Entanglement Holography

see also Bartek Czech's talk and 1604:03110!

Based on:

arXiv:1509.00113 + arXiv:1606.nnnnn (few days)

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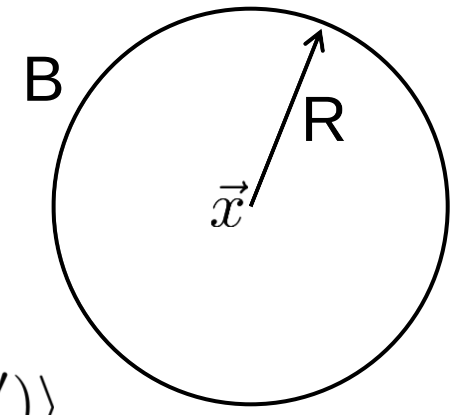
Basic idea:

(holography): Try to reorganize the degrees of freedom of quantum gravity in an interesting way. This may shed light on the (non)local nature of the underlying fundamental degrees of freedom. May help understand bulk reconstruction. Sheds new light on perturbative bulk computations.

(CFT): Try to reorganize the degrees of freedom of CFT’s in an interesting way. Define new quantum information theoretic quantities? May shed new light on the structure of correlation functions. May be useful to study which CFT’s have weakly coupled gravitational duals and for other applications.

Starting observation:



The first law of entanglement entropy states that

$$\delta S(B) = 2\pi \int_B d^{d-1}x' \; \frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R} \, \langle T_{tt}(\vec{x}') \rangle$$

This contains the “bulk-boundary” propagator of de Sitter space and hence

$$\left(\nabla_a \nabla^a - m^2\right) \, \delta S = 0 \qquad\qquad m^2 L^2 = -d$$

in the spacetime with the metric

$$ds^2 = \frac{L^2}{R^2}\left(-dR^2 + d\vec{x}^2\right)$$

Size of ball B = scale = time of de Sitter

Valid in any CFT

Similar equations but different perspective in:
Nozaki, Numasawa, Prudenziati, Takayanagi;
Bhattacharya, Takayanagi

This field equation provides a relation between the entanglement entropy for smaller regions versus that of larger regions (a la RG flow).

It is universal (i.e. state-independent).

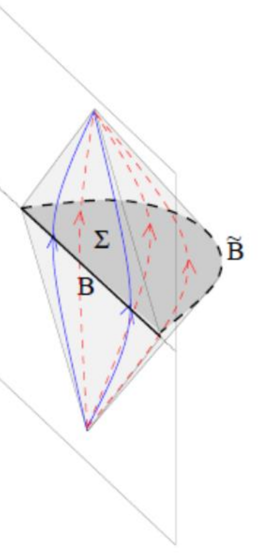
Initial data: one-point functions of the energy-momentum tensor.

Turns out to also work for higher spin fields if you generalize the first law:

Consider the following first law

$$\delta S^{(s)}(B) = (2\pi)^{s-1} \int_B d^{d-1}x' \left(\frac{R^2 - |\vec{x} - \vec{x}'|^2}{2R} \right)^{s-1} \langle W_{tt\ldots t}(\vec{x}') \rangle$$

which can be used to define a linearized higher spin generalization of entanglement entropy. It obeys a dS Klein-Gordon equation with mass



$$m^2 L^2 = -(s-1)(d+s-2)$$

(Cf Belin, Hung, Maloney, Matsuura, Myers, Sierens; Hijano, Kraus)

Why does de Sitter space appear?

