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Mutual information of two disjoint intervals in Tomonaga-Luttinger liquids: bosons vs. fermions

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Outline

- Mutual information of two disjoint intervals in Tomonaga-Luttinger liquids: case of bosons

  S. F., V. Pasquier, and J. Shiraishi,
  Phys. Rev. Lett. **102**, 170602 (2009)

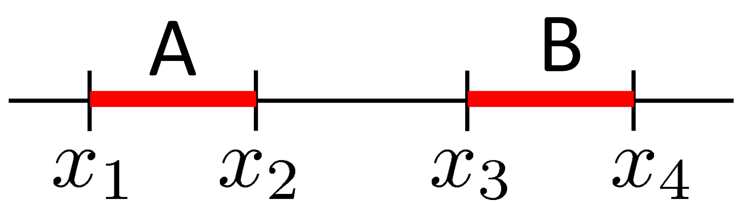
Correspondence:
Jordan-Wigner trans.
or bosonization

Non-local!

Different Mutual info.

- Case of interacting fermions

  Discussions with H. Katsura, A. Furusaki, S. Ryu, ...



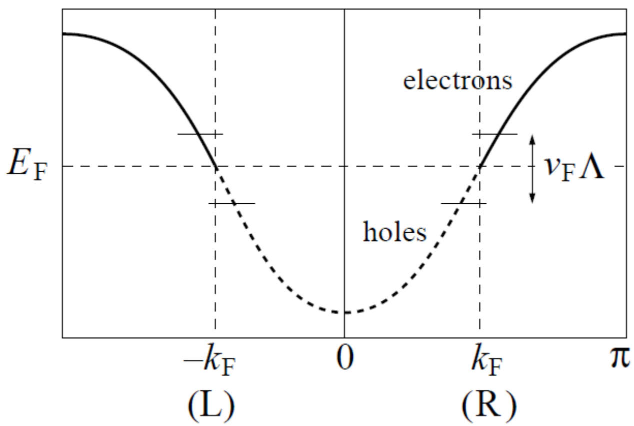
$$I_{A:B} \equiv S_A + S_B - S_{A\cup B}$$

Introduction

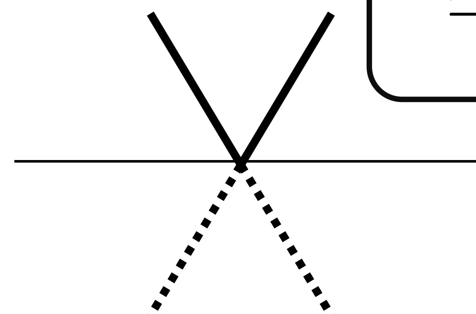
Wide variety of spatially 1D quantum critical (gapless) systems
⇔ Conformal field theory (CFT)

•1D conductor (e.g., Carbon nanotube)

c: central charge
$\simeq$ number of gapless modes



⇨



massless Dirac fermions (c=1)

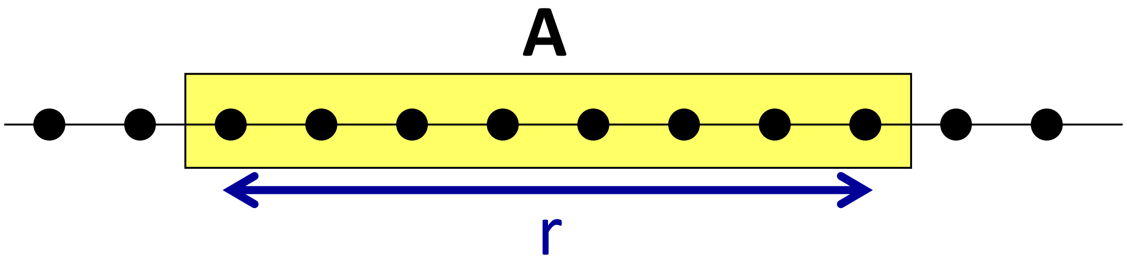
•spin-1/2 XXZ chain ⇨ Interacting Dirac fermions (c=1)

[Jordan-Wigner transformation]

•Ising chain in transverse field (critical point) ⇨ Majorana (c=1/2)

Q. Given a microscopic model, how can we address the information of underlying CFT?

Entanglement entropy (EE) in 1D



Holzhey, Larsen, & Wilczek, Nucl.Phys.B, 1994
Vidal, Latorre, Rico, & Kitaev, PRL, 2003
Calabrese & Cardy, J.Stat.Mech, 2004

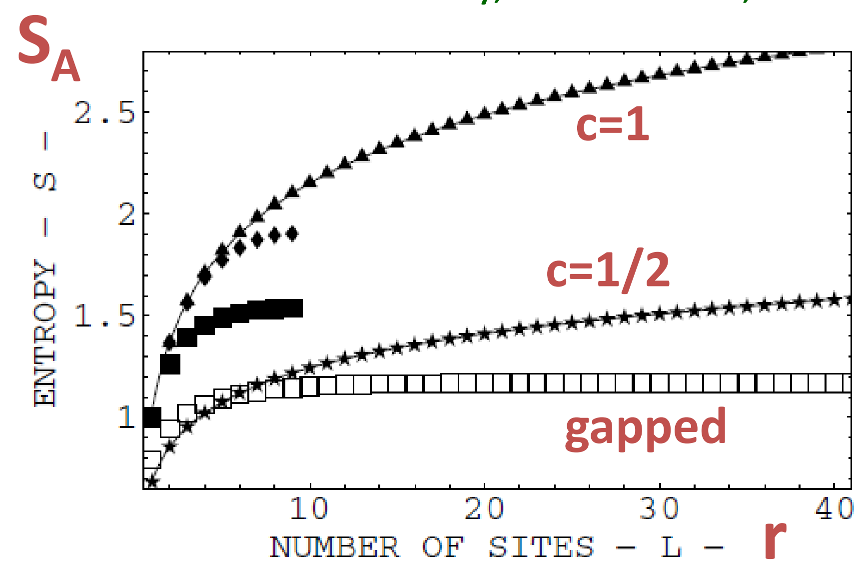
- Gapped (non-critical) system

$$S_A \to \text{const.}\ (r \to \infty)$$

- Critical system

$$S_A \simeq \frac{c}{3} \log r + s_1$$

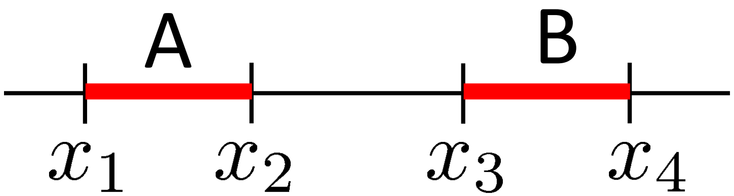
non-universal



Has become a standard tool in DMRG analyses of 1D systems (spin Bose metal with c=3: D. N. Sheng et al., PRB, 2009)

Q. How can we obtain more detailed info of CFT?

Ans.: Use two intervals. Calculate the mutual information.



$$I_{A:B} \equiv S_A + S_B - S_{A\cup B}$$

Information of CFT
beyond the central charge
(or "operator content")

- Original suggestion H. Casini and M. Huerta, Phys. Lett. B 600, 142 (2004)
- Calculations in Tomonaga-Luttinger liquids (TLL) with c=1

S. F., V. Pasquier, and J. Shiraishi,
Phys. Rev. Lett. 102, 170602 (2009)

Numerical and half-analytical calculations

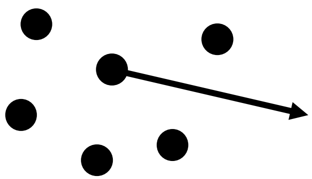
P. Calabrese, J. Cardy, and E. Tonni,
J. Stat. Mech (2009), P11001;
J. Stat. Mech.(2011), P01021.

Full analytical calculation

Mutual information is directly related to the TLL parameter K.

