Modular Hamiltonians and relative entropies for holographic states

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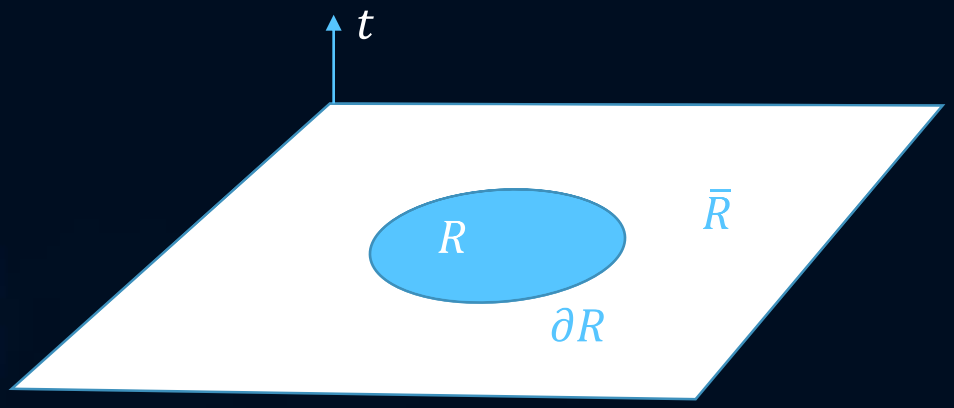
INTRODUCTION

Entanglement entropy

Divide the system in two subregions: $R, \bar{R}$

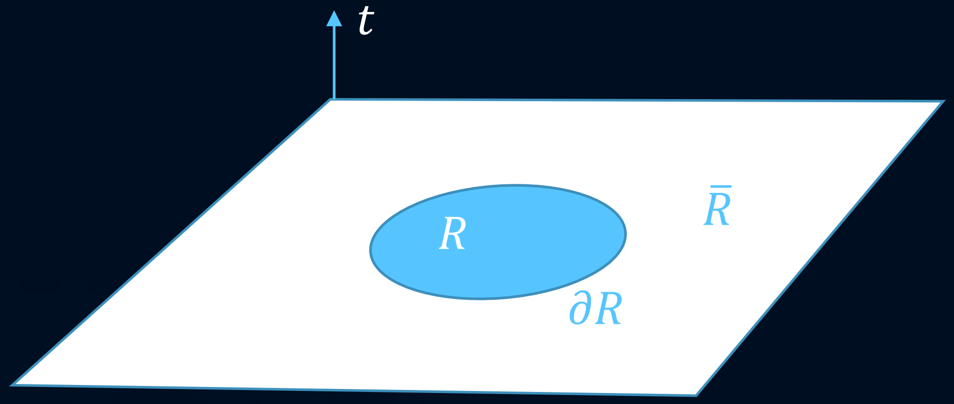


$$\rho = |\psi\rangle\langle\psi|$$

$$\rho_R = tr_{\bar{R}}\, \rho$$

Entanglement entropy

Divide the system in two subregions: $R, \bar{R}$

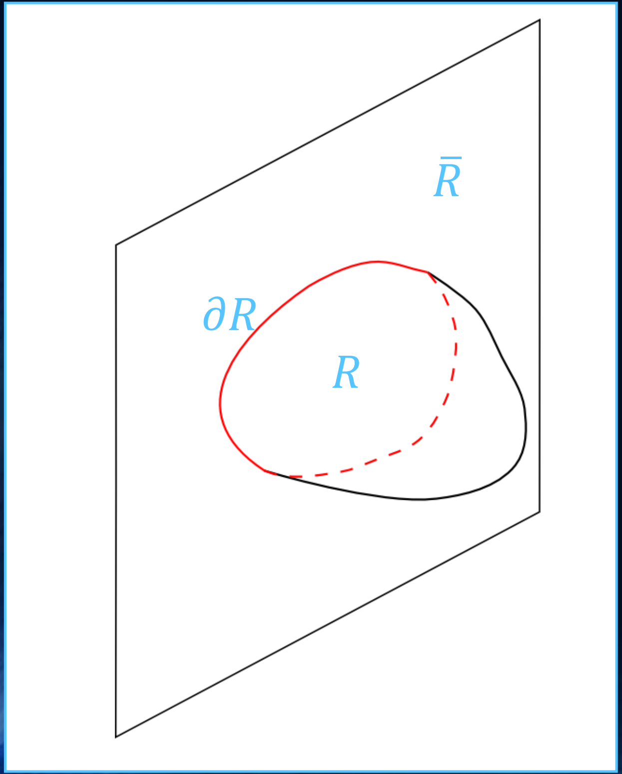


$$S_{EE} = -tr\, \rho_R \log \rho_R$$

This provides one way of characterizing the entanglement properties of the region R in a given state.

