Conformal bootstrap in Mellin space

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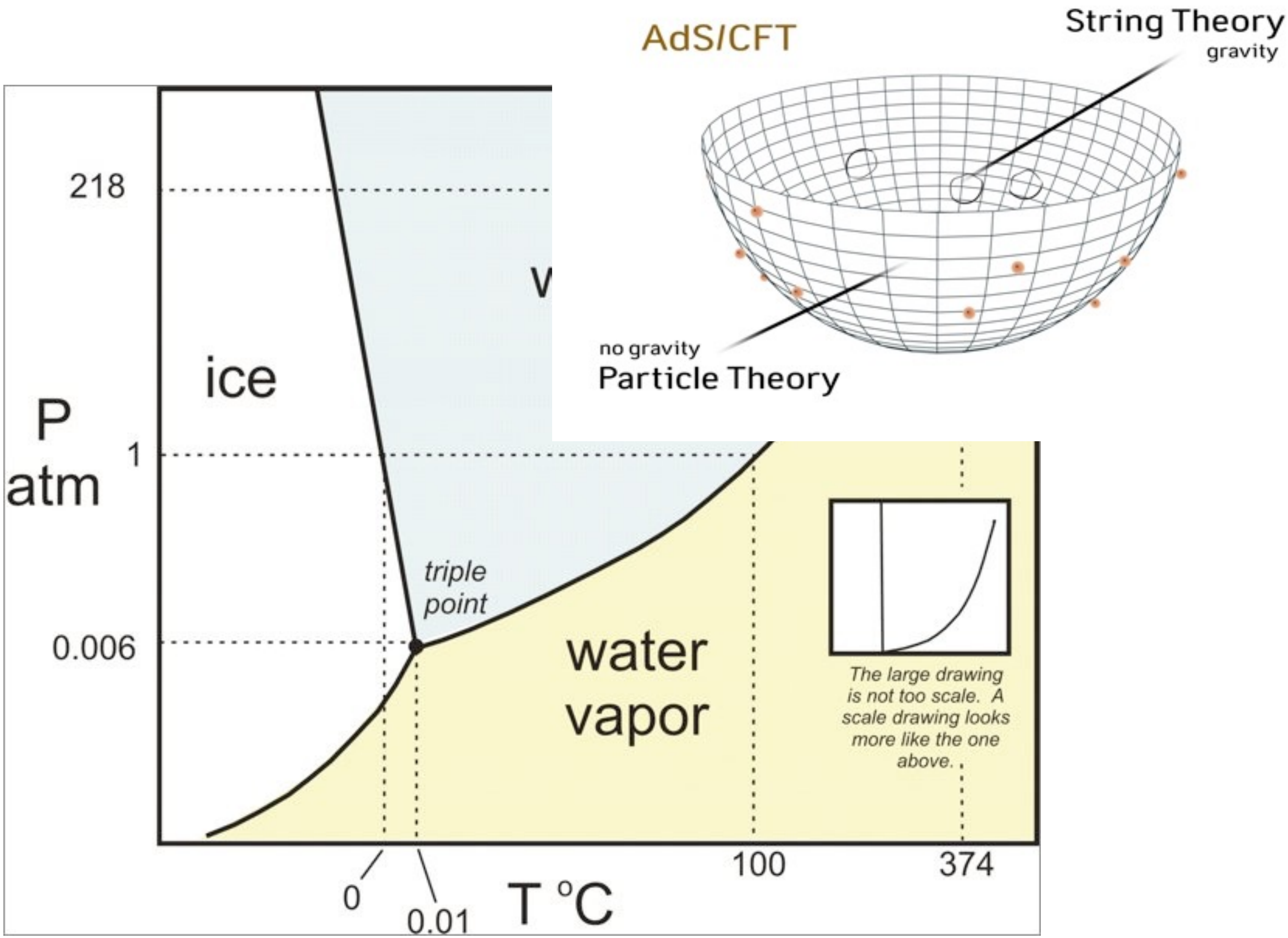
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The large drawing is not too scale. A scale drawing looks more like the one above.
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AdS/CFT
String Theory
gravity
no gravity
Particle Theory

In the late '60's, Migdal and Polyakov[64] developed a "bootstrap" formulation

of critical phenomena based on a skeleton Feynman graph expansion, in which all parameters including the expansion parameter inself would be determined self-consistently. They were unable to solve the bootstrap equations because of their complexity, although after the ε expansion about four dimensions was discovered, Mack showed that the bootstrap could be solved to lowest order in ε. If the 1971 renormalization group ideas had not been developed, the Migdal-Polyakov bootstrap would have been the most promising framework of its time for trying to further understand critical phenomena. However, the renormalization group methods have proved both easier to use and more versatile, and the bootstrap receives very little attention today.

Wilson-Nobel lecture 1982

I would like to quote from a 2003 interview with A.M. Polyakov [12], where the desire for a better 3D theory has been stated with clarity:

"Let me tell you what I think of the renormalization group. I think there are two types of useful equations. One type is human-made, they are invented by people. The other type reflects some 'pre-established harmony.' They can be discovered (uncovered) and not invented. Renormalization

group is clearly a human made thing. It's clearly a smart way of calculating things but it doesn't have a breathtaking quality of, say, the Dirac equation.

The example of the second kind is operator product expansions. They form some beautiful mathematical relations and I was dreaming in the 1970s to have some classification of fixed points based on the possible operator product expansions. The program was a little like classifying Lie algebras. In that case you start with the commutator relations which define the Lie algebra and then you classify all possible semi-simple algebras. You arrive at a stunningly beautiful theory (which was clearly discovered and not invented). I was working on that project in the 1970s and I still think it might have a chance. It was successful in two dimensions. We can classify possible fixed points in two dimensions using operator product expansions. That's what conformal field theories are about. And I think it's not excluded, that in 3 dimensions something like that is still possible. I was working for a while on this without much success in the 1970s and then I switched to other things.

I think the epsilon expansion ended the subject in the practical sense. You can calculate more or less what you want with good accuracy but aesthetically the subject is not closed yet. It's possible that there will be classification of fixed points in three dimensions, based on string theory, similar to what we have in two dimensions. But that's just dreams."

Polyakov interview 2003. Source: Rychkov, 2011

OPE

Operator Product Expansion

$$\phi(0)\phi(x) \sim \sum c_{\Delta,\ell}(x^2)^{\Delta_O/2-\Delta_\phi-\ell/2}x^{a_1}\cdots x^{a_\ell}O_{a_1\cdots a_\ell}(0)$$

- Convergent power series expansion (not asymptotic!)
- Radius of convergence set by closest operator insertion.
- Operator relation.

General philosophy of bootstrap

- No Lagrangian
- No Feynman diagrams
- No regularization, RG etc.
- Only conformal symmetry, crossing symmetry and unitarity.

Outline

- Critical exponents via epsilon expansion
- Successes of modern bootstrap
- Drawbacks of modern version
- A new (*take on old*) approach (ongoing work)

d>2, no supersymmetry

- Partly motivated by the 1974 seminal paper by Polyakov, 1505.00963 by Rychkov-Tan and 1510.07770 written with K. Sen (to join IPMU as postdoc).

- Mainly ongoing work with R. Gopakumar, A. Kaviraj, J. Penedones and K. Sen

- New method will be based on using “Witten blocks” which are naturally formulated in Mellin space insead of usual conformal blocks in position space. Think of flat space Mandelstam variables instead.

- Seems to make transparent the connection with usual QFT.

- May be useful to clarify the role of string theory.