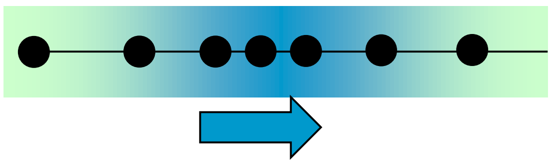
(Critical exponent in correlation functions: $\eta = 1/2K$)

Description of interacting particles



- Interacting fermions in D>1: Fermi liquid

  single-particle picture
  renormalized dispersion relation; finite lifetime



compressional wave

- Interacting fermions/bosons/spins in D=1:
  Tomonaga-Luttinger liquid (TLL)

  Density fluctuations
  $\longrightarrow$ bosonic description

$$\psi^\dagger(x) = \underbrace{[\rho(x)]^{1/2}}_{\text{density}} \underbrace{e^{-i\sqrt{\pi}\theta(x)}}_{\text{phase}}$$

Phenomenological bosonization

Haldane, 1981

$$\psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i\sqrt{\pi}\theta(x)}$$

density phase

$$\phi(x) = \sqrt{\pi}\int^x dx' \, [\rho(x') - \rho_0]$$

$$[\partial_x\phi(x), \theta(x')] = i\delta(x - x')$$

- Hamiltonian

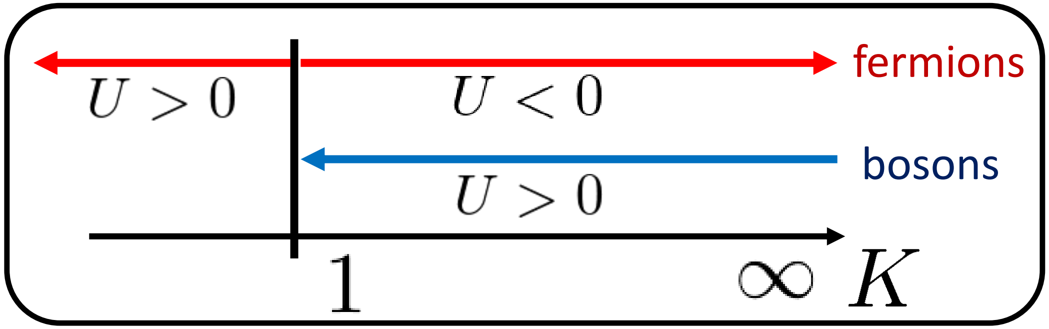
$$H = \int dx \frac{v}{2}\left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right]$$

v : velocity

K : TLL parameter

$$U\rho(x)^2 = \frac{U}{\pi}(\partial_x\phi)^2$$

Interaction



- Field redefinition

$$\Theta = \sqrt{K}\theta \qquad \Phi = \phi/\sqrt{K} \qquad [\partial_x\Phi(x), \Theta(x')] = i\delta(x - x')$$

$$H = \int dx \frac{v}{2}\left[(\partial_x\Theta)^2 + (\partial_x\Phi)^2\right]$$

No parameter in the Hamiltonian ?? (except for v)

Boson compactification conditions

In fact, a characteristic parameter exists in the boson compactification conditions.

Case of TLL of bosons (PBC on a ring of length L)

- Phase winding $\Theta(L) - \Theta(0) = \sqrt{\frac{K}{\pi}} \cdot 2\pi M = 2\pi \tilde{R} M$

- Excess number of particles (relative to GS) $M, \Delta N \in \mathbb{Z}$

$$\Phi(L) - \Phi(0) = \sqrt{\frac{\pi}{K}} \int_0^L dx' [\rho(x') - \rho_0] = \sqrt{\frac{\pi}{K}} \Delta N = 2\pi R \Delta N$$

$$\Phi \equiv \Phi + 2\pi R$$

$$\Theta \equiv \Theta + 2\pi \tilde{R}$$

Compactification radii:

$$R = \frac{1}{\sqrt{4\pi K}}, \quad \tilde{R} = \sqrt{\frac{K}{\pi}} \quad (R\tilde{R} = 1/2\pi)$$

Case of TLL of fermions (discussed later)

$$\Delta N = N_R + N_L \qquad M = \frac{N_R - N_L}{2} \qquad N_R, N_L \in \mathbb{Z}$$

Physical operators: vertex operators

$$\Phi \equiv \Phi + 2\pi R \qquad \Theta \equiv \Theta + 2\pi \tilde{R} \quad (R\tilde{R} = 1/2\pi)$$

- Vertex operators: $e^{in\Phi/R} \qquad e^{im\Theta/\tilde{R}}$ etc.

  n,m=integer

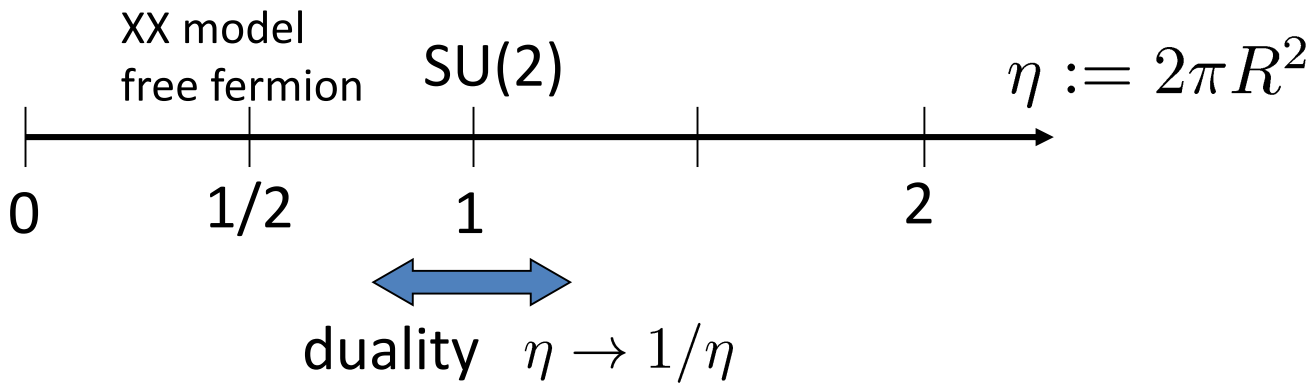
- Correlation functions:

$$\langle e^{in\Phi(x)/R} e^{-in\Phi(x')/R} \rangle = \frac{1}{|x-x'|^{n^2/(2\pi R^2)}} = \frac{1}{|x-x'|^{n^2/\eta}}$$

$$\langle e^{im\Theta(x)/\tilde{R}} e^{-im\Theta(x')/\tilde{R}} \rangle = \frac{1}{|x-x'|^{m^2/(2\pi \tilde{R}^2)}} = \frac{1}{|x-x'|^{m^2\eta}}$$

R or $\eta := 2\pi R^2$ controls power-law behavior of correlations.

