Flow equation for the large N scalar model and induced geometry

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in collaboration with

J. Balog (Wigner Research Center), T. Onogi (Osaka Univ.), P. Weisz (MPI, Munich)

base on

S. Aoki, J. Balog, T. Onogi, P. Weizs,
``Flow equation for the large N scalar model and induced geometries”,
arXiv:1605.02413[hep-th].

related work

S. Aoki, K. Kikuchi, T. Onogi,
``Geometries from field theories”
PTEP 2015(2015)10, 101B01 (arXiv:1505.00131[hep-th]).

www.mcs.anl.gov

We dedicate this work to the memory of
Peter Hasenfratz

Contents

- Motivation
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Motivation

Holography

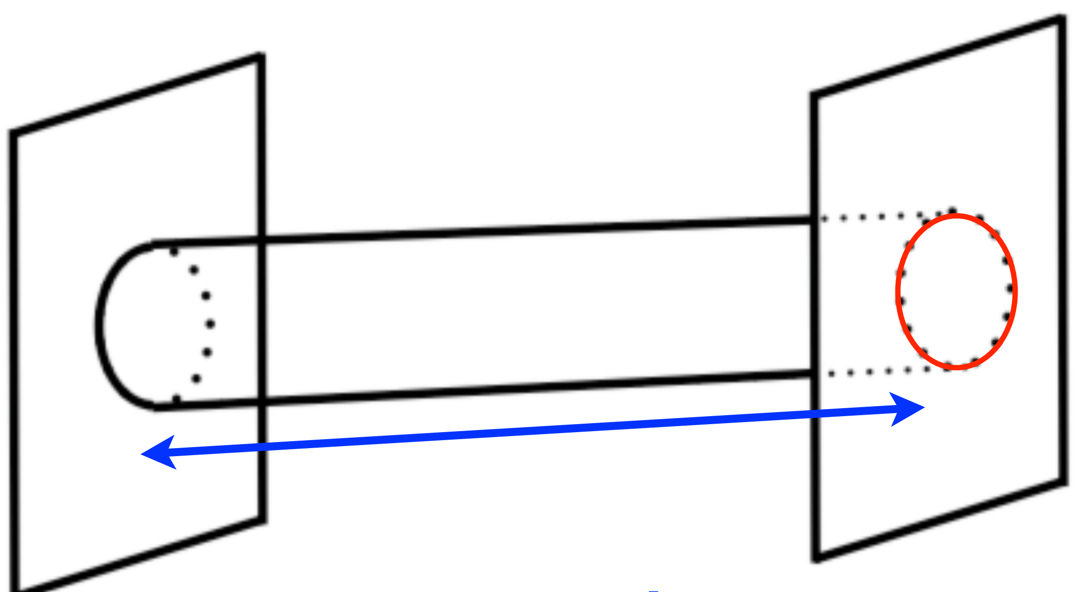
AdS/CFT correspondence ➡ **Gravity/Gauge**

Maldacena 1997

huge numbers of evidences but no proof

open string/closed string duality ?

D-brane



closed string = gravity

D-brane

open string

=

gauge theory

Different viewpoint

We propose a general method

Quantum Field Theory $\Rightarrow$ Geometry

cf. Geometry of classical gauge theories

covariant derivative $D_\mu = \partial + igA_\mu$ connection

field strength (curvature) $F_{\mu\nu} \propto [D_\mu, D_\nu]$

Apology: I am not an expert on String theory and related topics.
(I am mainly working on lattice QCD.)
So I can not answer your questions related to these.