Epsilon expansion: Review

Epsilon-expansion; Wilson; Wilson-Fisher; Polyakov;.Mack.....Rychkov, Tan

O(N) model
$$\int d^{4-\epsilon}x \; \left[(\partial_\mu \phi^i)^2 + \lambda (\phi^i \phi^i)^2 \right]$$

- Wilson-Fisher fixed point.
- 3d Ising model (critical point of boiling water) $N = 1, \epsilon = 1$
- 2d Ising model c= $\frac{1}{2}$ $\quad N = 1, \epsilon = 2$
- XY model $\quad N = 2, \epsilon = 1$

PHYSICAL REVIEW D VOLUME 7, NUMBER 10 15 MAY 1973

Quantum Field-Theory Models in Less Than 4 Dimensions*

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(Received 9 November 1972)

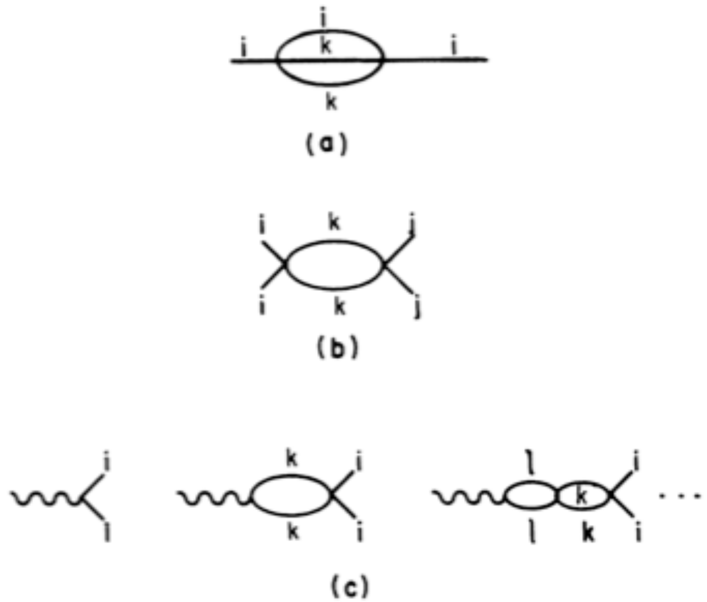


FIG. 1. (a) Diagram giving the lowest-order correction to the propagator. The two lines with internal index k form a loop; the sum over k gives a factor of N. (b) Diagram giving the leading correction to the four-point function (k is summed over). (c) Bubble graphs for vertex function involving ϕ^2 or $\overline{\psi}\psi$. The wavy line represents ϕ^2 or $\overline{\psi}\psi$; the straight lines refer to the elementary fields ϕ, ψ, or $\overline{\psi}$. The indices k and l are summed over.

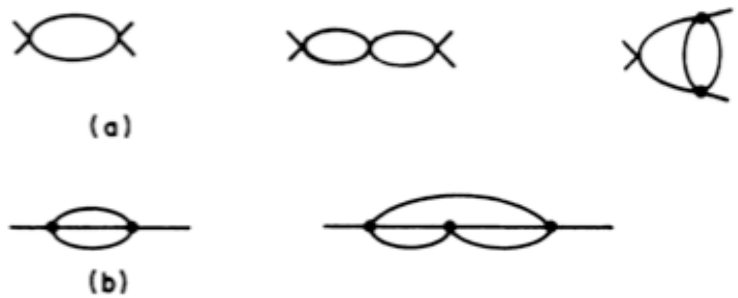


FIG. 2. (a) Diagrams determining u_R to order u_0^3. (b) Diagrams determining $D(q)$ to order u_0^2.

- Regularize, renormalize
- Locate fixed point of beta function
- Use Callan-Symanzik to determine anomalous dimension

$$\langle O(0)O(x)\rangle \propto \frac{1}{(x^2)^{\Delta}}$$

$$\Delta = bare + anom = (d-2)/2 + \gamma$$

- State of the art is epsilon^5 by Kleinert et al.
- Needs ~135 diagrams. Possibility of mistakes.
- Any way the series is asymptotic.
- QUESTION: Can the 3d Ising model at the critical point yield an analytic solution?

$$\Delta_\phi = \frac{d-2}{2} + \frac{N+2}{4(N+8)^2}\epsilon^2 + \frac{N+2}{4(N+8)^2}\left[6\frac{3N+14}{(N+8)^2} - \frac{1}{4}\right]\epsilon^3$$

Ising

N=1

$$d = 2 \to 0.11$$

$$\text{actual} = \frac{1}{8} = 0.125$$

$$d = 3 \to 0.519$$

$$\text{numerics} \approx 0.518$$

$$\text{experiments} \approx 0.521$$

XY

N=2

$$d = 3 \to 0.52,$$

$$\text{expt} \approx 0.506$$

Numerical results are from bootstrap based on methods pioneered by Rattazzi, Rychkov, Vichi and Tonni and used by El Showk et al. Often quoted as most accurate numerical estimates for the 3d Ising model at criticality.

$$\Delta_{\phi^2} = d - 2 + \frac{N+2}{N+8}\epsilon + \frac{N+2}{2(N+8)^3}(13N+44)\epsilon^2$$

Ising

$d = 2 \to 1.136$ $\quad$ $\text{actual} = 1$

$d = 3 \to 1.45$ $\quad$ $\text{numerics} \approx 1.41$ $\quad$ $\text{expts} \approx 1.41$

XY

$d = 3 \to 1.54$ $\quad$ $\text{expt} \approx 1.51$

seems good
but!!

$$\alpha = 2 - \frac{d}{d - \Delta_{\phi^2}}$$

$\to -0.055$ $\quad$ $\text{expt} \approx -0.013$

$O(\epsilon^5) \to -0.004$

International space station superfluid He experiment

discrepancy!

Higher spin operators: Wilson-Kogut

$$\Delta(\phi\nabla^\ell\phi) = d - 2 + \ell + \frac{N+2}{2(N+8)^2}\epsilon^2(1 - \frac{6}{\ell(\ell+1)})$$

$$N = 1 \to \gamma(\phi\nabla^\ell\phi) = \frac{\epsilon^2}{54}(1 - \frac{6}{\ell(\ell+1)})$$

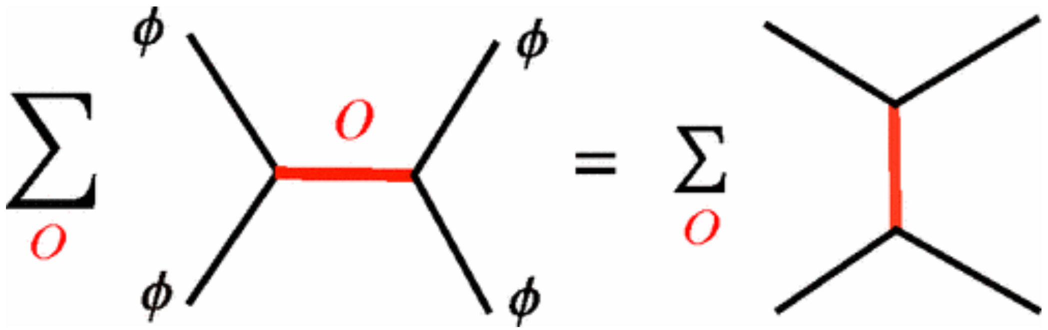
For large spin, we can use analytic bootstrap since the blocks are known in this limit for any dimension [Kaviraj, Sen, AS, 2015]

Find precise agreement with this.

Using a new approach, we can do better and get this result for any spin without Feynman diagrams!

Quick review of modern bootstrap

Can only be reproduced upon considering large spin operators on the RHS



"direct-channel"

"crossed-channel"

Crossing

$u \leftrightarrow v$

even spin

$$1 + \sum_{\tau,\ell} P_{\tau,\ell}\, g_{\tau,\ell}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \left(1 + \sum_{\tau,\ell} P_{\tau,\ell}\, g_{\tau,\ell}(v,u)\right)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{24}^2 x_{13}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{24}^2 x_{13}^2}$$

Conformal cross ratios

Twist
$\tau = \Delta - \ell$

$P_{\tau,\ell}$ — OPE x OPE

$g_{\tau,\ell}(u,v)$ — Dolan, Osborn; Blocks

- Blocks satisfy 2nd order partial differential equation in u,v (quadratic Casimir).