Boson compactification radius R in XXZ model

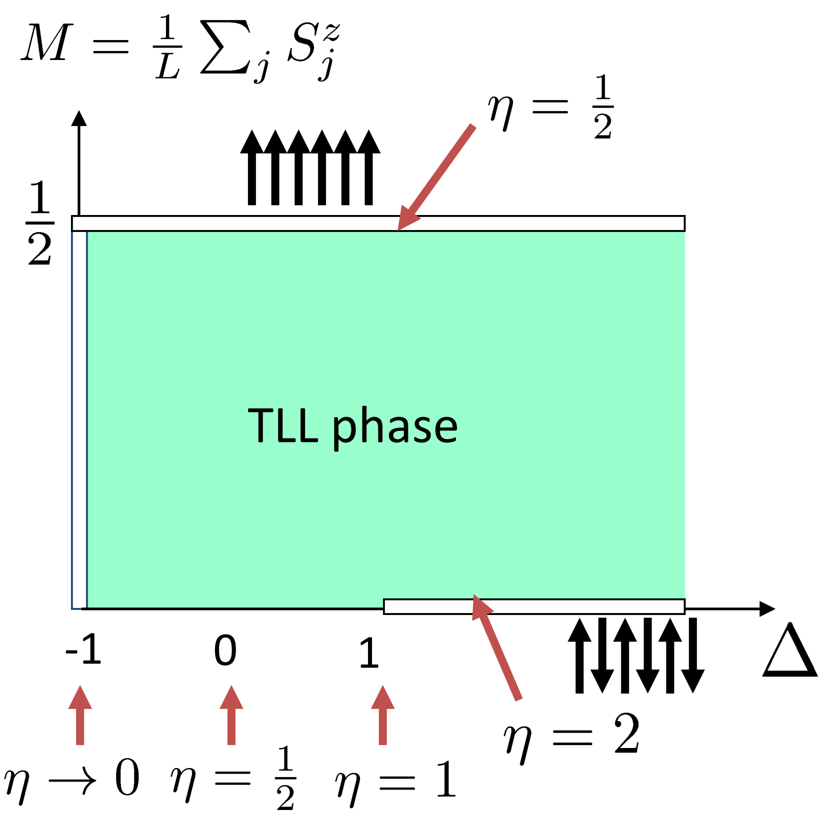


XXZ chain in a magnetic field

$$H = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) - h \sum_j S_j^z$$

Power-law decay of correlations:

$$\langle S_0^x S_r^x \rangle \sim \frac{(-1)^r}{r^\eta}, \quad \langle S_0^z S_r^z \rangle \sim \frac{(-1)^r}{r^{1/\eta}}$$



Mutual information

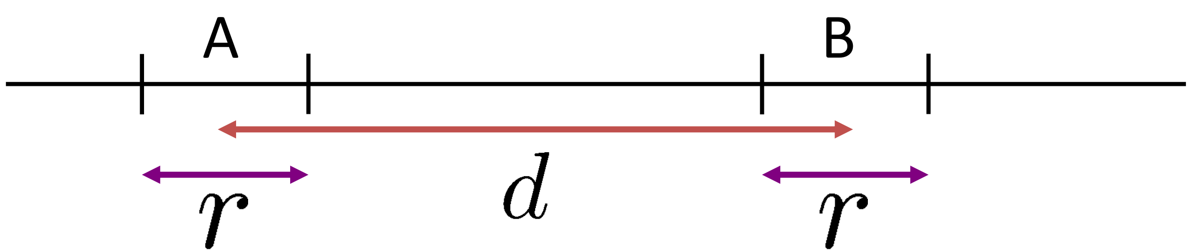
$$I_{A:B} := S_A + S_B - S_{A\cup B} \geq 0$$

Non-zero value signals the presence of a correlation:

$$\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle \neq 0 \text{ for some } O_A,\ O_B$$

Idea: Use this as a "region-region" correlation function!

Expected behavior in critical systems



- $r = \text{const.},\ d \to \infty : \quad I_{A:B} \longrightarrow 0$

  due to power-law decaying correlation

- $\dfrac{r}{d} = \text{const.},\ d \to \infty : \quad I_{A:B} \longrightarrow \text{nonzero}$

  What determines this value?

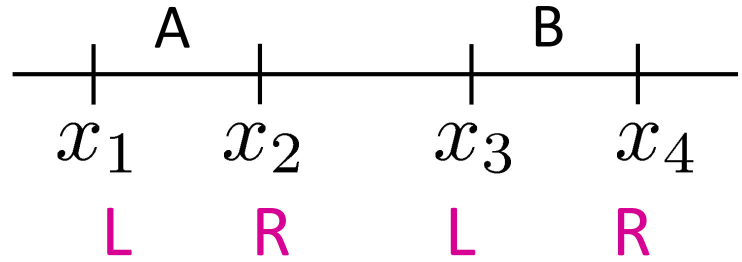
Calabrese-Cardy (initial) prediction

J.Stat.Mech, 2004

$$S_A = \frac{c}{3}\log(x_2 - x_1) + s_1$$

non-universal constant

Caution: This prediction is not always valid.



$$S_{A\cup B} = \frac{c}{3}\log\left(\frac{x_{21}x_{32}x_{43}x_{41}}{x_{31}x_{42}}\right) + 2s_1$$

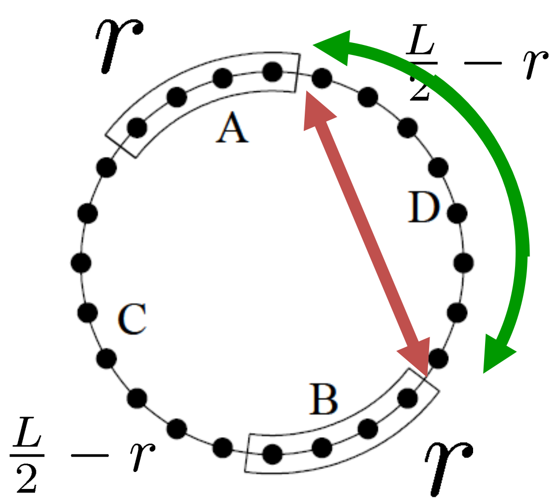
$$x_{ij} = x_i - x_j$$

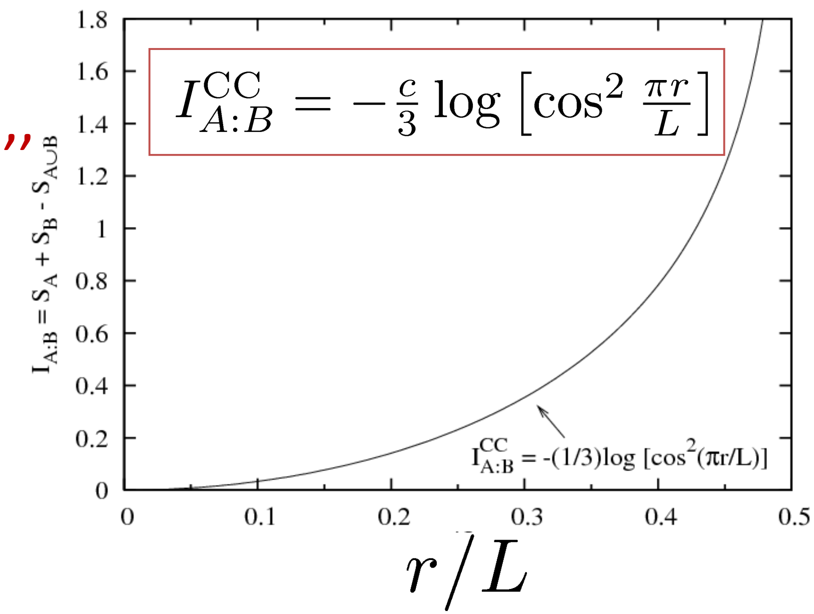
$$I^{\mathrm{CC}}_{A:B} = \frac{c}{3}\log\left(\frac{x_{31}x_{42}}{x_{32}x_{41}}\right)$$

Non-universal constants are canceled!
Invariant under global scale transformations.

finite chain of length L: $x_{ij} \longrightarrow \frac{L}{\pi}\sin\frac{\pi x_{ij}}{L}$

``chord distance''