Summary
of
Proposal and Results

Proposal

d-dim. field -> (d+1)-dim. field

$$\varphi^a(x) \quad \Rightarrow \quad \sigma^a(t,x)$$

d dimensions → d+1 dimensions

(d+1)-dim. field -> (d+1)-dim. induced metric

$$\hat{g}_{\mu\nu}(z) := h \sum_{a=1}^{N} \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z) \qquad z = (t,x)$$

(d+1)-dim. induced metric-> geometry

$$G_{\mu\nu}(z) := \langle G_{\mu\nu}(\hat{g}_{\mu\nu}(z)) \rangle$$

quantum average of Einstein tensor

⇒ ``geometry” of d+1 dimenional space

Results

In the large N limit, we show

$$\boxed{G_{\mu\nu}(z) \simeq -\Lambda g_{\mu\nu}(z) \qquad \Lambda < 0 \quad \text{(Euclidean) AdS space}}$$

in the following 2 limits:

UV limit $\quad t \to 0 \qquad \Lambda = -\dfrac{d(d-1)}{h(d-2)} \qquad d \geq 3$

IR limit $\quad t \to \infty \qquad \Lambda = -\dfrac{(d-1)}{h} \qquad d \geq 2$

Details I. Proposal

d-dim. field -> (d+1)-dim. field

(Gradient) Flow equation

large N index

action for d-dim. theory

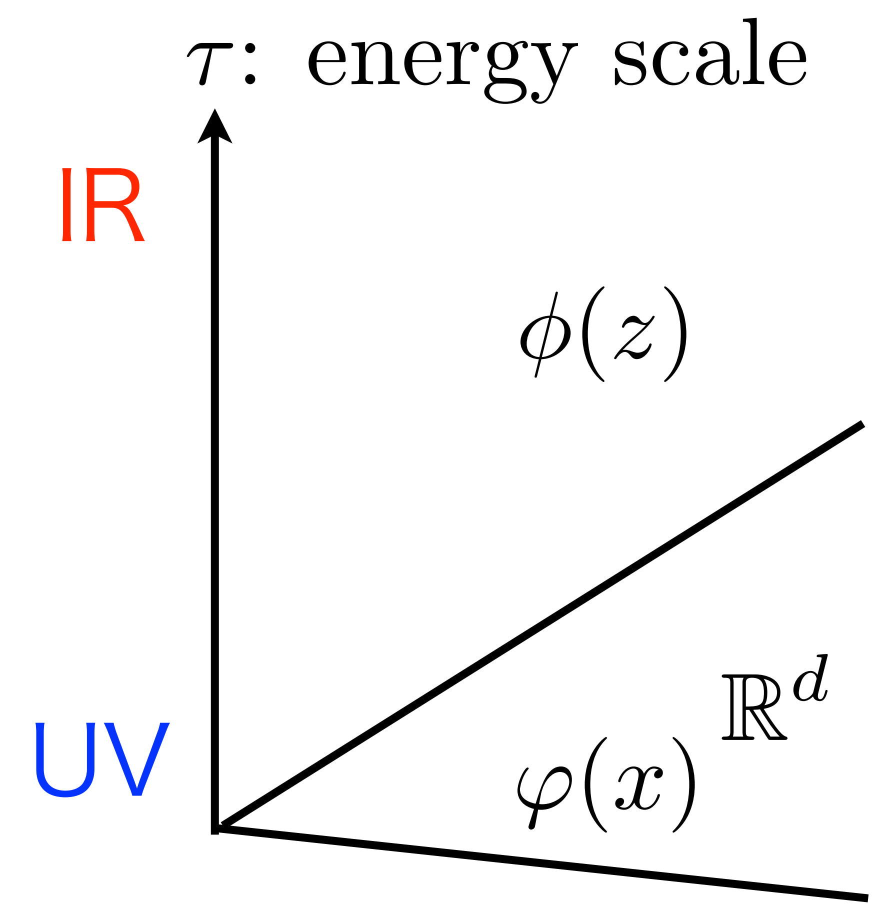
$$\frac{\partial}{\partial t}\phi^a(t,x) = -\left.\frac{\delta S_f(\varphi)}{\delta\varphi^a(x)}\right|_{\varphi^a(x)\to\phi^a(t,x)}$$

(d+1)-dim. field

d-dim. field

$$\phi^a(0,x) = \varphi^a(x)$$ **initial condition**

$$z = (\tau = \sqrt{t}, x) \in (\mathbb{R}^+, \mathbb{R}^d)$$



Remark

$\varphi(x)$ is the field in the path integral (NOT the operator).

What is the (gradient) flow equation ?