- Integral representation known in any dimensions. Infinite series representation known in terms of Gegenbauer polynomials in any dimensions.

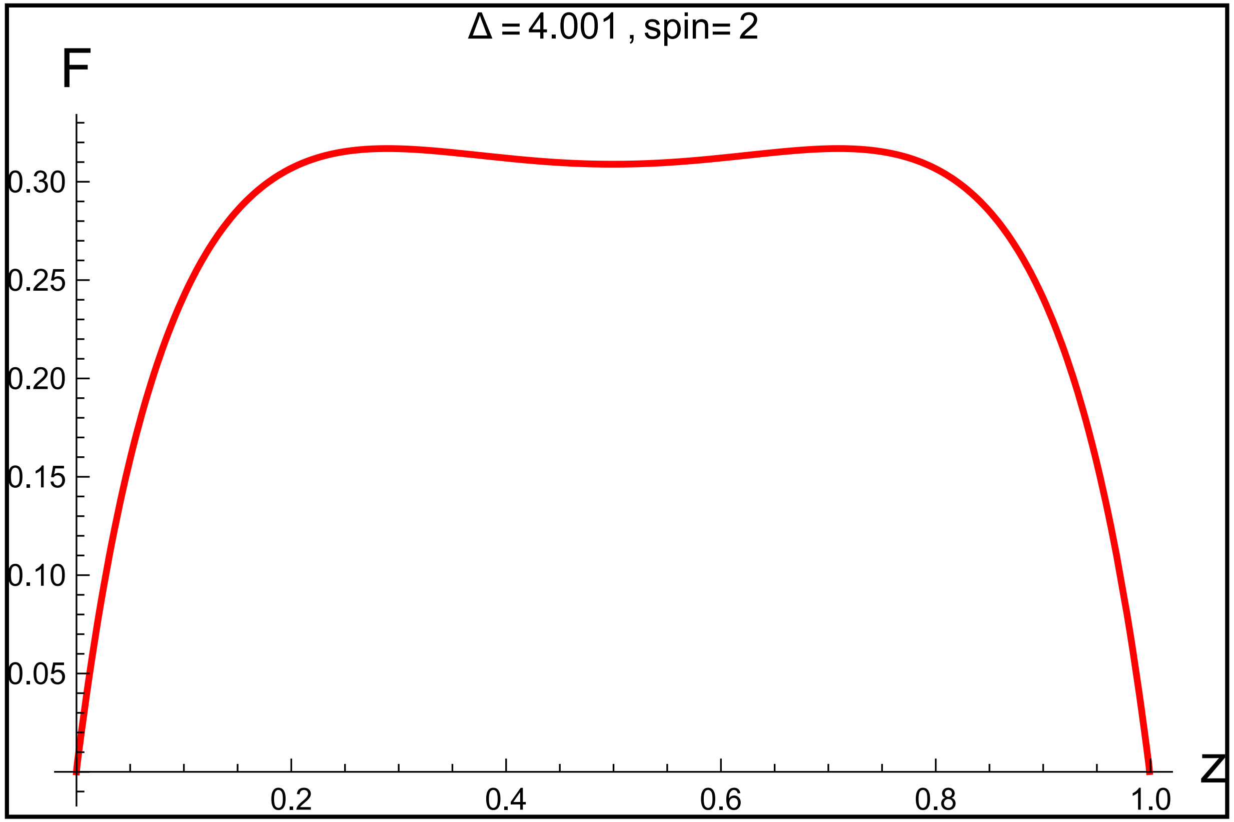
How is a bound possible?