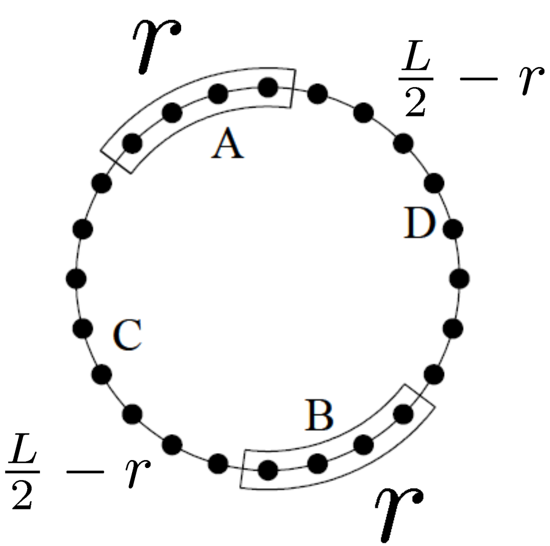


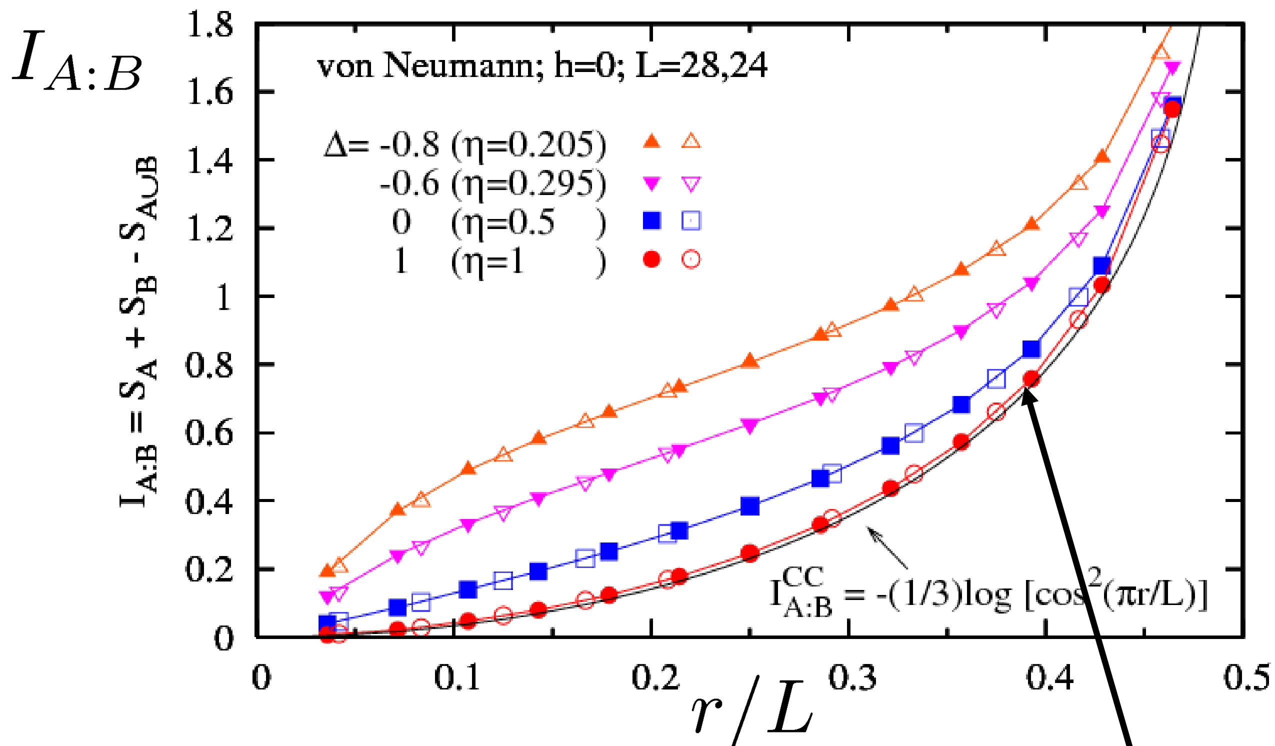
Numerical analysis

XXZ model: $$H = \sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z)$$

$-1 < \Delta \leq 1$: c=1 TLL phase

Exact diag., L=24, 28



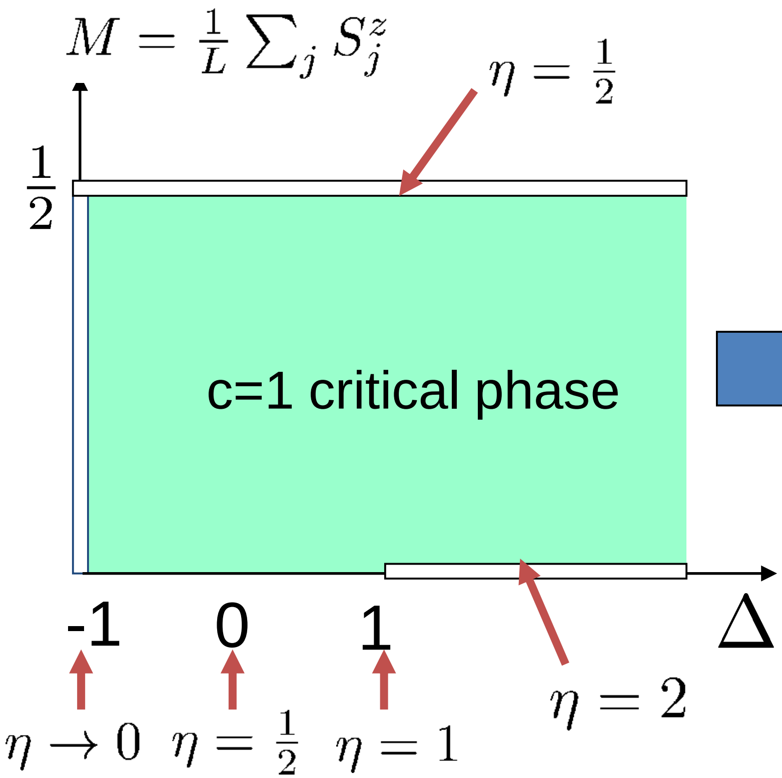


black line:
CC result

Deviation from CC result
when we go away from SU(2) point

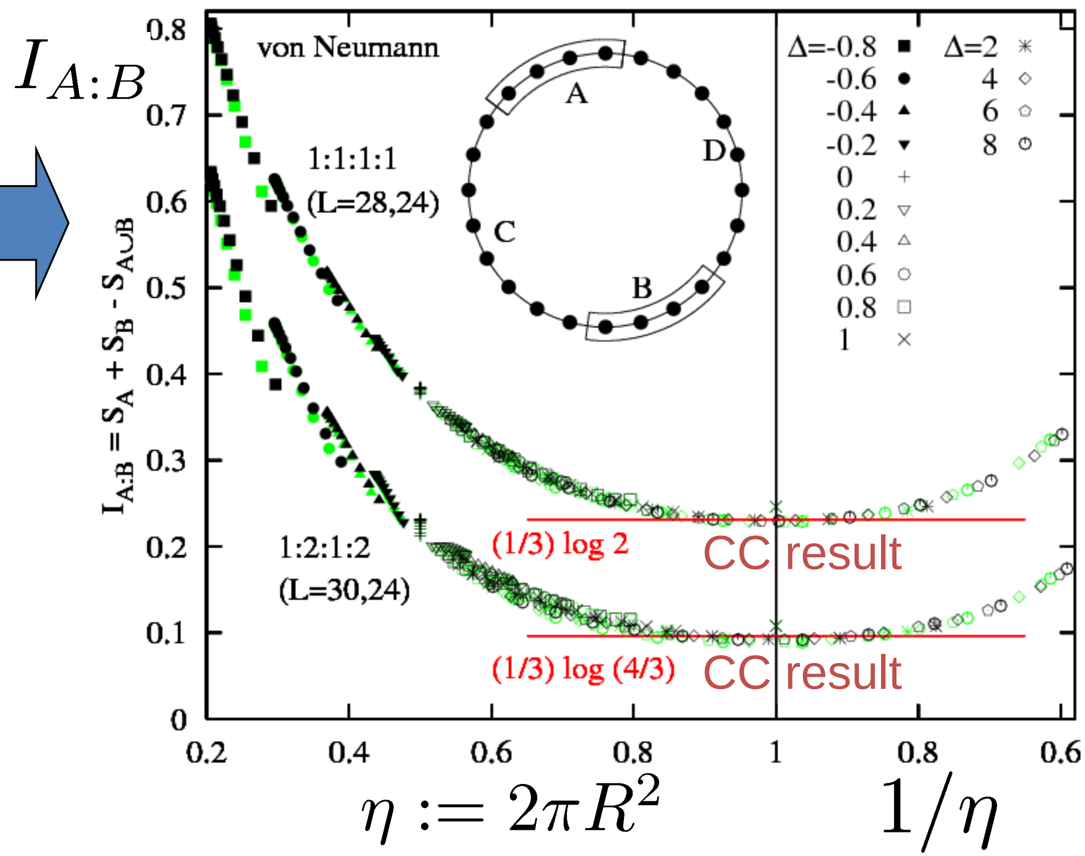
Dependence on the exponent η

$$H=\sum_{j=1}^{L}(S_j^xS_{j+1}^x+S_j^yS_{j+1}^y+\Delta S_j^zS_{j+1}^z)-h\sum_{j=1}^{L}S_j^z$$



fixed divisions

$r_A{:}r_C{:}r_B{:}r_D$ =1:1:1:1, 1:2:1:2



Direct relation between $I_{A:B}$ and η

CFT calculation of entanglement entropy

We now try to understand the deviation from CC result.