Holographic entanglement entropy

AdS/CFT: [Ryu-Takayanagi] proposed that if the space-time is static the entanglement entropy of R is given by a simple geometric object :



$$S_{EE} = \frac{A_{min}}{4G_N}$$

If time dependent , extremal surface [Hubeny-Rangamani-Takayanagi].

Simple to compute in gravity!

Geometry=entanglement?

Motivation

There are lots other quantum information quantities, do they have a nice bulk dual?

Modular Hamiltonian, $\log \rho$, for theories with a holographic dual.

Very complicated operator, simple expression in bulk perturbation theory?

$$\log \rho = I + \int_{bulk} \phi + \int_{bulk} g + \int_{bulk} \phi\phi + \cdots ?$$

QUANTUM INFORMATION PRELIMINARIES

Modular Hamiltonian

The modular Hamiltonian is just the logarithm of the density matrix:

$$K_\rho = -\log \rho$$

The term Hamiltonian is misleading, it depends on the state. It is a complicated operator.

For certain states/subregions, it is very simple:

- For half space in the vacuum, Rindler Hamiltonian: $K_R = \int dy \int_0^\infty dx\, x\, T_{00}$.
- For spherical regions in the vacuum of CFT's, is a simple integral of the stress tensor [Casini-Huerta-Myers] :

$$\mathrm{K} = \int d\Omega \int_0^L dr \sqrt{g}\, \frac{L^2 - r^2}{2\, L}\, T_{00}$$

- For free theories, it is bilocal on the fields [Casini-Huerta,...].

One of our main results will be that for theories with a gravitational dual, it is also simple.

Modular Hamiltonian

The modular Hamiltonian generates the "modular flow"

$$U(s) = e^{i\,K_R\,s}, O_R(s) = U(s)O_R U(-s)$$

It is very non-local in general, but if the modular Hamiltonian is local,

then it is just the usual time evolution (Rindler time, for example).

When acting on the state, K_R is not smooth.

$K_T = K_R - K_{\bar{R}}$ is free of divergences and annihilates the state.

Relative entropy

Consider two states, $\rho, \sigma,$ relative entropy:

$$S_{rel}(\rho|\sigma) = tr\ \rho \log \frac{\sigma}{\rho} = \Delta K_\sigma - \Delta S$$

$$\Delta K_\sigma = \langle K_\sigma \rangle_\rho - \langle K_\sigma \rangle_\sigma$$

σ is a reference state , it provides a measure of distinguishability.

If $\sigma = e^{-\beta H}$, then difference in free energies.

In holography, we know what ΔS is, understand ΔK_σ.

Properties of relative entropy

In field theories, relative entropy is finite and unambiguous.

It also satisfies some interesting properties:

- Positivity: $S_{rel}(\rho|\sigma) \geq 0$
- Monotonicity: $S_{rel}(\rho_R|\sigma_R) > S_{rel}(\rho_{\tilde{R}}|\sigma_{\tilde{R}}), \tilde{R} \subset R.$

First law

Positivity of relative entropy for small perturbations implies a first law:

$$\delta S = \delta K_\sigma$$

Can read the modular Hamiltonian if we know δS for all possible perturbations.

A simple example: if $S = \langle O \rangle + \tilde{S}$, then the first law guarantees that $K = O + \widetilde{K}$.

In this example, there won't be a contribution from O to the relative entropy!

Entanglement entropy and factorization

Entanglement entropy presents a series of ambiguities, Hilbert space doesn't quite factorize: divergent cutoff dependence (area law,...), gauge invariant operators are non-local,..

They are localized in $\partial \mathrm{R}$ and don't contribute to the relative entropy nor the mutual information.

Graviton relative entropy is well defined.