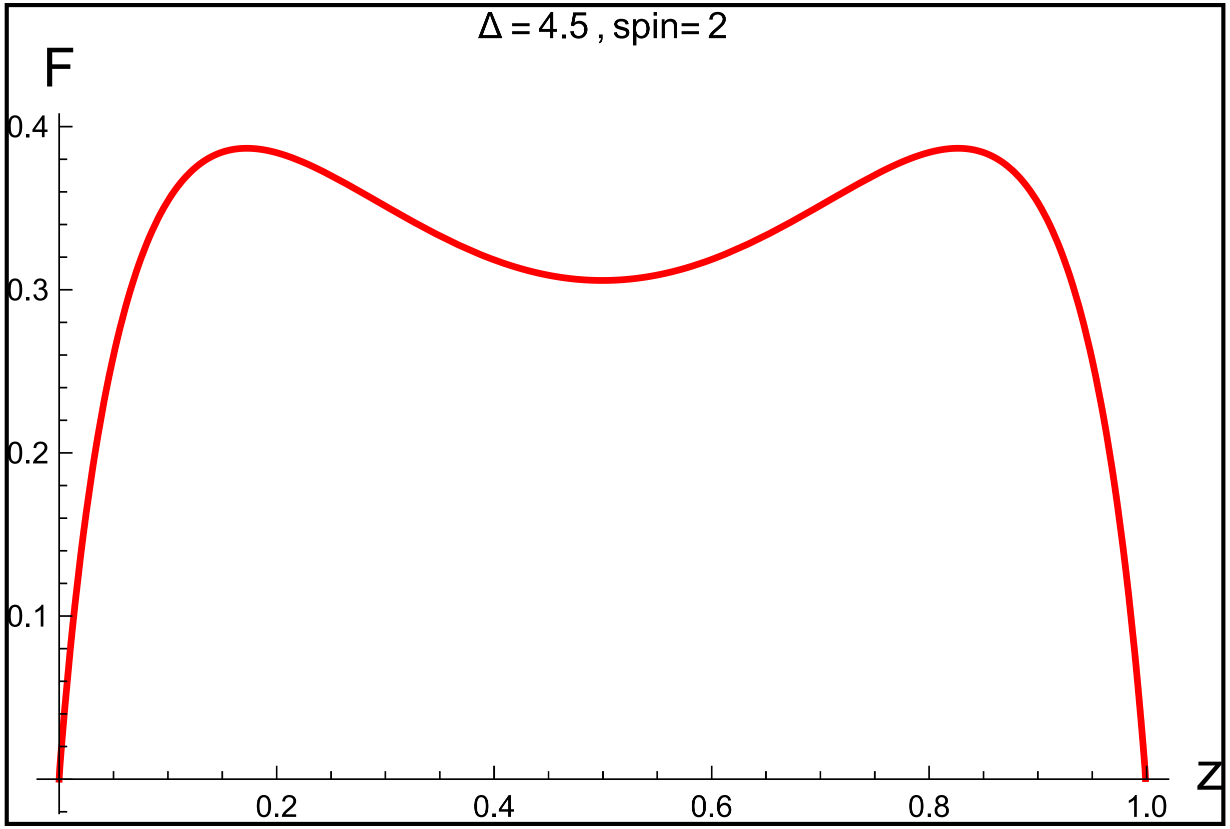
Rattazzi, Rychkov, Tonni, Vichi

$$\sum_{\Delta,\ell} C^2_{\Delta,\ell} F_{\Delta,\ell}(u,v) = 0$$

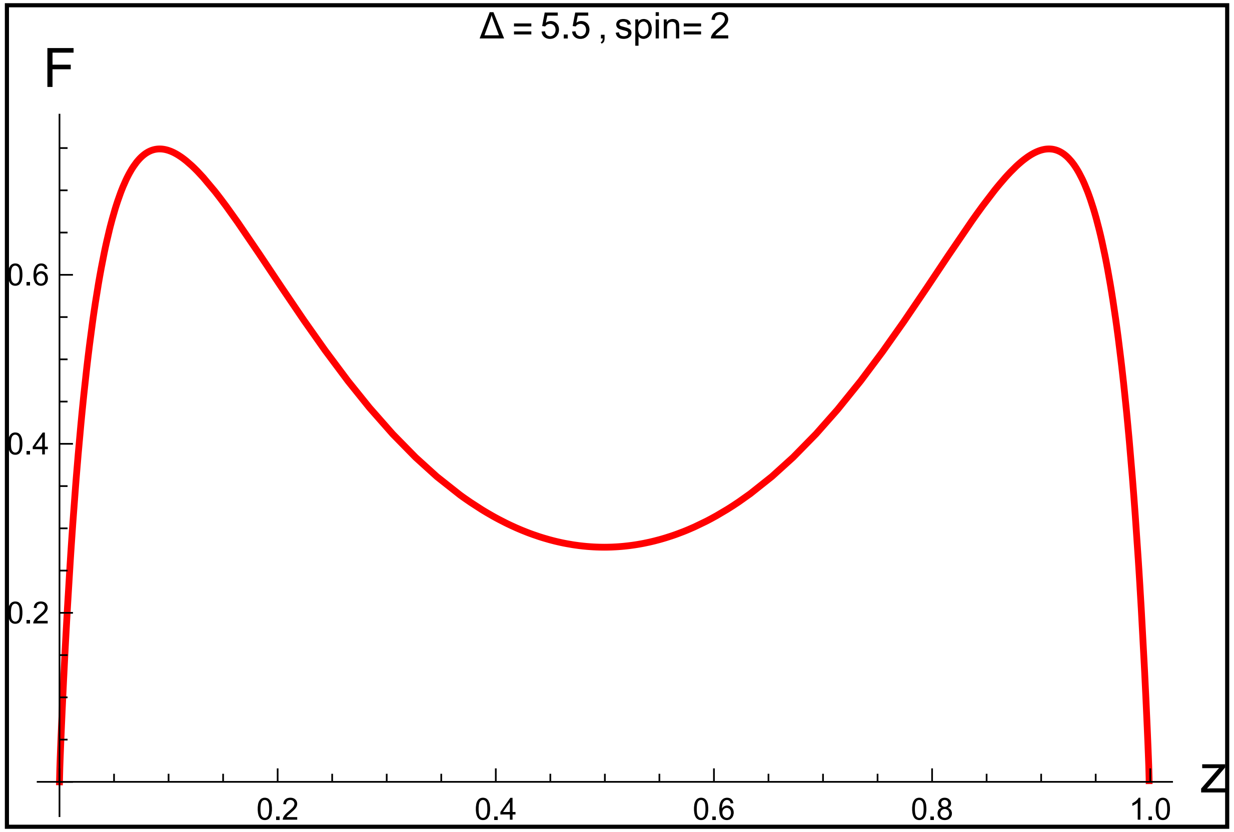
1. Expand around u=v=1/4
2. To satisfy equation derivatives also have to vanish.
3. Look at 2nd derivative of the F's.
4. All non-zero spin F's have minima (same sign).
5. Zero spin F's have opposite sign.

Non-zero spin typical



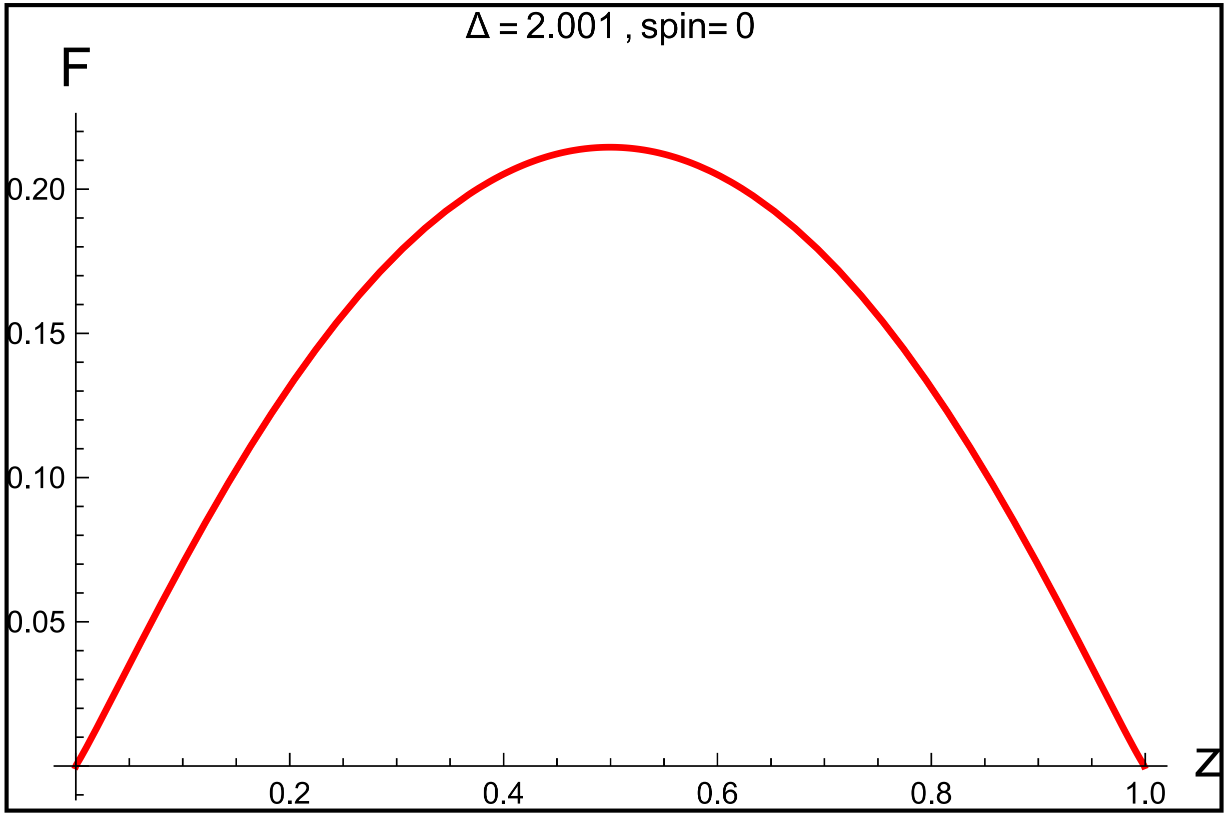


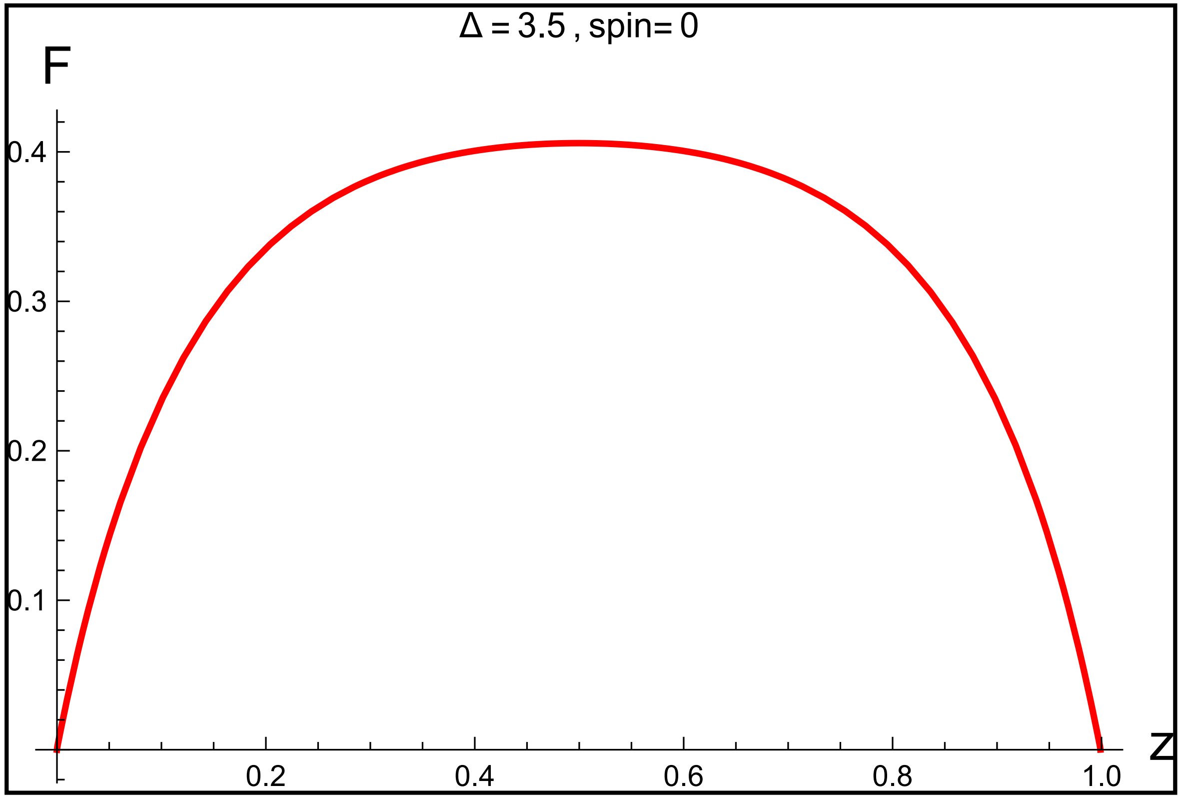
Δ = 4.5 , spin= 2
F
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1.0
z