We first follow the CFT calculation in the single-interval case.

Calabrese & Cardy, J.Stat.Phys., 2004

Cardy,Castro-Alvaredo,& Doyon, J.Stat.Phys., 2007

Then we consider how to extend it to the two-interval case.

<u>Starting point</u>: Replica trick

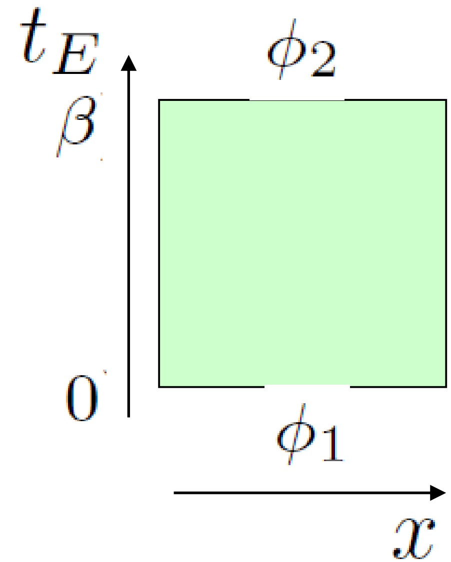
$$S_A = \lim_{n\to 1} \underbrace{\frac{-1}{n-1} \log(\mathrm{Tr}\ \rho_A^n)}_{\text{Renyi entropy } S_A^{(n)}}$$

We compute $\mathrm{Tr}\ \rho_A^n$ for integer n>1.

Then we take an analytic continuation n->1.

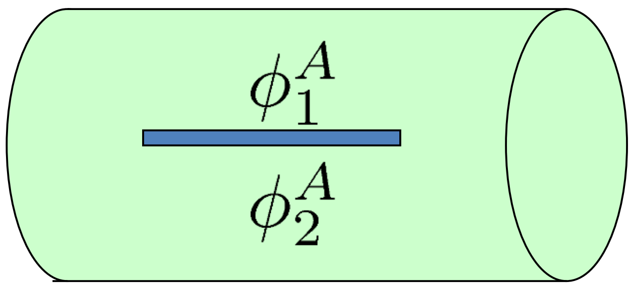
Path integral representation

$$\rho = \frac{1}{Z} e^{-\beta H}$$ finite-temperature total density matrix



$$\langle \{\phi_1(x)\} | \rho | \{\phi_2(x)\} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \exp\left[- \int dx dt_E \mathcal{L}[\phi] \right]$$

$$\langle \phi_1^A | \rho_A | \phi_2^A \rangle = \frac{1}{Z}$$

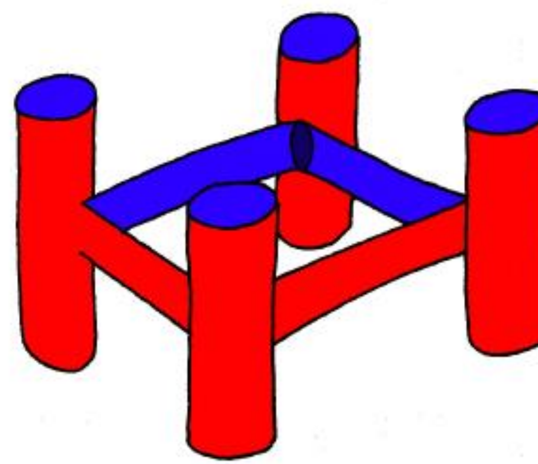


$$\mathrm{Tr}\, \rho_A^n = \frac{Z_{\mathcal{R}_n(A)}}{Z^n}$$

$\mathcal{R}_n(A)$

n-sheeted

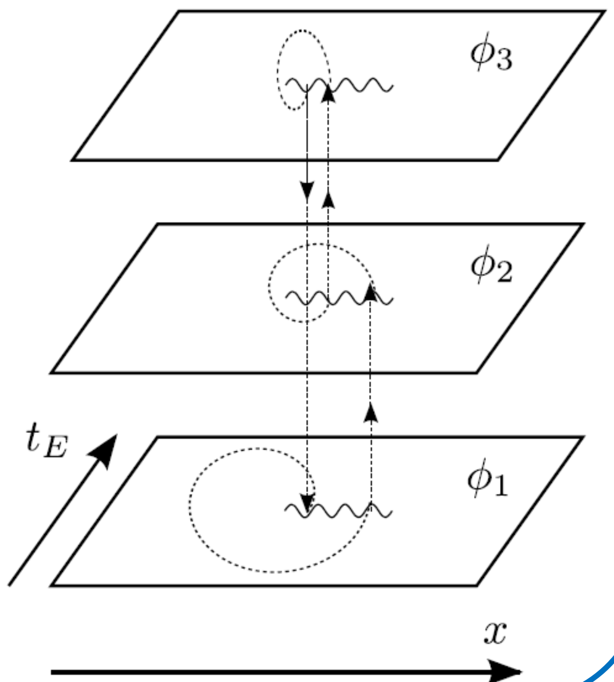
Riemann surface



Calabrese & Cardy, J.Stat.Phys.,2004

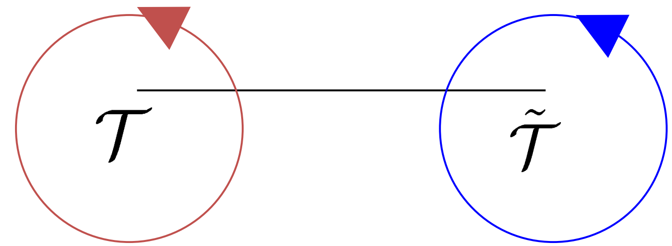
Branch-point twist field

$\phi(z|k)$ on $\mathcal{R}_n(A)$



n-component field on $\mathbb{C}$

$\Phi(z) = {}^t(\phi(z|1), \ldots, \phi(z|n))$



$\Phi(z) \to \sigma\Phi(z)$ $\quad$ $\Phi(z) \to \sigma^{-1}\Phi(z)$

$$\sigma = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}$$

cyclic permutation

$$\mathcal{L}_{\mathrm{tot}}[\Phi(x, t_E)] = \sum_k \mathcal{L}[\phi(x, t_E|k)]$$

$$\mathrm{Tr}\, \rho_A^n = \frac{Z_{\mathcal{R}_n(A)}}{Z^n}$$

$$= \frac{\int \mathcal{D}\Phi\ \mathcal{T}(x_1, 0)\tilde{\mathcal{T}}(x_2, 0) e^{-S_{\mathrm{tot}}[\Phi]}}{\int \mathcal{D}\Phi\ e^{-S_{\mathrm{tot}}[\Phi]}} = \langle \mathcal{T}(x_1, 0)\tilde{\mathcal{T}}(x_2, 0)\rangle_{\mathbb{C}}$$