GRAVITY

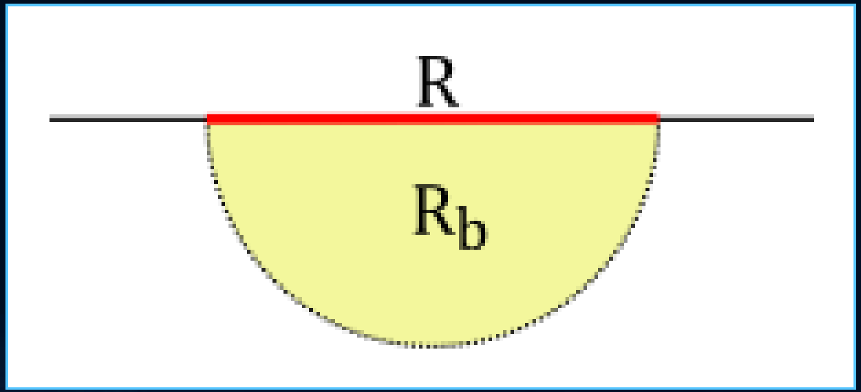
Holographic entanglement entropy

If one includes bulk quantum corrections (1/N corrections), one has to consider bulk entanglement entropy [Faulkner,AL,Maldacena]:

$$S_{EE} = S_{bulk}(R_b) + S_{local}$$

Entanglement wedge , R_b, region between R and the extremal surface.

R_b is in some sense the region dual to R.



Holographic entanglement entropy

The other terms are localized in the extremal surface.

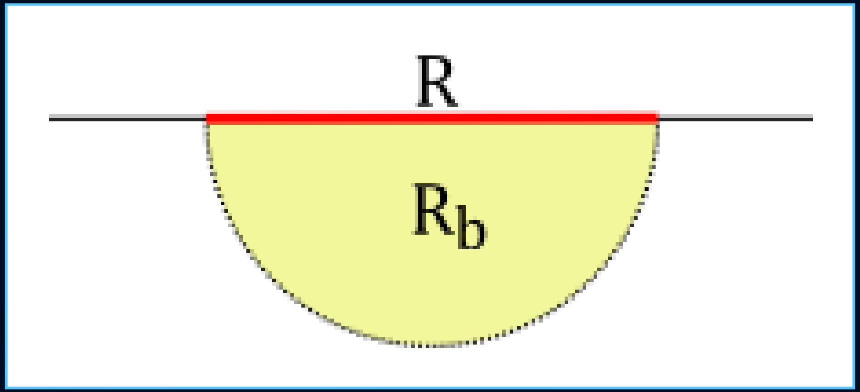
$$S_{local} = \frac{A_{ext}(g_0 + g_{G_N})}{4G_N} + \langle S_{...} \rangle$$

Backreaction of the metric due to the quantum fields, $g_{G_N} \sim O(G_N)$.

$S_{...}$: expectation value of an operator integrated over the extremal surface.

Both terms in S_{local} are localized in the extremal surface.

Bulk entanglement entropy is non-local.



Bulk perturbation theory

Modular Hamiltonian for a simple family of states?

Reference state: $(g_\psi, |\psi\rangle)$.

Consider excitations over this state accessible in bulk perturbation theory.

States "close to $|\psi\rangle$", small backreaction.

We will focus in $O\left(G_N^0\right)$ differences in the entropy: not yet clear how to go further.

Fixed background with gravitons as free matter.

Examples of states that one can consider

States where we don’t need to take into account the backreaction to compute the bulk entanglement entropy $\delta g \sim O(G_N)$.

One can consider coherent states of scalars or gravitons to second order in λ, $e^{i\,\lambda \int dx\, \Pi_c(x)\hat{\Phi}(x)}|\psi\rangle$. We can think of these as classical fields turned on.

One could also consider a squeezed state of gravitons or scalars, $e^{i\,\lambda\, a^{\dagger}a^{\dagger}}|\psi\rangle$, or any other multiparticle state.

Bulk perturbation theory

We want to think of operators in this small Hilbert space.