Free theory $$\frac{\partial}{\partial t}\phi^a(t,x) = \left(\Box - m^2\right)\phi^a(t,x)$$

$$\Rightarrow \quad \phi^a(t,x) = \frac{e^{-m^2 t}}{(4\pi t)^{d/2}}\int d^d y\, e^{-(x-y)^2/t}\varphi^a(y)$$

Heat kernel

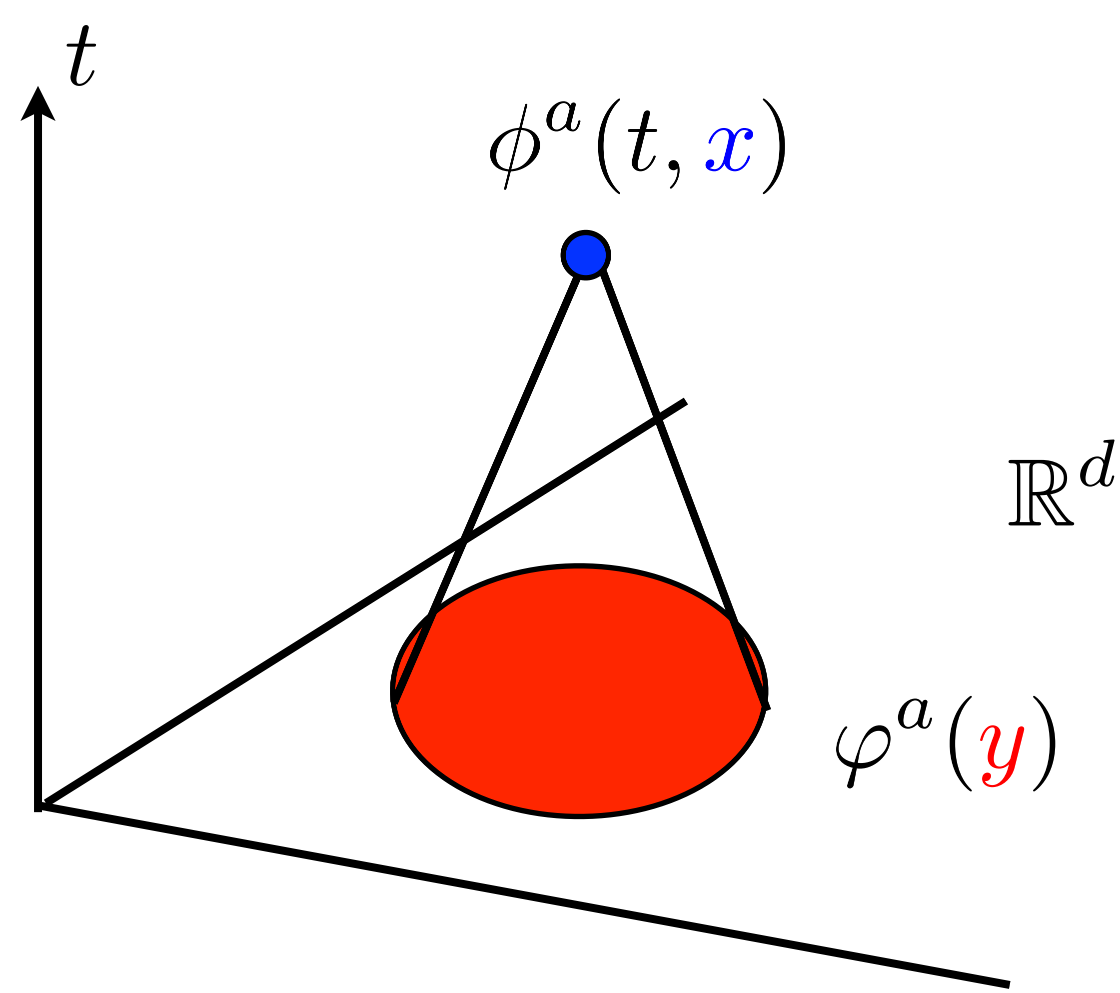
Lattice QCD

introduced to smooth out UV fluctuations of gauge fields

Narayanan-Neuberger 2006, Luescher 2010

flow gauge field is UV finite

Luescher-Weisz 2011



cf. Ricci flow $$\frac{d}{dt}g_{ij} = -2R_{ij}$$

used to prove Poincare conjecture by Perelman

Normalized flow field

$$\sigma^a(z) := \frac{\phi^a(z)}{\sqrt{\langle \phi^2(z) \rangle}}$$

Non-Linear Sigma Model (NLSM) normalization

Quantum average

d-dimension

$$\langle \mathcal{O}(\varphi) \rangle := \langle \mathcal{O}(\varphi) \rangle_S = \frac{1}{Z} \int \mathcal{D}\varphi \; \mathcal{O}(\varphi) e^{-S(\varphi)}, \qquad Z := \int \mathcal{D}\varphi \; e^{-S(\varphi)}$$

Flow equation

integrate out UV modes

Normalization

renomalization of field

define ``Renormalization Group" transformation

Remarks

One may take different nomalization conditions instead of NLSM.

$S \neq S_f$ is allowed. If $S = S_f$, we call it **"gradient flow"**.

(d+1)-dim. field -> (d+1)-dim. metric-> geometry

$$\sigma^a(z) : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^N$$

h: constant with mass dimension -2

$$\hat{g}_{\mu\nu}(z) := h \sum_{a=1}^{N} \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z)$$

Induced metric on a $d+1$ dim. manifold $\mathbb{R}^+ \times \mathbb{R}^d$ from a manifold in $\mathbb{R}^N$, defined by $\sigma^a(z)$ with $\langle \sigma^2(z) \rangle = 1$

any correlation functions can be calculated using