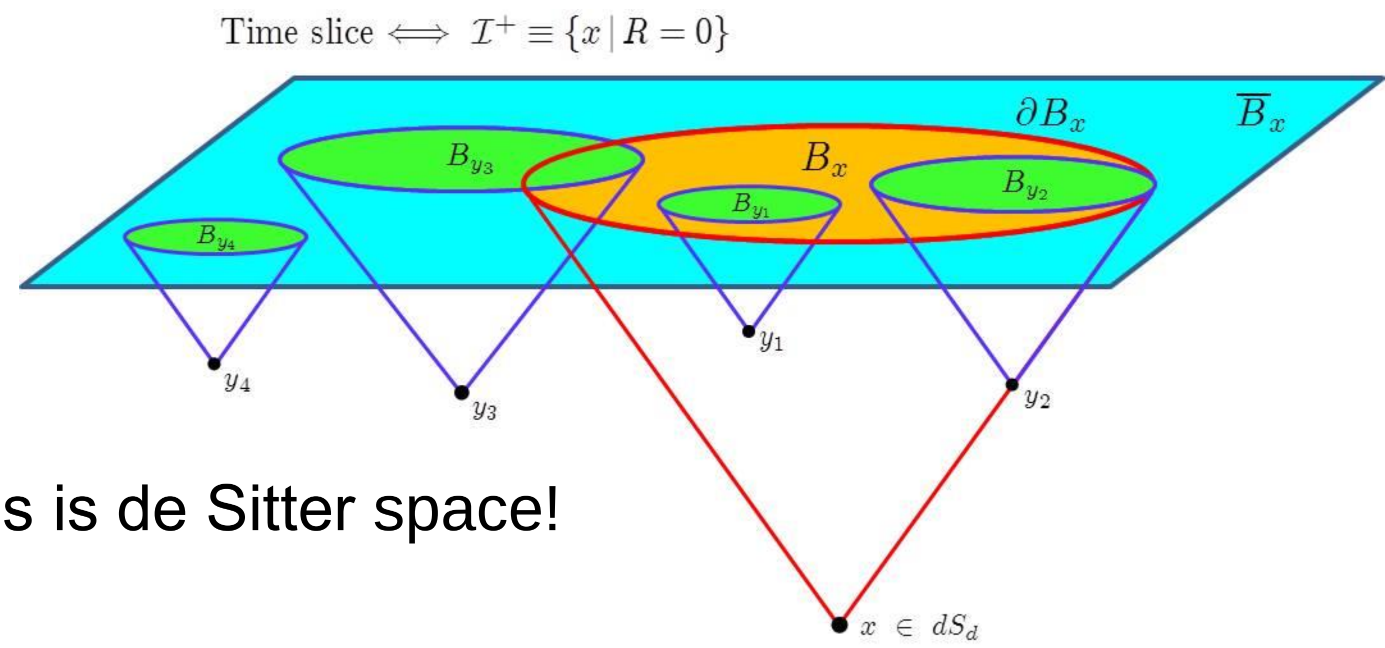
kinematic space, MERA,…

Euclidean $\mathrm{AdS_d}$ is the equation

$$\langle X, X\rangle \equiv -X_0^2 + X_1^2 + \ldots + X_d^2 = -1$$

A "spherical" minimal surface is given by

$$\langle X, U\rangle = 0, \qquad \langle U, U\rangle = +1$$



The space of U's is de Sitter space!

This was all at a constant time slice.

Generalize to non-constant time slices:
Lorentzian AdS_{d+1} is the equation

$$\langle X, X\rangle \equiv -X_0^2 - X_1^2 + X_2^2 + \ldots + X_{d+1}^2 = -1$$

A "spherical" minimal surface is given by

$$\{\langle X, U\rangle = 0 \cap \langle X, V\rangle = 0\}, \qquad \langle U, U\rangle = +1, \langle V, V\rangle = -1$$