Cardy,Castro-Alvaredo,& Doyon, J.Stat.Phys.,2007

Entanglement entropy as two-point correlation functions

$$\mathrm{Tr}\ \rho_A^n = \frac{Z_{\mathcal{R}_n}}{Z^n} \propto \langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle = x_{21}^{-2\Delta_n}\bar{x}_{21}^{-2\bar{\Delta}_n}$$

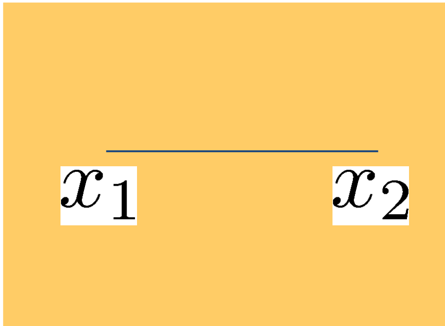
$$x_i = \bar{x}_i$$
$$x_{ij} = x_i - x_j$$

conformal dimensions: $\Delta_n = \bar{\Delta}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$

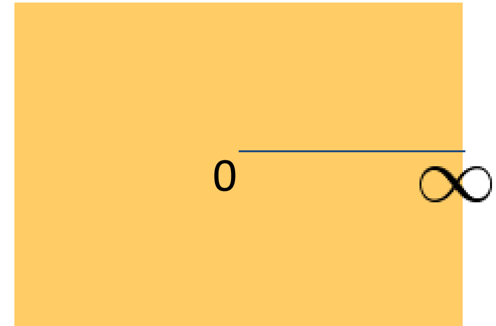
$$S_A^{(n)} = \frac{-1}{n-1}\log(\mathrm{Tr}\ \rho_A^n) = \frac{1+n}{6n} c \log x_{21} + s_n$$

$$n \rightarrow 1 \qquad S_A = \frac{c}{3}\log x_{21} + s_1$$

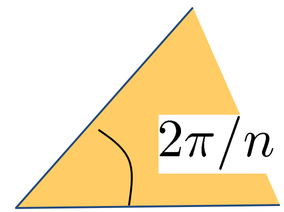
Why is the single-interval case so simple?



$$\zeta = \frac{z - x_1}{z - x_2}$$

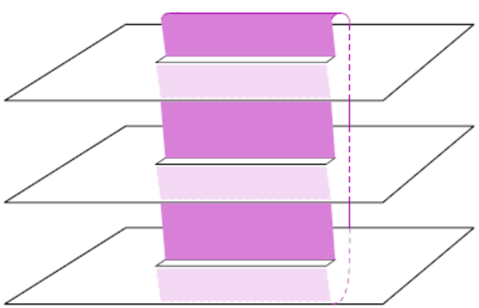


$$w = \zeta^{1/n}$$

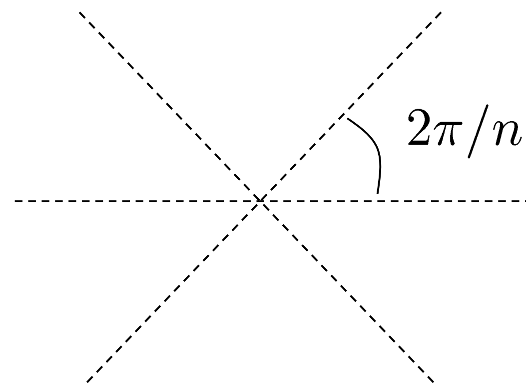


$w \in \mathbb{C}$

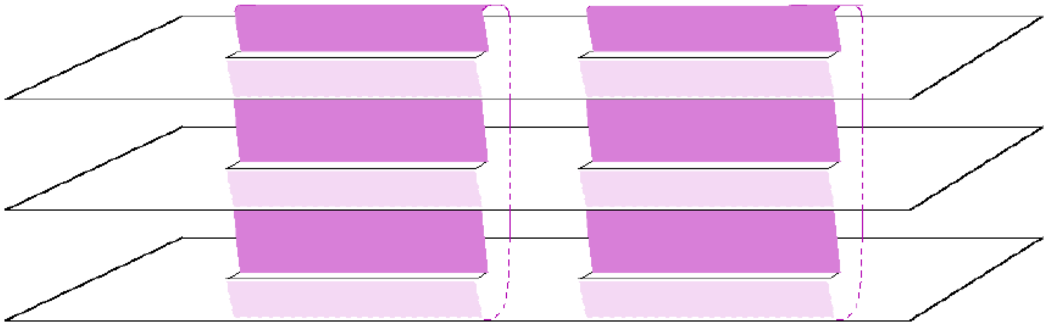
$z \in \mathcal{R}_n(A)$



$$w = \left(\frac{z - x_1}{z - x_2}\right)^{1/n}$$



Two-interval case: 4-point function



$$\mathcal{T}(x_1) \quad \tilde{\mathcal{T}}(x_2) \quad \mathcal{T}(x_3) \quad \tilde{\mathcal{T}}(x_4)$$

Consider $w = \frac{x_{21}}{x_{42}} \frac{x_4 - z}{z - x_1}$ $: (x_1, x_2, x_3, x_4) \to (\infty, 1, x, 0)$

cross ratio

$$x = \frac{x_{21} x_{43}}{x_{31} x_{42}}$$

$$\mathrm{Tr}\, \rho^n_{A\cup B} \propto \langle \mathcal{T}(x_1) \tilde{\mathcal{T}}(x_2) \mathcal{T}(x_3) \tilde{\mathcal{T}}(x_4) \rangle$$

$$\propto \left(\frac{x_{31} x_{42}}{x_{21} x_{32} x_{43} x_{41}} \right)^{2\Delta_n} \left(\frac{\bar{x}_{31} \bar{x}_{42}}{\bar{x}_{21} \bar{x}_{32} \bar{x}_{43} \bar{x}_{41}} \right)^{2\bar{\Delta}_n} F_n(x, \bar{x}; \eta)$$

$$S_{A\cup B} = \underbrace{\frac{c}{3} \log \left(\frac{x_{21} x_{32} x_{43} x_{41}}{x_{31} x_{42}} \right) + 2 s_1}_{\text{Calabrese-Cardy result}} \underbrace{- \lim_{n\to 1} \frac{1}{n-1} \log F_n(x, x; \eta)}_{\text{new part}}$$

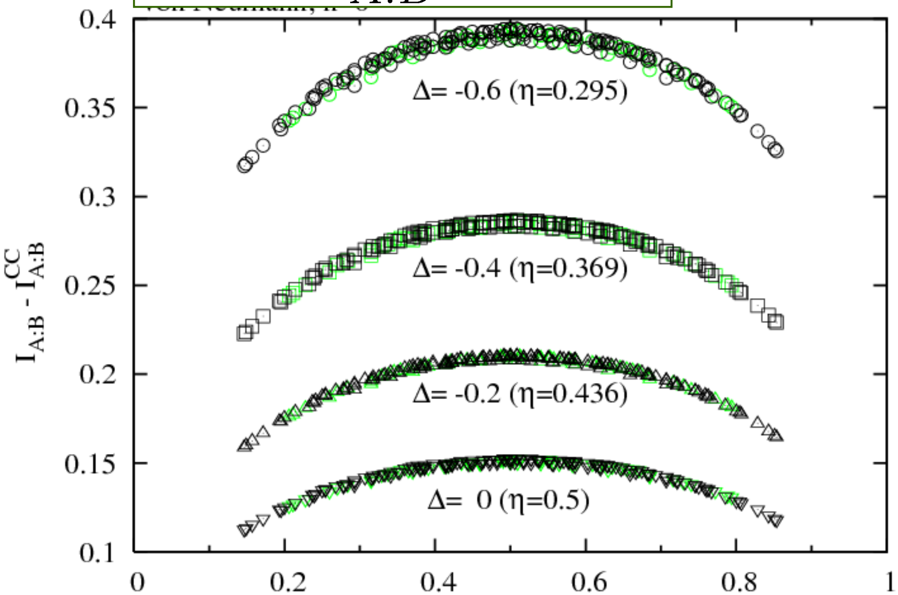
Cross-ratio-dependent part

$$I_{A:B} = \underbrace{I_{A:B}^{\mathrm{CC}}}_{\text{Calabrese-Cardy result}} + \underbrace{\lim_{n\to 1} \frac{1}{n-1} \log F_n(x, x; \eta)}_{\text{new part}}$$