Consider two states in this small Hilbert space, ρ, σ.

$$A_{ext}\left(g_\rho = g_\sigma + \delta g\right) = A_{ext}(g_\sigma) + \int_{RT} \#\, \delta g \; + \int_{RT} \#\, \delta g\, \delta g + \cdots$$

Extremal area: expectation value of operator in bulk perturbation theory

$$A_{ext}\left(g_\rho\right) = \langle \hat{A}_{ext} \rangle_\rho$$

$$\hat{A}_{ext} = A_{ext,\sigma} I + \int_{RT} \#\, \hat{g}_\sigma \; + \int_{RT} \#\, \hat{g}_\sigma\, \hat{g}_\sigma$$

With $\langle \hat{g}_\sigma \rangle_\rho \; = \delta g$.

Gauge where the surface stays in the same position as we change the state.

The dual of the modular hamiltonian

For states in the small Hilbert space we have

$$S_{EE}(\rho) = \frac{\langle \hat{A}_{ext} \rangle_\rho}{4G_N} + \langle S_{...} \rangle_\rho \; + S_{bulk,\rho}(R_b)$$

Apply the first law to perturbations of σ :

$$K_\sigma = \frac{\hat{A}_{ext}}{4G_N} + S_{...} + K_{bulk,\sigma}$$

Note: $K_{bulk,\sigma}$ is bilocal on the (free) fields, so K_σ has a simple expression in terms of bulk quantum fields.

$O(G_N^{-1})$ discussed in [Jafferis-Suh]

APPLICATIONS

Consequences

Some interesting consequences of the previous formula are :

- Relative entropy is bulk relative entropy

$$S_{rel}(\rho|\sigma) = S_{bulk,rel}(\rho|\sigma)$$

- Modular flow of bulk local operators is bulk modular flow

$$[\phi_{R_b}, K_{bdy}] = [\phi_{R_b}, K_{bulk}]$$

- The operator $K_T = K_{T,bulk}$, it clearly annihilates the vacuum. It is non trivial that the difference between two simple operators localized in different regions annihilates the vacuum.

Relative entropy in the literature

Sphere in the vacuum of a CFT

$$K_{bdy} = 2\pi \int_R \frac{L^2 - r^2}{2L} T_{00}$$

Wald's *gravitational* "first law" is linear in the metric. Can write it as an operator equation:

$$K_{bdy} = \int_R \xi_{bdy}.T = A_{lin}(\hat{g}) + \int_{\mathrm{R}_b} \xi.G(\hat{g})$$

Integrate by parts Einstein tensor, properly integrated in the entanglement wedge: $\int_{\mathrm{R}_b} \xi.G(\delta g)$.

If matter doesn't couple with curvature, then $\int_{\mathrm{R}_b} \xi.G(\hat{g}) = \int_{\mathrm{R}_b} \xi.T_{bulk} = K_{bulk}$.

More generally, $S_{\dots} + K_{bulk} = \int_{\mathrm{R}_b} \xi.G(\hat{g})$.

Relative entropy in the literature

[Blanco-Casini-Hung-Myers,Lashkari-Raamsdonk, Lashkari-Raamsdonk-Rabideau-Sabella-Garnier,Lin-Marcolli-Ooguri-Stoica] studied the relative entropy of perturbative bulk states compared with the vacuum to second order. Our formula might look surprising at first.

Gravitational calculation (no QFT in the bulk), coherent states in the bulk.

In this symmetric configuration, the bulk modular Hamiltonian is the canonical energy.

$$S_{rel}(\rho|\sigma) = S_{bulk,rel}(\rho|\sigma) = \Delta K_{bulk} = E_{can}$$

Which agrees with their results.

Positivity of relative entropy = positivity of bulk relative entropy.

Relative entropy and state distinguishability

Bulk and boundary relative entropies are the same $\rightarrow$ perturbing the bulk region R_b can be seen from the boundary region R ....

This suggests that there is a mapping between these two regions that one should understand (entanglement wedge reconstruction [Almheiri-Dong-Harlow]).

Often called subregion-subregion duality: one wants to understand to what extent a bulk subregion encodes a given boundary subregion.

[Dong-Harlow-Wall] showed using the relative entropy that $\phi(R_b)$ is morally an operator in region R.

Entanglement wedge reconstruction

Can our methods help to find explicit expression for the operators in R_b in terms of operators in R? What does this really mean?

[HKLL]: In bulk perturbation theory one can think of a local bulk operator as a sum over boundary operators. Basically one just uses Green's theorem:

$$\phi(X) = \int dt \int_{bdy} dx \, G(X|x,t) O(x,t) + O(G_N)$$

It is sometimes useful to think of it as a sum over Heisenberg operators.