The space of spherical minimal surfaces is

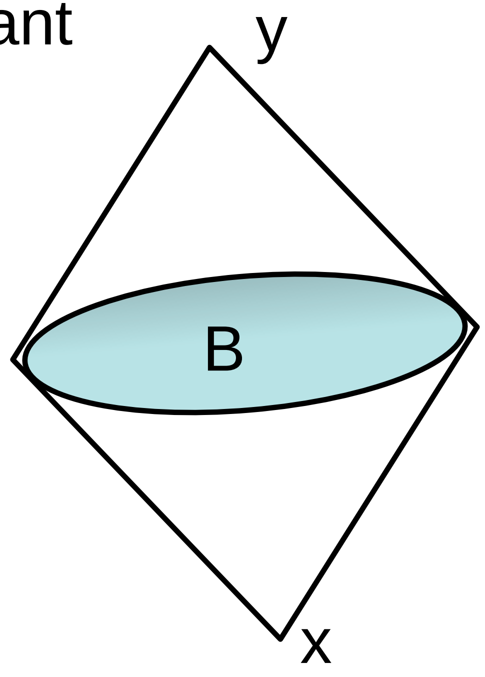
$$M = \frac{SO(2,d)}{SO(1,d-1)\times SO(1,1)}$$

This is a space of dimension 2d with metric with signature (d,d).

Spherical minimal surfaces in Lorentzian signature can be characterized by two points x,y that bound a causal diamond.

There is a unique conformally invariant metric on the space of two points M.

The geodesic distance between two pairs of points is a simple function of their cross ratio.



For d=2: $M = dS_2 \times dS_2$

$$ds^2 = \frac{1}{(x-y)^2}\left(\eta_{\mu\nu} - 2\frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2}\right) dx^\mu dy^\nu$$

Do field equations persist beyond the linearized approximation?

Consider EE in 2 dimensions in a gravitationally excited states

Consider a general metric of the form

$$ds^2 = \frac{dw^2 + dx^+ dx^-}{w^2} - \frac{6}{c} T(x^+) dx^{+2} - \frac{6}{c} \bar{T}(x^-) dx^{-2} + \frac{36}{c^2} w^2 T(x^+) \bar{T}(x^-) dx^+ dx^-$$

then the entanglement entropy is equal to

$$S_{EE} = \frac{c}{6} \log \left(\frac{(f_+(x_1^+) - f_+(x_2^+))(f_-(x_1^-) - f_-(x_2^-))}{\epsilon^2 \sqrt{\partial_+ f_+(x_1^+) \partial_+ f_+(x_2^+) \partial_- f_-(x_1^-) \partial_- f_-(x_2^-)}} \right)$$

with

$$T(x^+) = \frac{c}{12} \frac{\partial_+^3 f_+}{\partial_+ f_+} - \frac{3}{2} \left(\frac{\partial_+^2 f_+}{\partial_+ f_+} \right)^2$$

Roberts, 1204:1982
Asplund, Callebaut, Zukowski, 1604:06287

Entanglement entropy can be holomorphically factorized

$$S_{EE} = S + \bar{S}$$

These obey, in any background,

$$\frac{\partial}{\partial x_1^+}\frac{\partial}{\partial x_2^+}S = \frac{c}{6\epsilon^2}e^{-12S/c} \qquad \frac{\partial}{\partial x_1^-}\frac{\partial}{\partial x_2^-}\bar{S} = \frac{c}{6\epsilon^2}e^{-12\bar{S}/c}$$

This is like a pair of Liouville equations, suggesting $S, \bar{S}$ define metrics on de Sitter space in conformal gauge. Connection to quantum gravity on de Sitter space?

For this particular case, $M=dS_2 x dS_2$ corresponding to left- and right movers.

A more interesting test: take a higher spin theory in AdS3, for example with massless fields of spins 2 and 3.

Using the Chern-Simons formulation of the theory, and the relation between entanglement entropy and Wilson lines (JdB, Jottar; Ammon, Castro, Iqbal) we can compute both the ordinary entanglement entropy and its spin three generalization for arbitrary spin 2,3 backgrounds.

Both spin-2 and spin-3 entanglement entropy factorize in left- and right-movers.