functional integral in d-dimensions

$$\langle \hat{g}_{\mu\nu}(z) \rangle := \langle \hat{g}_{\mu\nu}(z) \rangle_S ,$$

geometry

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \hat{g}_{\mu_2\nu_2}(z_2) \rangle := \langle \hat{g}_{\mu_1\nu_1}(z_1) \hat{g}_{\mu_2\nu_2}(z_2) \rangle_S ,$$

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle := \langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle_S ,$$

quantum corrections

key properties

1 $\hat{g}_{\mu\nu}(z) \propto \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z)$ may give finite results for $\tau \neq 0$

Flow: a heat kernel type smearing $\tau \to 0$ is UV while $\tau \to \infty$ is IR

Finiteness as QFT is NOT guaranteed in general but true in the large N limit.

cf. d dimensional induced metric $g_{\mu\nu}(x) \sim \partial_\mu \varphi(x) \partial_\nu \varphi(x)$ is badly divergent

2 metric becomes classical in the large N limit

$$\langle \hat{g}_{\mu\nu}(z_1) \hat{g}_{\alpha\beta}(z_2) \rangle = \langle \hat{g}_{\mu\nu}(z_1) \rangle \langle \hat{g}_{\alpha\beta}(z_2) \rangle + O\left(\frac{1}{N}\right)$$

large N factorization

$$\Rightarrow \quad \langle G_{\mu\nu}(\hat{g}_{\mu\nu}) \rangle = G_{\mu\nu}(\langle \hat{g}_{\mu\nu} \rangle) + O\left(\frac{1}{N}\right)$$

classical geometry after quantum averages d-dim. quantum field theory

large N limit

(d+1)-dim. classical metric

Geometry in d+1 dimensions

Details II. Results

Large N Model

φ^4 model

$$S(\mu^2, u) = N \int \mathrm{d}^d x \left[\frac{1}{2} \partial^k \varphi(x) \cdot \partial_k \varphi(x) + \frac{\mu^2}{2} \varphi^2(x) + \frac{u}{4!} \left(\varphi^2(x) \right)^2 \right]$$