Can one represent $\phi(X)$ with support only in a boundary subregion?

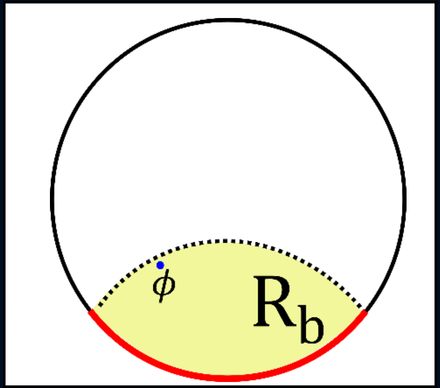
Entanglement wedge reconstruction

Rindler reconstruction [HKLL,Morrison,Almheiri-Harlow-Dong]: for one interval/sphere if one has a bulk operator in the entanglement wedge, it can be localized in R:

$$\phi(X_{R_b}) = \int d\tau \int_R \, dx \, G'(X_{R_b}|x,\tau) O(x,\tau) + O(G_N)$$

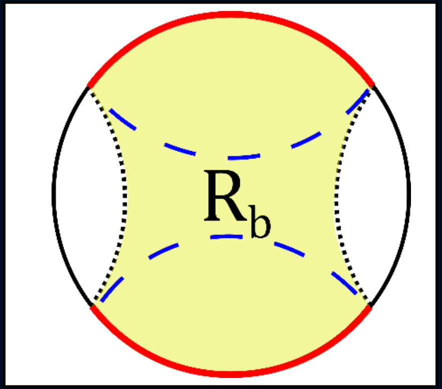
Where τ corresponds to Rindler time. Sum over non-local modular flowed operators at $t = 0$.

$\phi(X_{R_b})$ is in causal contact with R, it makes sense that a boundary observer in R can reconstruct this operator with simple boundary operators.



Entanglement wedge reconstruction

Reconstruct bulk operators in the entanglement wedge as boundary operators in a subregion for two intervals?



Causal wedge: Rindler wedge of the individual intervals (region in causal contact with the boundary region). The previous should work there.

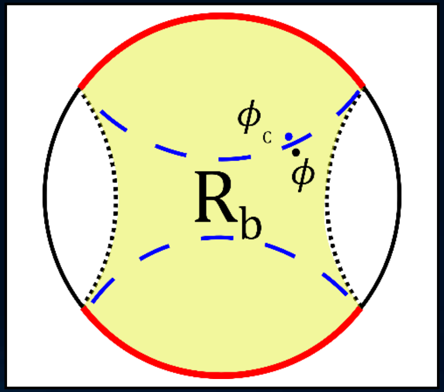
Part of R_b not in causal contact with the boundary causal domain. $\phi(X_{R_b})$ can't be just a superposition of simple operators localized in each interval.
Modular flow generates operators which are smeared over the two intervals.

Entanglement wedge reconstruction

Simplest proposal that has enough non-locality:

$$\phi(X_{R_b}) = \int ds \int_R \; dx \, G''(X_{R_b}|x,s) O(x,s) + O(G_N)$$

Modular flow seems important for reconstruction. It is not clear if it is enough.



CONCLUSIONS

Conclusions

- In gravity, given a reference state, the modular Hamiltonian is the modular Hamiltonian in the bulk plus terms which are localized in ∂R_b.
- Bulk and boundary relative entropies are equal, which has interesting consequences.
- Bulk and boundary modular flows are equivalent.
- More entanglement=entanglement than entanglement=geometry. However, for large superpositions of states (like the TFD) or $S_{EE,bulk} \sim O(G_N^{-1})$, separating geometry (area) and entanglement should be more fuzzy.