Result

$$
\begin{aligned}
\Sigma_1 &= (\gamma_2(x_1) - \gamma_2(x_2))\gamma_1'(x_1) - (\gamma_1(x_1) - \gamma_1(x_2))\gamma_2'(x_1) \\
\Sigma_2 &= (\gamma_2(x_1) - \gamma_2(x_2))\gamma_1'(x_2) - (\gamma_1(x_1) - \gamma_1(x_2))\gamma_2'(x_2) \\
\Phi_1 &= (\gamma_1'(x_1)\gamma_2''(x_1) - \gamma_2'(x_1)\gamma_1''(x_1)) \\
\Phi_2 &= (\gamma_1'(x_2)\gamma_2''(x_2) - \gamma_2'(x_2)\gamma_1''(x_2))
\end{aligned}
$$

$$
S_{EE}^2 = \log\left(\frac{\Sigma_1\Sigma_2}{\Phi_1\Phi_2}\right), \qquad S_{EE}^3 = \log\left(\frac{\Sigma_2^{1/2}\Phi_1^{1/6}}{\Sigma_1^{1/2}\Phi_2^{1/6}}\right).
$$

These obey in any background

$$\frac{\partial^2}{\partial x_1 \partial x_2} S^2_{EE} = \exp\left(-\frac{S^2_{EE}}{2} + 3S^3_{EE}\right) + \exp\left(-\frac{S^2_{EE}}{2} - 3S^3_{EE}\right)$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} S^3_{EE} = -\frac{1}{2}\exp\left(-\frac{S^2_{EE}}{2} + 3S^3_{EE}\right) + \frac{1}{2}\exp\left(-\frac{S^2_{EE}}{2} - 3S^3_{EE}\right)$$

These equations are those of SL(3) Toda theory, suggesting we are looking at higher spin gravity in de Sitter space in conformal gauge?

Lesson:

For pure gravitational backgrounds found a closed equation for entanglement entropy.

For spin-2,3 backgrounds found closed equations for spin 2,3 entanglement entropy.

Suggests:

In backgrounds/states with several fields turned on, we can perhaps find a closed set of equations if we include an entanglement-ish variable for each field which is excited.

How to define these entanglement-ish quantities $S_\phi(x,y)$?

First define to first order for perturbations around the ground state. To leading order all quantities $S_\phi(x,y)$ vanish unless ϕ corresponds to the metric.

Proposal: for scalar operators

$$\delta S_\phi(x,y) = C \int_{D(x,y)} d^d\xi \left(\frac{|x-\xi||\xi-y|}{|x-y|} \right)^{\Delta_{\mathcal{O}}-d} \langle \mathcal{O}(\xi) \rangle$$

$\mathcal{O}$: CFT operator dual to ϕ

where

$D(x,y)$: causal diamond bounded by x and y.

Features:

- It again obeys a Klein-Gordon equation on M.
- It is reminiscent of the first law of entanglement entropy, except that the integral is over a full causal diamond.
- The kernel that appears in the integral intertwines the SO(2,d) actions on AdS and M.
- It has an OPE interpretation

$$A(z)B(w) = \frac{C(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_C}} + \frac{\partial C(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_C-1}} + \dots$$

resum all derivatives and take A=B:

$$(z-w)^{2\Delta_A} A(z)A(w) = C' \int_{D(z,w)} d^d x \left(\frac{|z-x||x-w|}{|z-w|} \right)^{\Delta_c - d} C(x)$$

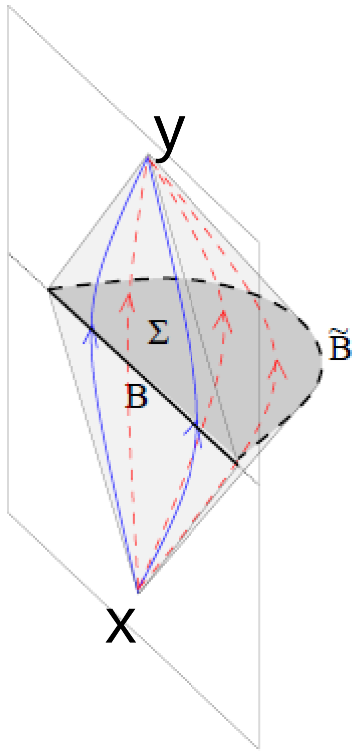
which has exactly the same form!