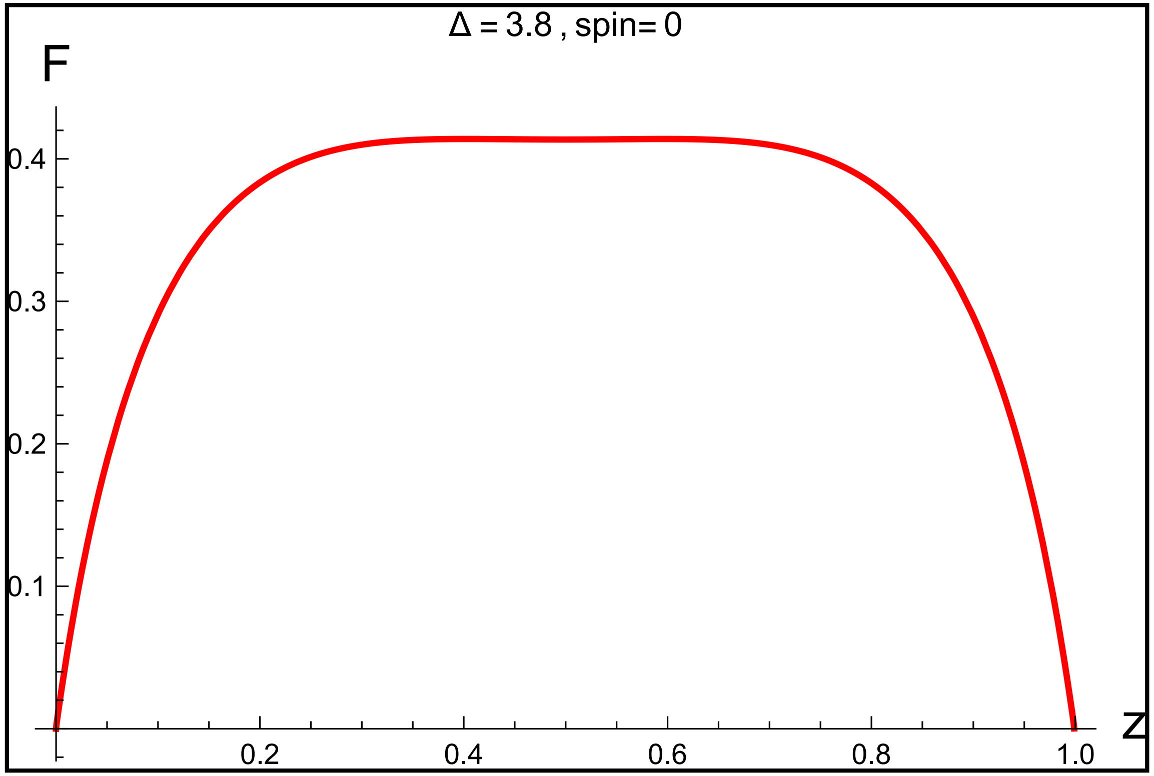
Δ = 5.5 , spin= 2
F
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
z

Zero spin





Δ = 3.5 , spin= 0
F
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1.0
z



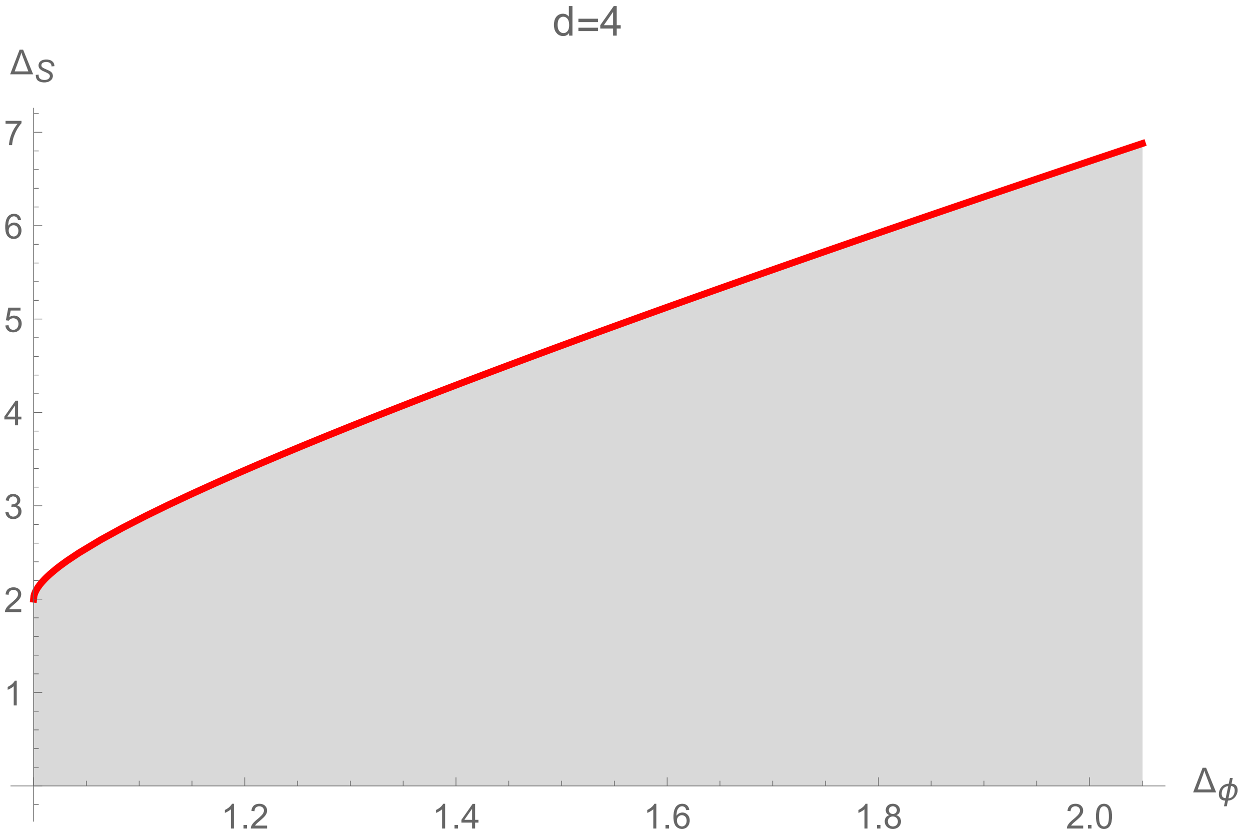
Beyond some value,
again minima

- Conclude that for the bootstrap equation to hold, we must have another scalar in the spectrum <u>whose dimension is below some number</u>.

- Can refine this analysis using Linear programming and Semi-definite programming. Lots of literature.

- Can also get bounds on OPE coeffs.