$u = 0$: free, $u \to \infty$: NLSM

$\varphi^2(x) \equiv \varphi(x) \cdot \varphi(x) = \sum_{a=1}^N \varphi^a(x) \varphi^a(x)$

large N limit

$$\langle \varphi^a(x) \varphi^b(y) \rangle = \delta^{ab} \frac{1}{N} \int \mathrm{d}p \, \frac{\mathrm{e}^{ip(x-y)}}{p^2 + m^2}$$

mass renormalization

$$\mu^2 = m^2 - \frac{u}{6} Z(m) \qquad Z(m) = \int \mathrm{d}p \, \frac{1}{p^2 + m^2} \geq 0, \qquad \mathrm{d}p \equiv \frac{\mathrm{d}^d p}{(2\pi)^d},$$

$$Z(m) \to \infty \ (d > 1)$$

Flow field

Flow equation

$$\frac{\partial}{\partial t}\phi^a(t,x) = -\left.\frac{\delta S(\mu_f^2, u_f)}{\delta\varphi^a(x)}\right|_{\varphi\to\phi} = \left(\Box - \mu_f^2\right)\phi^a(t,x) - \frac{u_f}{6}\phi^a(t,x), \quad \phi^a(0,x) = \varphi^a(x)$$

Solution in the large N limit

$$\phi(t,p) = f(t)\mathrm{e}^{-p^2 t}\varphi(p).$$

$$\dot{f}(t) = -\mu_f^2 f(t) - \frac{u_f}{6} f^3(t)\zeta_0(t), \qquad \zeta_0(t) = \int \mathrm{d}p\, \frac{\mathrm{e}^{-2p^2 t}}{p^2+m^2}, \qquad \zeta_0(0) = Z(m).$$

2-pt function

$$\langle \phi^a(t,x)\phi^b(s,y)\rangle = \frac{\delta^{ab}}{N}\frac{Z(m_f)}{\sqrt{\zeta(t)\zeta(s)}}\int \mathrm{d}p\, \frac{\mathrm{e}^{-(t+s)p^2}\mathrm{e}^{ip(x-y)}}{p^2+m^2}.$$

divergent !

Normalized field

$$\sigma^a(t,x) = \frac{\phi^a(t,x)}{\sqrt{\langle \phi^2(t,x) \rangle}},$$

$$\langle \phi^2(t,x) \rangle = \frac{Z(m_f)}{\zeta(t)} \int \mathrm{d}p \, \frac{\mathrm{e}^{-2tp^2}}{p^2+m^2} = Z(m_f) \frac{\zeta_0(t)}{\zeta(t)} \to Z(m_f)$$

2-pt function for nomalized field

$$\langle \sigma^a(t,x) \sigma^b(s,y) \rangle = \frac{\delta^{ab}}{N} \frac{1}{\sqrt{\zeta_0(t)\zeta_0(s)}} \int \mathrm{d}p \, \frac{\mathrm{e}^{-(t+s)p^2} \mathrm{e}^{ip(x-y)}}{p^2+m^2}.$$

UV finite and independent bare parameters,
depends only on the renomalized mass

$$\zeta_0(t) = \int \mathrm{d}p \, \frac{\mathrm{e}^{-2p^2 t}}{p^2+m^2}$$

Induced metric

VEV of the metric

$$g_{\mu\nu}(z) := \langle \hat{g}_{\mu\nu}(z) \rangle = \begin{pmatrix} g_{\tau\tau}(\tau) & 0 \\ 0 & g_{ij}(\tau) \end{pmatrix}$$

$$g_{\tau\tau}(\tau) = \frac{h\tau^2}{16} \frac{\mathrm{d}^2 \log \zeta_0(t)}{\mathrm{d}t^2}, \qquad g_{ij}(\tau) = -\delta_{ij} \frac{h}{2d} \frac{\mathrm{d} \log \zeta_0(t)}{\mathrm{d}t}.$$

$$\zeta_0(t) = \frac{m^{d-2} \mathrm{e}^{2m^2 t}}{(4\pi)^{d/2}} \Gamma(1 - d/2, 2m^2 t).$$

incomplete gamma function

IR limit $m\tau \gg 1$

$$g_{\tau\tau}(\tau) = \frac{hd}{2\tau^2}, \qquad g_{ij}(\tau) = \frac{h\delta_{ij}}{\tau^2}$$