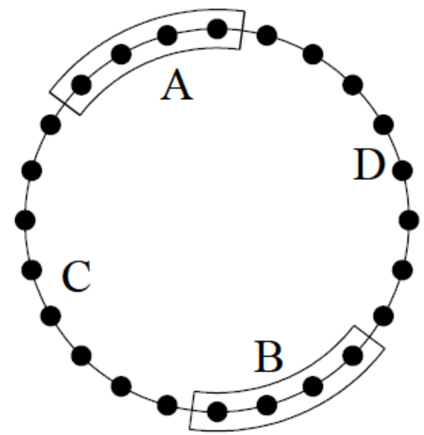
cross ratio

$$x = \frac{x_{21}x_{43}}{x_{31}x_{42}}$$

$I_{A:B} - I_{A:B}^{\mathrm{CC}}$ versus x

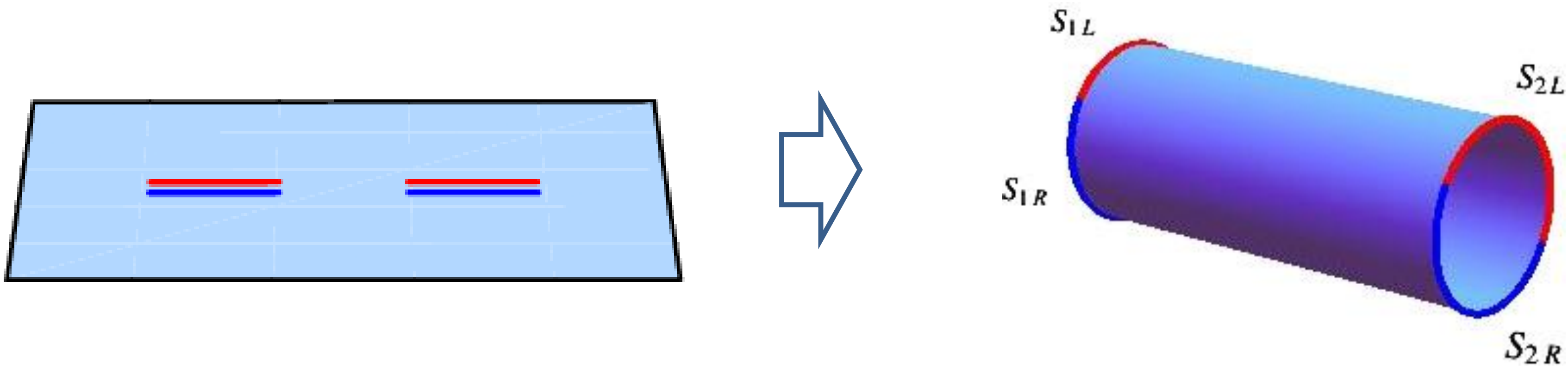


$$x = \frac{\sin\frac{\pi r_A}{L}\sin\frac{\pi r_B}{L}}{\sin\frac{\pi(r_A+r_C)}{L}\sin\frac{\pi(r_C+r_B)}{L}}$$

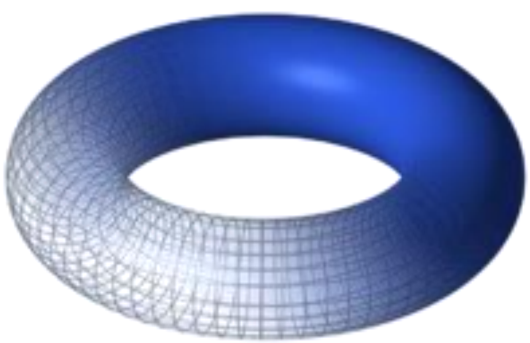


How to calculate F_n?

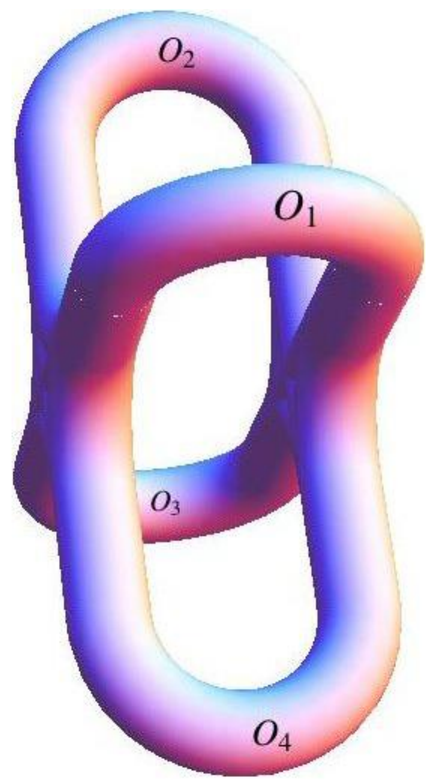
Non-trivial topologies



n=2
genus=1



n=4
genus=3



In general,
genus=n-1

n=2 case

Renyi mutual information: $I_{A:B}^{(n)} = I_{A:B}^{CC(n)} + \frac{1}{n-1} \ln F_n(x)$

- General result:

$$F_2(x) = [x(1-x)]^{c/6} Z_{\text{torus}}(\tau, -\tau)/\mathcal{N} \qquad x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

$$\lim_{x \to 0} F_2(x) = 1$$

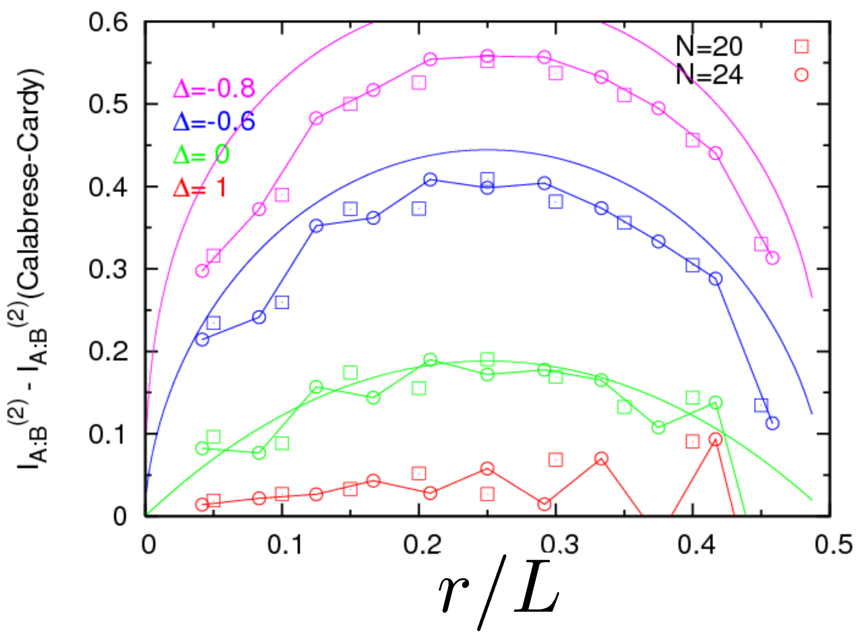
- Case of compactified boson

$$F_2(x) = \frac{\theta_3(\eta\tau)\theta_3(\tau/\eta)}{[\theta_3(\tau)]^2}$$

Coincides with the old results for Z2 twist field correlations

Al.B. Zamolodchikov,
Sov.Phys.JETP, 1986;
Nucl.Phys.B, 1987
Dixon,Friedan,Martinec,Shenker,
Nucl.Phys.B, 1987