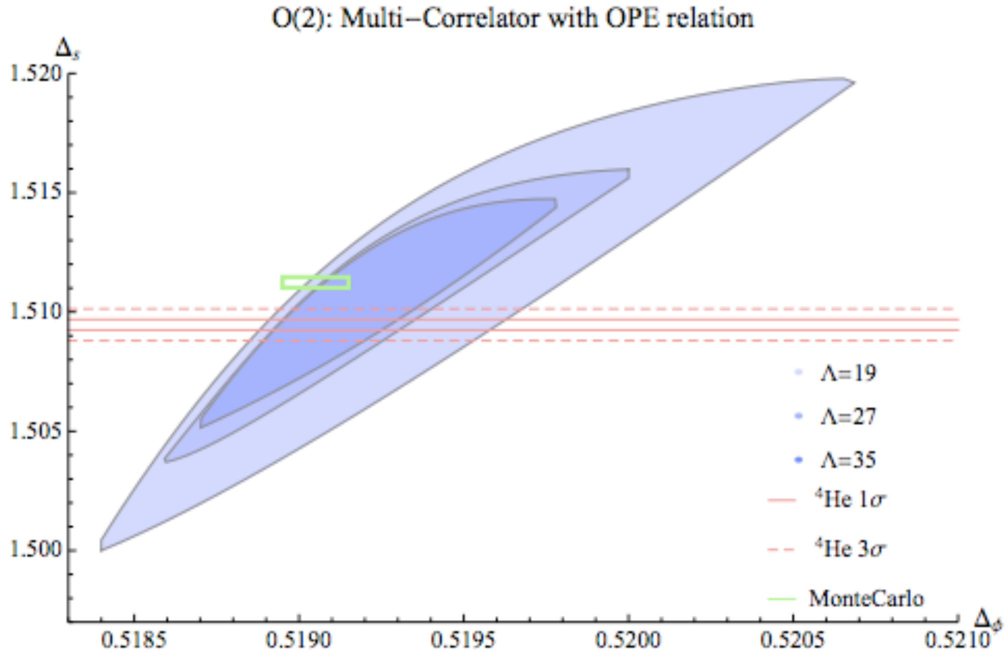
Bounds



$$\Delta_\phi = \frac{1+\eta}{2}, \quad \Delta_{\phi^2} = 3 - \frac{1}{\nu}, \quad \Delta_{\phi^4} = 3 + \omega$$

year	Method	ν	η	ω
1998	ϵ-exp	0.63050(250)	0.03650(500)	0.814(18)
1998	3D exp	0.63040(130)	0.03350(250)	0.799(11)
2002	HT	0.63012(16)	0.03639(15)	0.825(50)
2003	MC	0.63020(12)	0.03680(20)	0.821(5)
2010	MC	0.63002(10)	0.03627(10)	0.832(6)
this work		0.62999(5)	0.03631(3)	0.8303(18)

"most precise" 1403.4545, El Showk et al



Numerics are not yet accurate to resolve the disagreement with experiments

Kos, Poland, Simmons-Duffin, Vichi, 2015

Drawbacks

- Very hard to get any analytic result about low lying spectrum due to different powers of u, v appearing in each channel.
- Cannot reproduce Wilson-Fisher results except for large spin operators. [Sen, AS, 2015]
- Eqns depend on u,v (cannot simply match powers).
- Does not seem like a promising line for an analytic solution to the 3d Ising model at criticality.

- Conformal symmetry (not bootstrap) can be used to get critical exponents at leading order in epsilon—Rychkov, Tan, 2015.
- Can also do this numerically.
- Hence the current version of bootstrap may not be the most optimal way (at least analytically) of setting up the problem.

Motivating the new approach

- Polyakov in 1974 suggested a Lagrangian free approach to criticality based on unitarity and dispersion relations. This approach has not been examined carefully in the literature (at all, although it keeps getting cited in modern times).
- This approach gave the correct leading order (in epsilon) anomalous dimensions for certain operators.
- The general equations proved too hard to solve and this program was abandoned.

- Recently, Sen and I extended Polyakov's dispersion relation method to an order higher.
- This prompted us to look at his approach more closely.
- I will give a modern (our) version of his paper. It will turn to be more elegant in terms of Mellin space.

Mellin space

Position space correlators for identical scalars in CFT are functions of 2 conformal cross ratios u,v

Mellin transform correlator to s,t space

around
u=0, v=1

$$\int ds dt\, u^s v^t \mathrm{Measure}(s,t)\mathrm{Amplitude}(s,t)$$

Crossing symmetry (u interchange with v) now becomes a symmetry in terms of s,t

Amplitude for given spin exchange can be shown to go like s^{spin}

- Mellin amplitudes have been around for a while.

- Mack in 2009 emphasised their importance in CFTs. Further exemplified by Penedones; Paulos; Fitzpatrick, Kaplan, Penedones, van Rees, Raju…..

- Topic of study in recent times in the context of AdS/CFT. Mellin representation often are much simpler than their position space counterparts.

- Mellin space is the AdS analog of flat space momentum space.

$$\phi(s) = \int_0^{\infty} x^{s-1} f(x) dx$$

Mellin transform

$$f(x) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} x^{-s} \phi(s) ds$$

Inverse Mellin transform

$$\text{Measure}(s,t) = \Gamma(-t)^2\Gamma(s+t)^2\Gamma(\Delta_\phi - s)^2$$

s-contour is closed on the right.

Double pole implies possible log u terms in position space

Absent in OPE. Hence these must cancel.

Conformal partial wave decomposition

$$\mathrm{Amplitude}(s,t) \sim \sum_{\Delta,\ell} \int d\nu \mathcal{M}^{i}_{\ell}(\nu) Poly_{\nu}(s,t)\, c_{\Delta,\ell}$$

$$\mathcal{M}^{(s)}_{\ell}(\nu) = \frac{\Gamma(\frac{\Delta_1+\Delta_2-h+\ell+\nu}{2})\Gamma(\frac{\Delta_1+\Delta_2-h+\ell-\nu}{2})\Gamma(\frac{\Delta_3+\Delta_4-h+\ell+\nu}{2})\Gamma(\frac{\Delta_3+\Delta_4-h+\ell-\nu}{2})}{(\nu^2-(\Delta-h)^2)\Gamma(\nu)\Gamma(-\nu)(h+\nu-1)_{\ell}(h-\nu-1)_{\ell}}$$

Coincides for d=4 spectral function “derived” by Polyakov in 1974. Turns out to be same as what is used for the partial wave decomposition of Witten diagrams used in 1209.4355 by Costa, Goncalves, Penedones which was not realized.

- Idea now is to look at the crossing symmetric form of the amplitude in s,t space.

- Demand that it has no $(s-\Delta_\phi)^i$ with i=0,1 which would be incompatible with OPE.

For OPE to hold, in position space we (schematically have)

$$\sum_{\Delta,\ell} \sum_{m_1,m_2} c_{\Delta,\ell} u^{m_1}(1-v)^{m_2}(\alpha_{m_1,m_2} \log u + \beta_{m_1,m_2}) = 0$$

- This must hold for each m_1, m_2.
- Alpha's, Beta's are functions of conformal dimension and spin.
- Get constraints on the spectrum and OPE coefficients.
- Boils down to residue computations which can be done on Mathematica.

- However, in position space each channel is a sum over both spin and dimensions.

- This makes life a bit complicated. In particular the s-channel each power of (1-v) involves summing over a finite number of spins.

- In Mellin space we have a simplification.