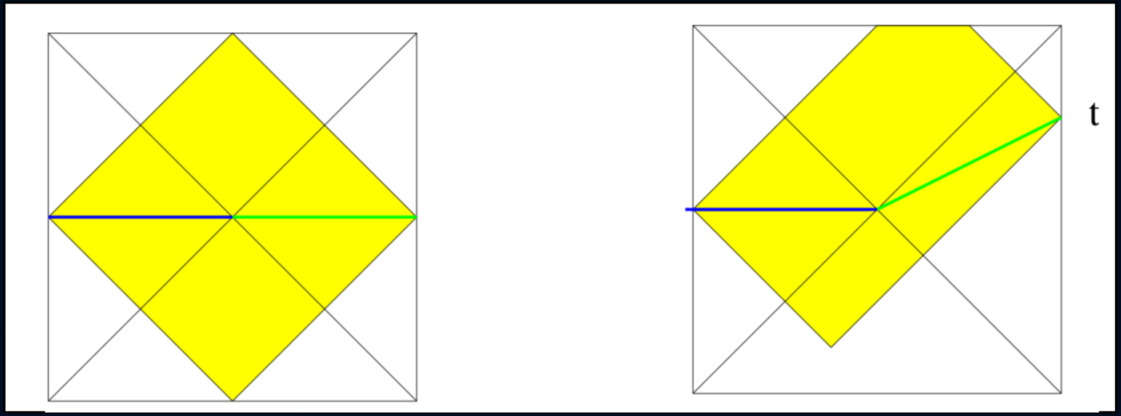
Open questions

- Can one do something similar for non-extremal surfaces in the bulk? Is it meaningful to consider $\hat{A} + K_{bulk}$ for regions other than the entanglement wedge? Boundary interpretation?
- Does the area term ever contribute (directly) to the relative entropy? Is this entropy distillable?
- Can one understand the G_N corrections perturbatively? How does one compare different states with different geometries and entanglement wedges…?
- Can one be explicit about reconstructing the entanglement wedge?
- ….

THANK YOU!!!

The action of the modular hamiltonian

Consider the TFD state, $|\Psi_{TFD}\rangle$. One can act with $e^{i\,K_R t}$ in the boundary to shift the time slice.



This state will be smooth from the bulk point of view.

However $K_R = \hat{A} + K_{R,bulk}$, each of the individual pieces is singular when acting on the state.

It is unclear if one could act individually with each of the pieces…

In general, the action of the modular Hamiltonian should be smooth from the bulk perspective.

Graviton contribution

We can consider a coherent state of gravitons such that $\langle h \rangle = \sqrt{G_N}\ \lambda\ \delta h(x)$

To first order in λ, there will be a linear change in the area. This can be seen as coming from S_{wald} or $\hat{A}$. The relative entropy will be zero.

To second order in λ, the Einstein equations will read

$$E(\delta g) = \langle T_{grav}\ (h, h) \rangle, \delta g = O(G_N\ \lambda^2)$$

The entropy will have a contribution from $A(\delta g) + \langle A(h, h) \rangle$.

The last term can be thought of as coming from S_{wald} or $\hat{A}$.

These localized entropy terms are also in the modular Hamiltonian.

$$\int \xi . \langle T_{grav}\ (h, h) \rangle - \langle A(h, h) \rangle = E_{can}$$

Something similar happens with a free scalar coupled with curvature

$$\int \xi . \langle T_{grav} \rangle - 4\pi\alpha\ \int \phi^2 = \int \xi . \langle T_{can} \rangle = E_{can}$$

Entanglement entropy and factorization

For conformal scalar fields and spherical EE, there is a similar ambiguity: lattice vs CFT calculation [Casini-Mazzitelli-Teste,Lee-AL-Perlmutter-Safdi].

Not related with factorization, but with boundary conditions.

These two ways of computing it have different modular hamiltonians:

$$K_{CFT} = \int_R \xi . T_{CFT}, \qquad K_{lat} = \int_R \xi . T_{can},$$

They differ by a boundary term in the entangling surface.

Same relative entropy.

Entanglement ambiguities

Gauge fields in the lattice [Casini-Huerta-Rosabal]: multiple possibilities for algebra associated with region R. Different choices give different entropies, this ambiguity is localized in ∂R.

In the continuum, for example: set boundary conditions for the electric field in ∂R. [CHR,Donelly-Wall] compute the entropy for each possible field configuration and sum over all of them.

There is no unique prescription, but relative entropy is independent of these choices.

Entanglement ambiguities

Take home message:

Entanglement entropy ambiguities are localized in ∂R and they don't contribute to the relative entropy.

We expect the situation is similar for gravitons.

Set some gauge invariant boundary conditions in ∂R.