(kernel =3pt function with “shadow field” – Ferrara, Grillo, Parisi, Grotto, 1972)

Features (continued):

It has an interesting bulk interpretation in AdS/CFT:

$$\delta S(x,y) = C'' \int_{\tilde{B}} d^{d-1}u \;\; \phi(u)$$



Entanglement wedge reconstruction is complicated for local operators – for these Integrated operators the reconstruction is apparently much nicer/more natural.

Morrison 1403.3426
Freivogel, Jefferson, Kabir, 1602:04811

Similar related bulk quantities:

(i) the integral of $\phi(x)$ along a geodesic from x to y (now spatially separated)
(ii) the integral of $\phi(x)$ over various codimension minimal surfaces in Euclidean signature

Connection to OPE persists for geodesics. Explains geodesic Witten diagrams of Hijano, Kraus, Perlmutter, Snively 1508.00501. Conformal blocks related to propagator on M.

$$\langle A(x_1)B(y_1)C(x_2)D(y_2)\rangle_{\mathcal{O}-\text{channel}} \sim \langle \delta S_\phi(x_1,y_1)\delta S_\phi(x_2,y_2)\rangle \sim \langle \int_{\tilde{B}_1}\phi \int_{\tilde{B}_2}\phi\rangle$$

First law can still be written down but typically involves an integral over the entire boundary with murky convergence properties.

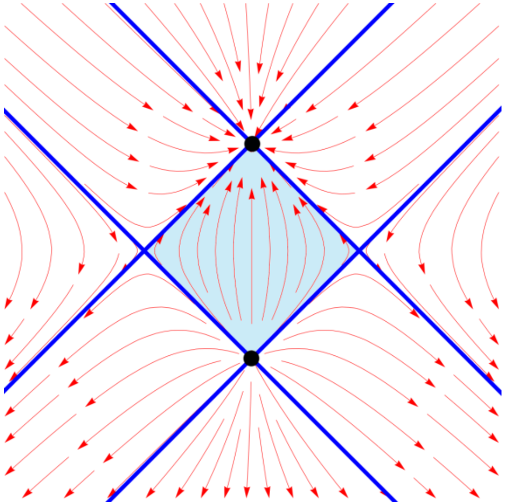
Everything follows essentially from conformal symmetry.

Generalization to higher spin fields (symmetric traceless tensors)

$$\delta S(x,y) \sim \int_{D(x,y)} d^d\xi \; |K|^{\Delta_{\mathcal{O}}-\ell-d}\, K^{\mu_1}\cdots K^{\mu_\ell}\, \langle \mathcal{O}_{\mu_1\ldots\mu_\ell}(\xi)\rangle \,.$$

where K^{μ} is the conformal Killing vector associated to the ball

$$K^{\mu} = \frac{(y-\xi)^{\mu}(x-\xi)^2 - (x-\xi)^{\mu}(y-\xi)^2}{(y-x)^2}$$



In the massless limit $\Delta_{\mathcal{O}} = \ell + d - 2$ and this reduces to the previous answer for conserved currents involving an integral over B(x,y) only.

Bulk dual less obvious: how to canonically integrate a tensor field over a minimal surface?

For (massless or massive) vector field A_μ

$$\delta S(x,y) = \int_{\tilde{B}} *F$$

Interactions?

