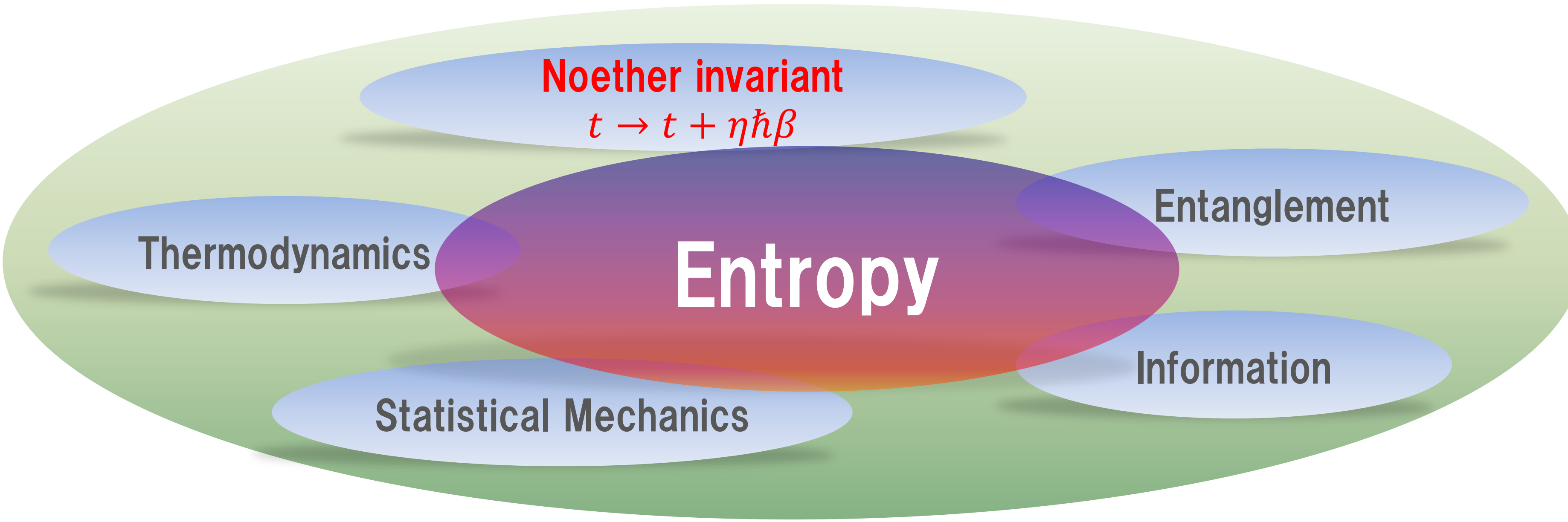
$$v \to v + \eta\hbar\beta_H(M(v), J(v))$$

changing so slowly that the Killing eq holds.

⇒Is our symmetry a holographic dual description of this symmetry?

Various aspects of entropy

This provides a new and unique characterization of entropy.



<u>There remains many questions:</u>

- What is "$t \to t + \eta\hbar\beta$"?
- Can we construct the QM version?
- Why does $\hbar$ appear in the classical framework?
- How is this symmetry related to $t + i\hbar\beta$?
- Can we study 2nd law from this viewpoint?

Thank you very much!