Amp(s,t) can be written as an expansion in conformal partial waves in s,t

$$s - channel \sim \sum_{\Delta,\ell} q_{\Delta,\ell}(s) Q_{\ell,0}^{2s+\ell}(t)$$

continuous
Hahn polynomials, 1209.4355

look at leading u-power
inconsistent with OPE

$$Q(t) \sim {}_3F_2(-\ell, \Delta - 1, s+t; \frac{\Delta-\ell}{2}, \frac{\Delta-\ell}{2}; 1)$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(a+s+t)\Gamma(b+s+t)\Gamma(-t)\Gamma(-t-a-b) Q_{\ell,0}^{2s+\ell}(t,a,b) Q_{\ell',0}^{2s+\ell'}(t,a,b) = \kappa_\ell(s)\delta_{\ell,\ell'}\,,$$

Askey, Andrews, Roy

where,

$$\kappa_\ell(s) = \frac{4^\ell \ell!}{(2s+\ell-1)_\ell^2} \frac{\gamma_{\ell+s,a}\gamma_{\ell+s,b}}{(2s+2\ell-1)\Gamma(2s+\ell-1)}\,,$$

with $\gamma_{x,y} = \Gamma(x+y)\Gamma(x-y)$.

- We can expand now the t,u channels in terms of the continuous Hahn polynomials.

- The advantage of doing this is that the s-channel now has contribution only from a single spin (may have multiple operators of different dimensions).

- This is a huge simplification.

$$q_\ell^{(s)}(s=\Delta_\phi) = -\sum_\Delta \frac{\Gamma(2\Delta_\phi+\ell-d/2)c_{\Delta_\ell}}{(\ell-\Delta+2\Delta_\phi)(\ell+\Delta+2\Delta_\phi-d)}$$

$$\begin{aligned} q^{(t)}_{\Delta,\ell}(s) =& \kappa_\ell(s)^{-1}\sum_{\Delta',\ell'}\int dt d\nu\ c_{\Delta',\ell'}\Gamma(a_s+s+t)\Gamma(b_s+s+t)\Gamma(\lambda_2-t-\frac{1}{2}(\Delta_2+\Delta_3))\Gamma(\bar\lambda_2-t-\frac{1}{2}(\Delta_2+\Delta_3)) \\ &\times \mathcal{M}^{(t)}_{\ell'}(\nu)\gamma_{\lambda_1,a_t}\gamma_{\bar\lambda_1,b_t}\mathfrak{P}^{(t)}(s-\frac{1}{2}(\Delta_3+\Delta_4),t+\frac{1}{2}(\Delta_2+\Delta_3),\nu,a_t,b_t)Q^{2s+\ell}_{\ell,0}(t,a_s,b_s)\,. \end{aligned}$$

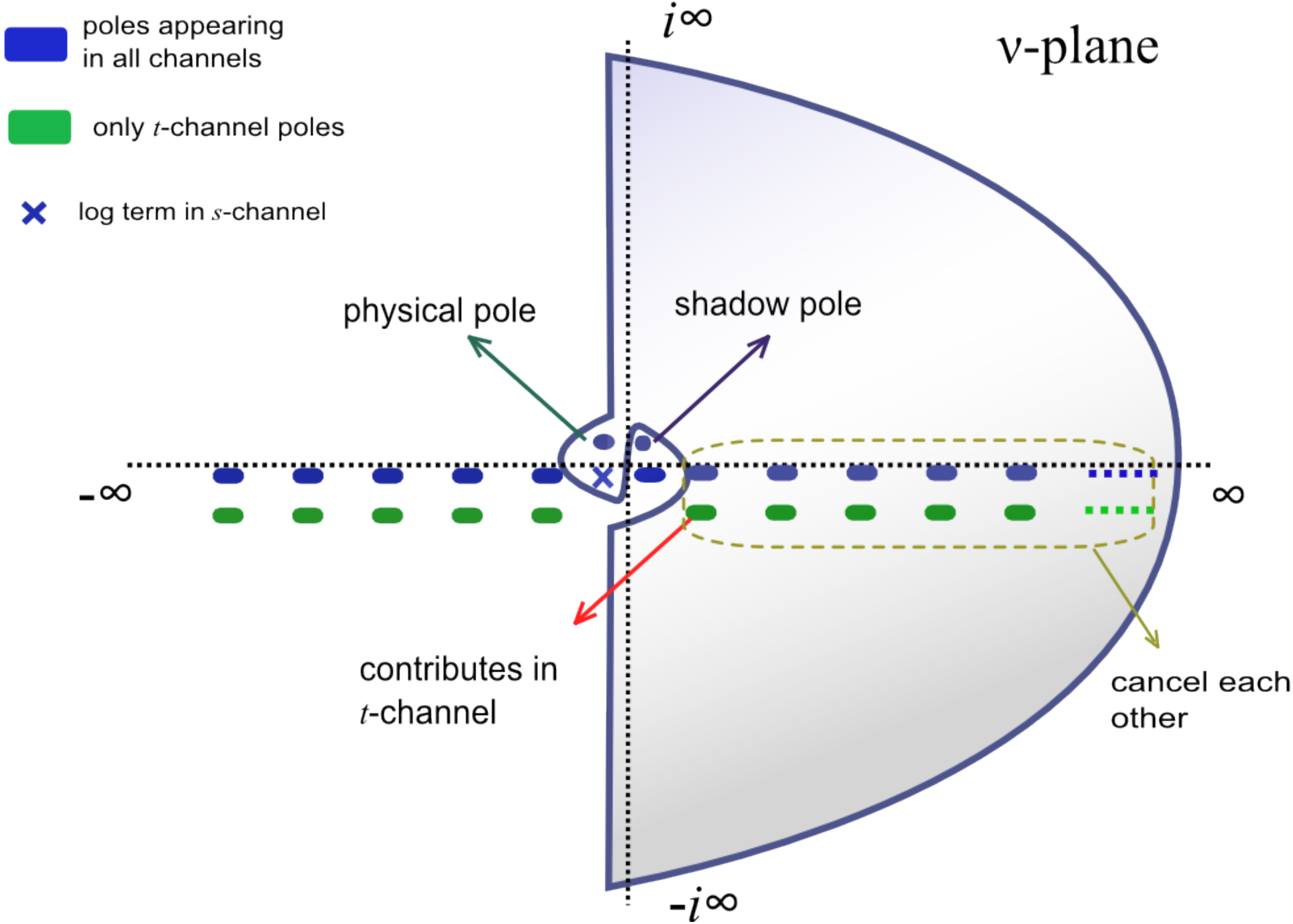
MACK POLYNOMIAL

- Equations simplify further at least in 3 cases that we have studied and agree with known answers.

- Epsilon expansion (can reproduce Wilson-Kogut for all double field operators including arbitrary spin)

- Large spin asymptotics

- Large N expansion (w A. Kaviraj and P. Dey) in O(N)



Epsilon expansion simplification upto second order

Large spin universal results

Position space version: Fitzpatrick et al, Komargodski-Zhiboedov, Kaviraj, Sen, AS; Alday-Zhiboedov

$$Q^{2s+\ell}_{\ell,0}(t) = \frac{2^{\ell}\ell^{-s-t}\Gamma(s+\ell)^2\Gamma(-1+s-t+\ell)}{\Gamma(-t)^2\Gamma(-1+2s+2\ell)}$$

Large spin limit is controlled by leading pole in t-variable. If poles are separated, we have a systematic expansion in inverse spin.

anom. dim. of large spin operator

$$\phi\partial\cdots\partial\phi$$

$$\gamma_\ell = -\frac{C_m 2^{2-\ell_m}\Gamma\left(\Delta_\phi\right)^2\Gamma\left(2\ell_m+\tau_m\right)}{\Gamma\left(\Delta_\phi-\frac{\tau_m}{2}\right)^2\Gamma\left(\ell_m+\frac{\tau_m}{2}\right)^2}\left(\frac{1}{\ell}\right)^{\tau_m}$$

Sign is negative always!