For scalar fields with e.g. a ϕ^3 interaction we tried to include higher order corrections to $\delta S(x,y)$ to cancel right hand side in

$$(\nabla_a \nabla^a - m^2) S(x,y) = C \int_{\tilde{B}} \phi^2$$

and replace it by something like $(\left(\int_{\tilde{B}} \phi\right)^2$ to get local field equations on the space of causal diamonds.

This does not quite work.

This is perhaps related to the existence of constraints.

The functions $\delta S(x,y)$ are not completely unconstrained (up to the field equation). It is a function of 2d variables whereas the CFT one-point functions depend on only d variables.

In d=2 for example, for scalars on has

$$(\nabla_L^2 - \nabla_R^2)\phi = 0$$

in addition to the field equation.

We should therefore look for interacting theories for constrained fields. The Liouville example moreover shows that the constraints get modified at higher order.

For a scalar in d dimensions the constraints read as follows

$$L_{[AB}L_{CD]}\delta S(x,y) = 0$$

with L_{AB} the generators of the conformal group.

These arise because for a scalar operator $\mathcal{O}(x)$

$$L_{[AB}L_{CD]}\mathcal{O}(x) = 0$$

Not very well known but true in any CFT that for example

$$P_{[\mu}M_{\nu\rho]}\mathcal{O}(x) = 0$$

$$(2\{M_{\mu\nu}, D\} - \{P_\mu, K_\nu\} + \{P_\nu, K_\mu\})\mathcal{O}(x) = 0$$

The constraints

$$L_{[AB}L_{CD]}\delta S(x,y)=0$$

become a set of second order differential operators when written explicitly

$$F^{ABCD}_{\mu\nu}(x,y)\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}\delta S(x,y)=0$$

This is reminiscent of the section constraints of exceptional field theory. Would be interesting to explore further. To be continued..

Coimbra, Strickland-Constable, Waldram
1112.3989
Berman, Cederwall, Kleinschmidt, Thompson,
1208.5884

Interactions: gravity

What about gravity where a non-linear definition is available?

Metric on space of causal diamonds is Kähler-like.
Denote $\bar{x}^\mu = y^\mu$ then

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K$$

With Kähler potential

$$K = \log(-(x-\bar{x})^2)$$

It has constant positive scalar curvature.

Speculation: Field equation for S is constant scalar curvature for the modified Kähler potential

$$K = \log(-(x-\bar{x})^2) + (S - S_{\text{vac}})$$

At the linearized level this yields the right field equation.

In d=2 it reproduces Liouville field equations.

Does it hold in general gravitational backgrounds?

Other ideas to define $\delta S(x, y)$ beyond linear order?

- Use replica trick and generalization of twist fields as suggested by OPE? Modular Hamiltonian vs $\mathcal{O}$?
- Try to construct order by order in perturbation theory using constraints (what criteria to use?).
- Study non-linear corrections to the first law (what criteria to use?)
- Use structure of conformal blocks, OPE's, etc?
- Use map to hyperbolic black hole, relate to partition functions of black holes with scalar fields?

Constructive approach:

Beach, Lee, Rabideau, van Raamsdonk, 1604.05308

1. Compute perturbative corrections to ordinary entanglement entropy due to other fields.

2. Demand that these corrections arise from a local theory on M.

3. This should fix most of the theory on M, viewing entanglement entropy as being related to the conformal factor of a metric on M.

4. Finally, we construct $S_\phi(x,y)$ perturbatively so that it agrees with the local theory on M.

Summary/open problems

- Found evidence for local interacting theories that describe the evolution of various entanglement-like quantities as one changes scale. *What is the fundamental meaning of this?*
- Generalization to arbitrary fields (charged fields, fermions, …)?
- Right way to think about interactions? Generalized twist fields?
- Not clear why this works? Is this a fundamental property of arbitrary CFT's, or only those with a weakly coupled gravitational dual?
- Is this a natural basis for QG observables?
- Does this shed light on a possible holographic dual description of de Sitter space?
- What is the meaning of the space M?
- Relation to OPE/conformal blocks/CFT data?