$u = \sqrt{d/2}\,\tau$ $\Rightarrow$

$$\mathrm{d}s^2 = \frac{hd}{2u^2}(\mathrm{d}u^2 + \mathrm{d}x^2)$$

Euclidean AdS metric

UV limit $m\tau \ll 1$

log correction appears in d=2

$$g_{\tau\tau}(\tau) \simeq h \begin{cases} \sqrt{\frac{2}{\pi}}\frac{m}{4\tau}, & d=1 \\ -\frac{1}{\tau^2\log(m^2\tau^2)}, & d=2 \\ \frac{d-2}{2}\frac{1}{\tau^2}. & d\geq 3 \end{cases} \quad , \quad g_{ij}(\tau) \simeq h\delta_{ij} \begin{cases} \sqrt{\frac{2}{\pi}}\frac{m}{\tau}, & d=1 \\ -\frac{1}{\tau^2\log(m^2\tau^2)}, & d=2 \\ \frac{d-2}{d}\frac{1}{\tau^2}, & d\geq 3 \end{cases}$$

Euclidean AdS metric

Einstein Tensor

$$G_{\tau\tau}(\tau) = -\Lambda_\tau(m\tau) g_{\tau\tau}(\tau), \qquad G_{ij}(\tau) = -\Lambda_d(m\tau) g_{ij}(\tau),$$

$$\Lambda_\tau(m\tau) = -\frac{d(d-1)}{2h} \left. \frac{\frac{\mathrm{d}\log Y(x)}{\mathrm{d}x}}{Y(x)} \right|_{x=m^2\tau^2/2}, \quad Y(x) = 1 + \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(1-d/2, x).$$

$$\Lambda_d(m\tau) = \Lambda_\tau(m\tau) + \delta\Lambda(m\tau), \qquad \frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} = \frac{2}{d}\left[\frac{\frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right)}{\frac{\mathrm{d}\log Y(x)}{\mathrm{d}x}} - 2 \right]_{x=m^2\tau^2/2}$$

$\delta\Lambda(m\tau) = 0$ $\Rightarrow$ $G_{\mu\nu}(\tau) = -\Lambda_\tau(m\tau) g_{\mu\nu}(\tau)$

$\Lambda_\tau(m\tau) = \Lambda = \text{constant}$ $\Rightarrow$ cosmological constant

IR limit $m\tau \gg 1$

$$G_{\mu\nu}(\tau) \simeq -\Lambda_{\mathrm{IR}} g_{\mu\nu}(\tau) \qquad \Lambda_{\mathrm{IR}} = -\frac{d-1}{h}$$

Euclidean AdS metric

UV limit $m\tau \ll 1$

$d \geq 3$

$$G_{\mu\nu}(\tau) \simeq -\Lambda_{\mathrm{UV}} g_{\mu\nu}(\tau) \qquad \Lambda_{\mathrm{UV}} = -\frac{d(d-1)}{h(d-2)}$$