$$\delta\tau_{\rm m}\log\ell \gg 1$$

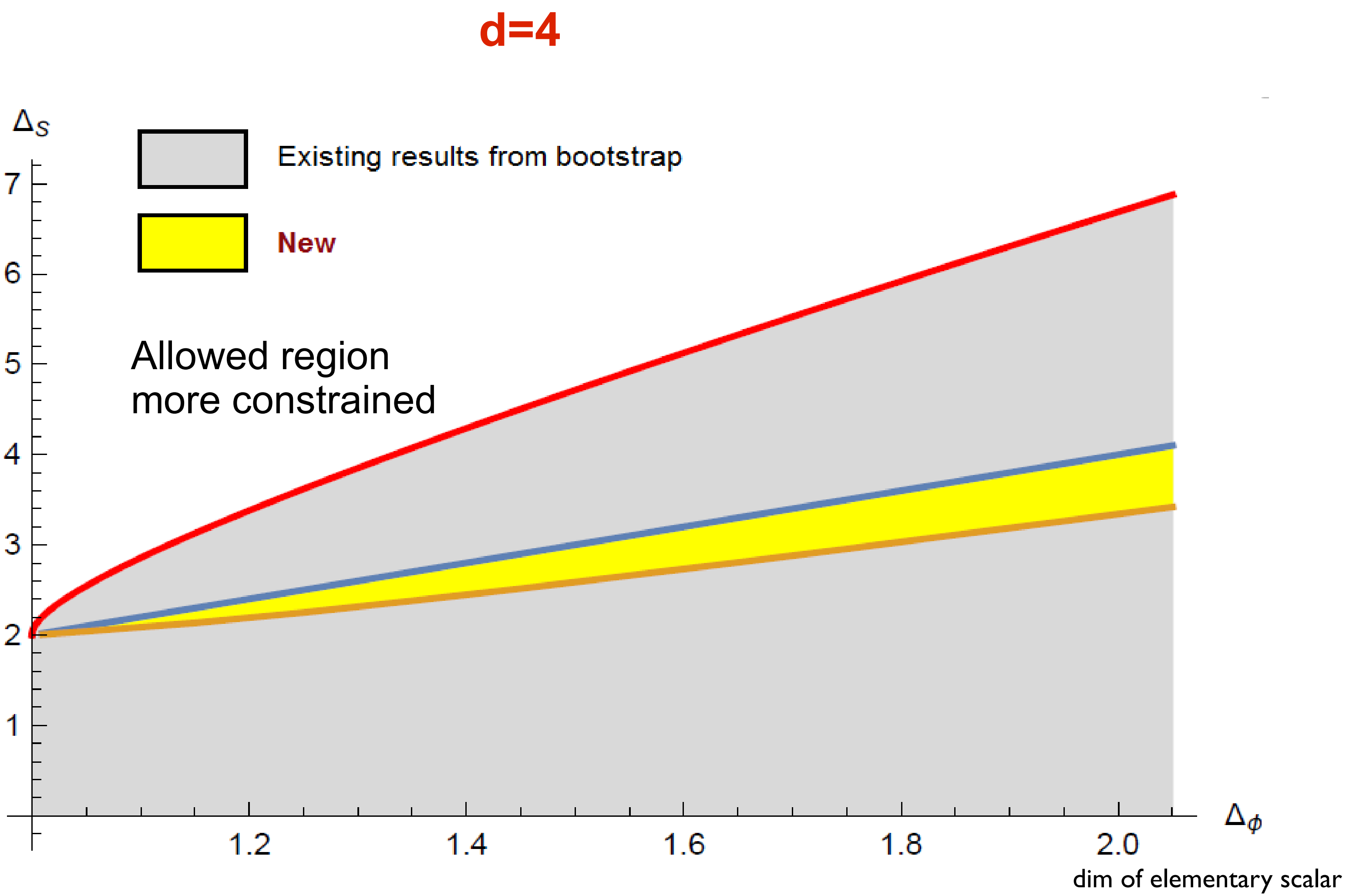
C_m : OPE leading exchange

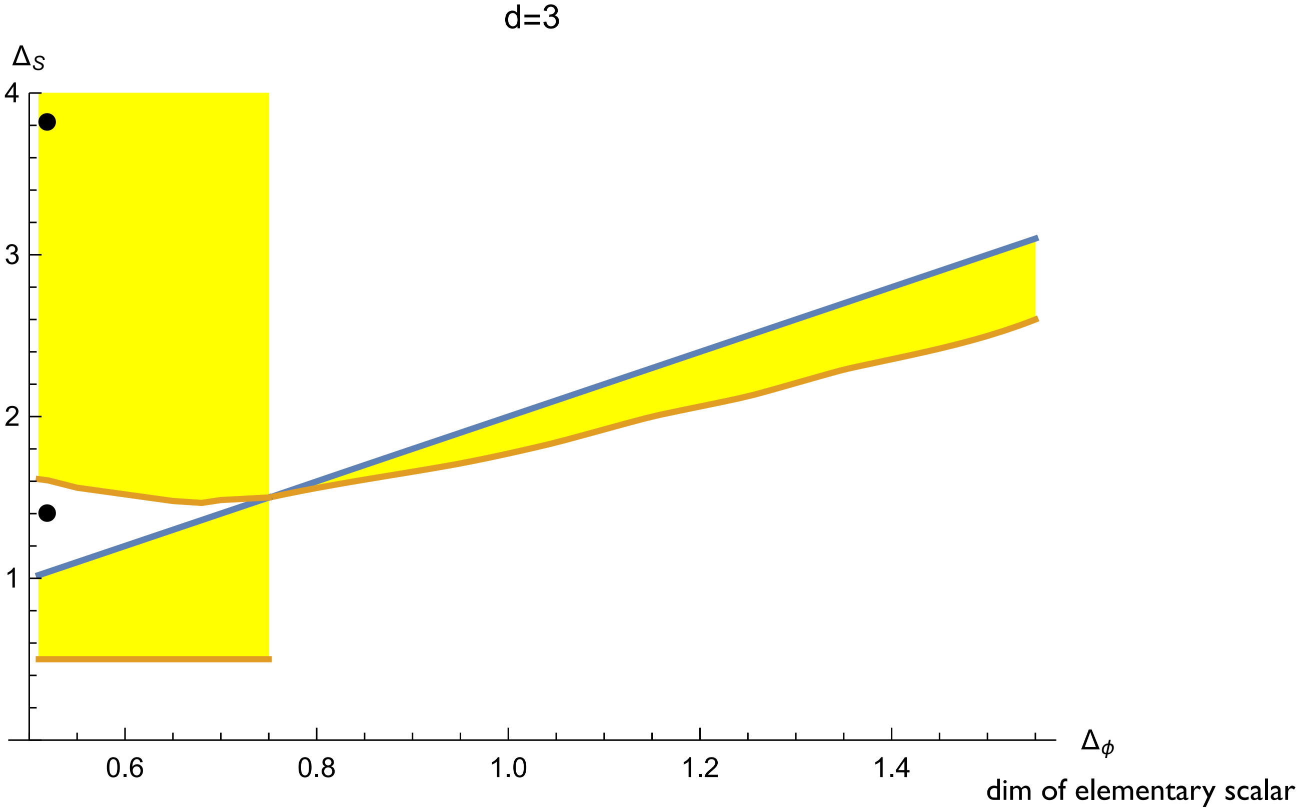
τ_m : Twist

- Do not need large N, not tied with gauge/gravity duality. Should hold for any CFT.
- Can also reproduce exactly from AdS/CFT hinting at a universal sector both in gravity and CFT.

More powerful numerics

(0,0) eqn
only





The single equation already predicts the existence of a higher order scalar in the 3d Ising model of dimension greater than something.

Summary

- New approach appears well suited to make connection with perturbative qft results analytically.
- Since method is non-perturbative, it is a good starting point to investigate expansion in other parameters, e.g., specific heat critical exponent.
- Will need to invent new maths for subleading u powers.
- Will be very interesting to develop numerics further.

- In the spirit of this conference, it will be interesting to ask what the connection with entanglement is.

- For heavy, heavy, light, light correlators, the continuous Hahn polynomials may lead to insights (that have been possible in AdS3/CFT2.)

Thank you!