Euclidean AdS metric

$d = 2$

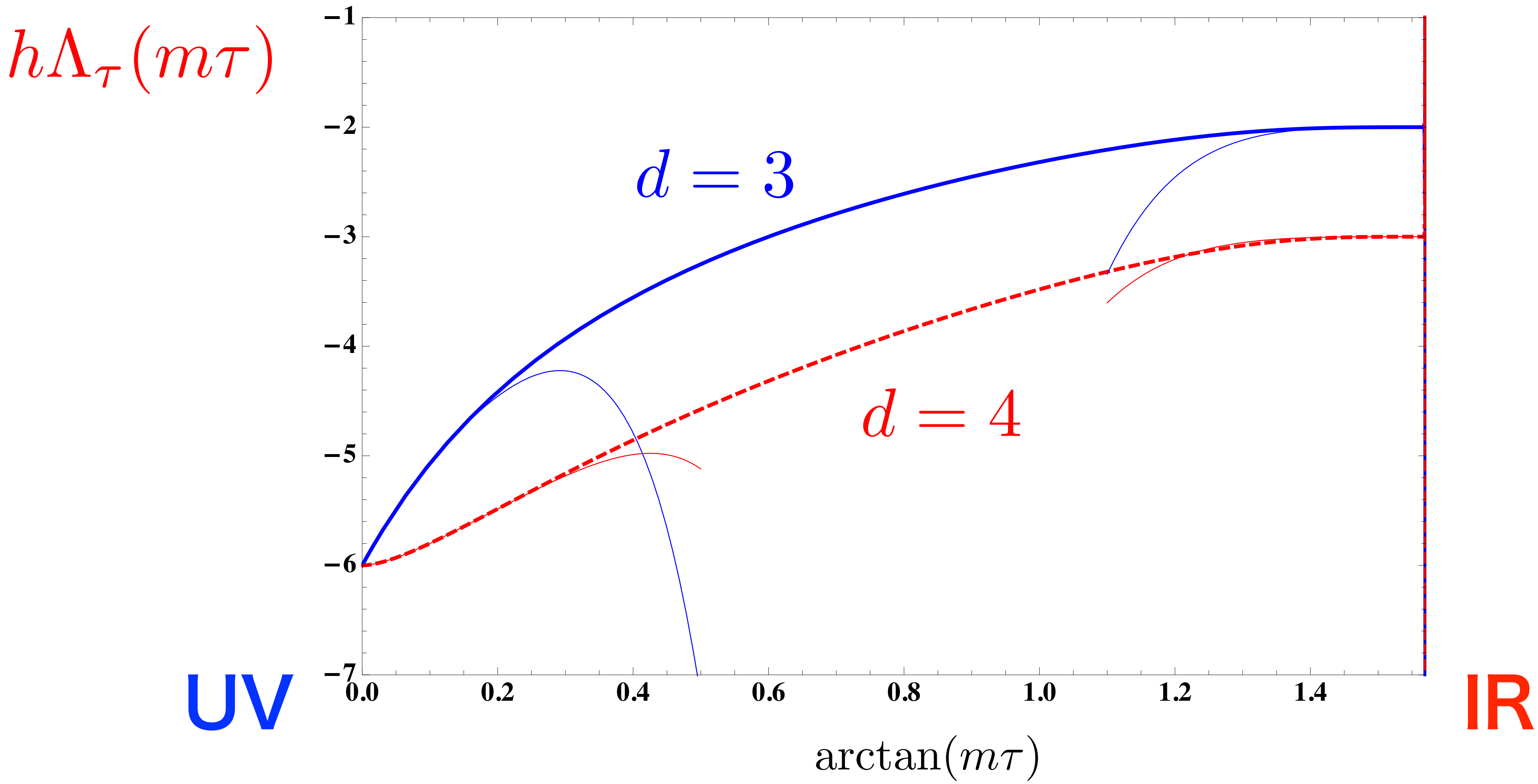
$$G_{\mu\nu}(\tau) \simeq -\Lambda(m\tau) g_{\mu\nu}(\tau) \qquad \Lambda(m\tau) \simeq \frac{\log(m^2\tau^2)}{h}$$

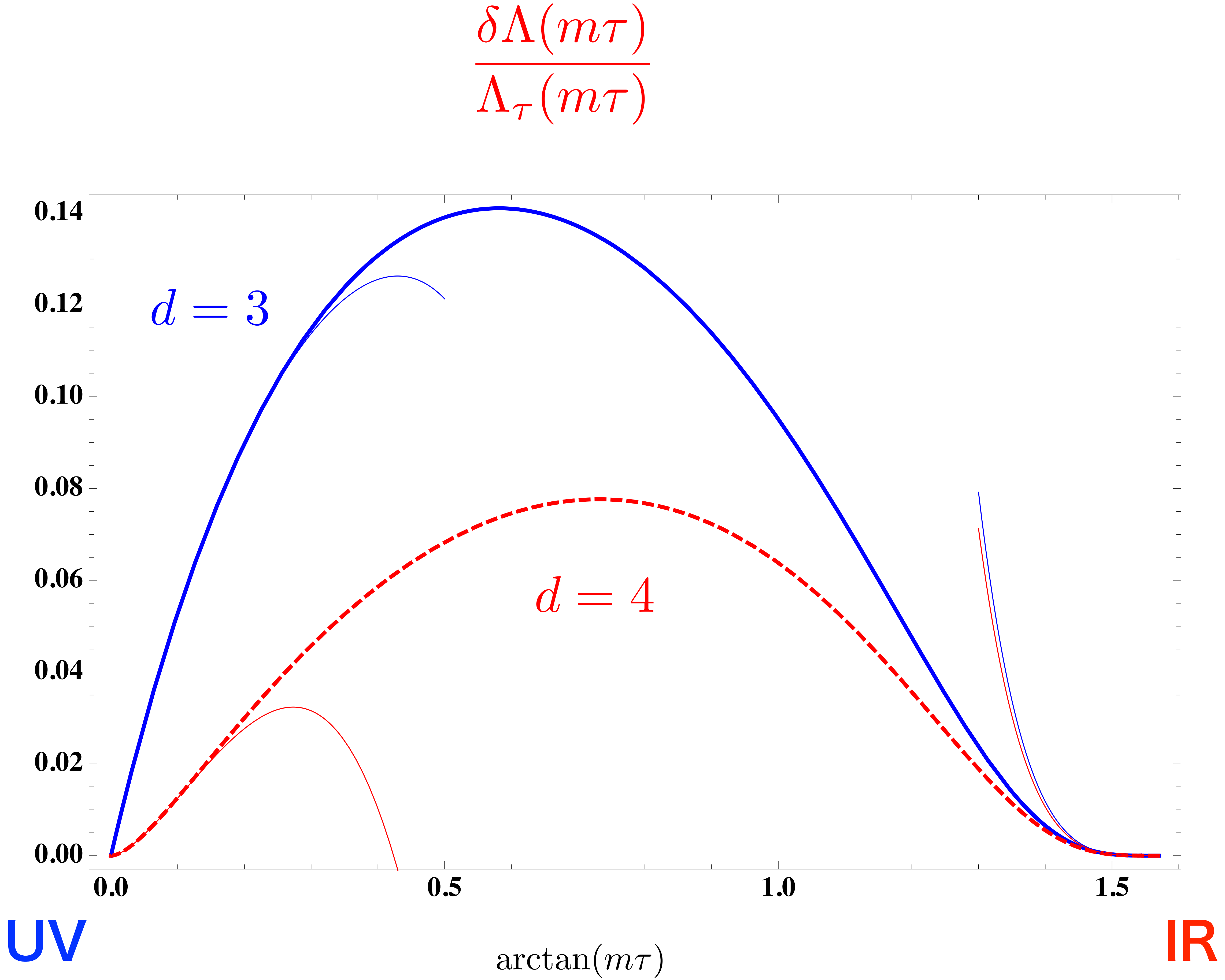
From UV to IR

AdS radius

$$R^2 := -\frac{1}{\Lambda}$$

$$\Rightarrow \quad R_{\mathrm{UV}}^2 = -\frac{h(d-2)}{d(d-1)} = \frac{d-2}{d} R_{\mathrm{IR}}^2 < R_{\mathrm{IR}}^2$$





$\frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)}$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
$d = 3$
$d = 4$
0.0
0.5
1.0
1.5
UV
$\arctan(m\tau)$
IR

Discussions

Summary

- Proposal: d-dim. QFT -> (d+1) dim. induced metric
  - using flow equation
- Properties in the large N limit
  - UV finiteness
  - metric becomes classical
- Results: large N scalar model
  - IR -> Euclidean AdS at d>1
  - UV -> Euclidean AdS at d > 2

Questions

- this approach meaningful ?
- relation to holography ?
  - higher spin theories ?
  - non CFT -> geometries ?
- quantum fluctuations in the large N expansion

$$\langle g_{\mu_1\nu_1}(z_1) g_{\mu_2\nu_2}(z_2)\rangle_c = O\left(\frac{1}{N}\right)$$

- Other models ?
  - fermions, gauge fields -> ?
  - finite Temperature -> black hole ?