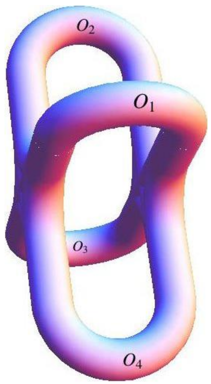
Agrees relatively well with numerics, but numerics show oscillations



Full analytical results for n>1

Calabrese, Cardy, Tonni, J. Stat. Mech (2009) P11001

Calculation of the partition fn.
for a surface of genus n-1



n=4
genus=3

$$F_n(x) = \frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2}$$

Θ :Riemann-Siegel theta function

Γ: $n \times n$ matrix produced from β_y

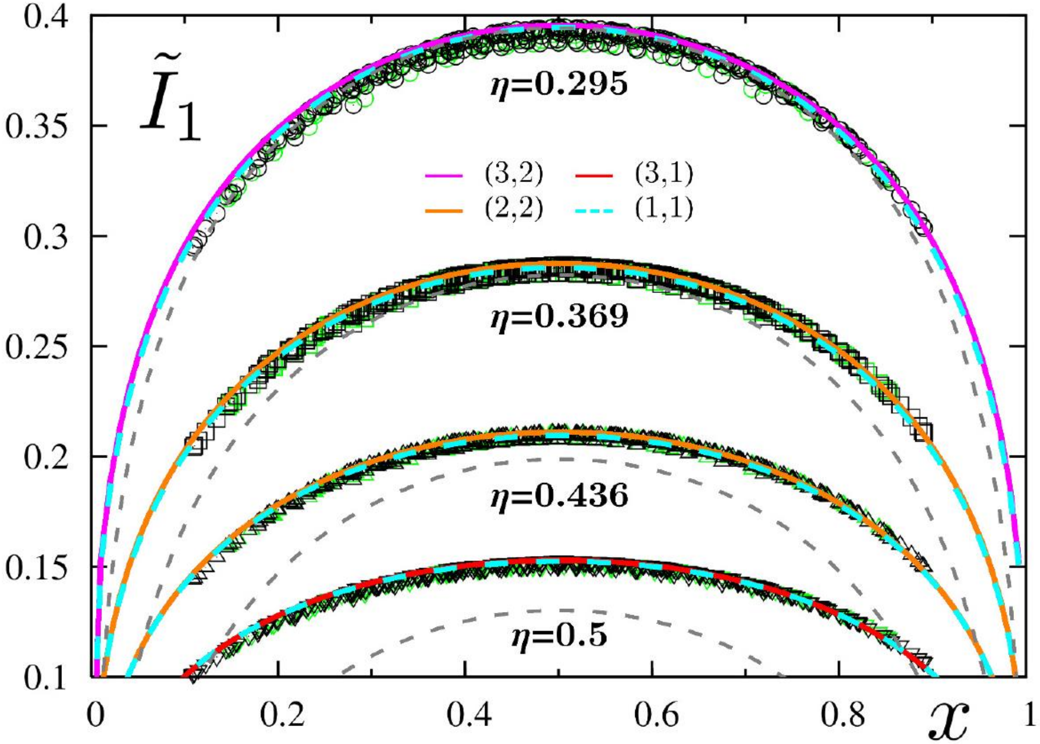
$$\beta_y = \frac{F_y(1-x)}{F_y(x)} \qquad F_y(x) \equiv {}_2F_1(y, 1-y; 1; x)$$

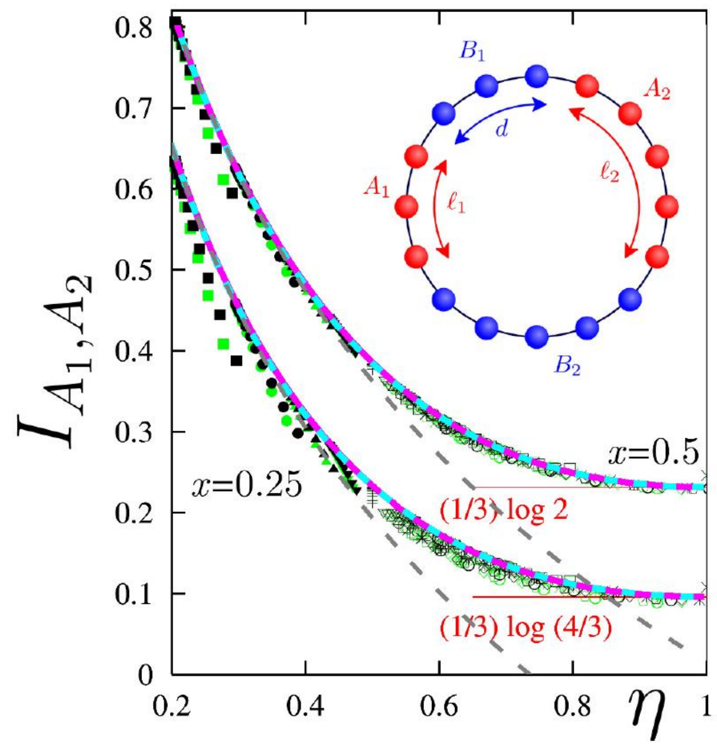
hypergeometric fn.

How to perform the analytical continuation n->1?

Numerical extrapolation n->1

De Nobili, Coser, and Tonni, J. Stat. Mech. (2015) P06021





Excellent agreement with exact diag. results!

What can we do with this result ?

- New method for determining R using the GS wave fn.

Mutual information $\longleftrightarrow$ Compactification radius R

Direct relation

- General critical systems in 1D

$$I_{A:B} - I_{A:B}^{\mathrm{CC}} = \lim_{n\to 1} \frac{1}{n-1} \log F_n(x,x) =: f(x)$$

fingerprints of different CFTs

<u>Calculations in different CFTs</u>

•Ising criticality (c=1/2) Alba, Tagliacozzo, Calabrese, PRB, 2010

•Z_2 orbifold (c=1) Alba, Tagliacozzo, & Calabrese, J. Stat. Mech. 2011
Calabrese, Cardy, & Tonni, J. Stat. Mech., 2011

•AdS / CFT Headrick, PRD, 2010

Question from many researchers

S. Ryu, H. Casini, M. Huerta, V. E. Korepin, H. Katsura, ...

- Field-theoretical calculation for the free fermion case

  H. Casini, C. D. Fosco, and M. Huerta, J. Stat. Mech., 2005

  Simple expressions of twist operators (in terms of vertex ops.) are available.

  $$I_{A:B} = I^{CC}_{A:B}$$

  Some researchers have also checked this numerically with correlation matrix method.

  Why doesn't the result in the XY case agree with this?

- Ans.: Jordan-Wigner transformation relating the two models are non-local!

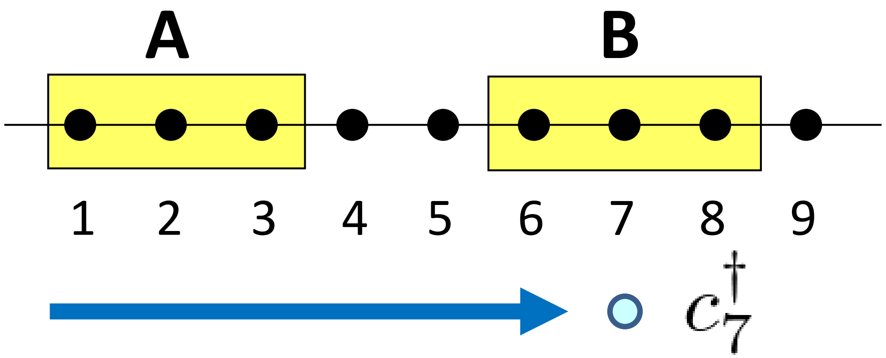
  $$S_j^+ = c_j^\dagger \exp(i\pi \sum_{i<j} n_i)$$

  Equivalence of the two models does not apply to the two-interval EE.

  Then, what is the behavior of mutual information in interacting fermions?

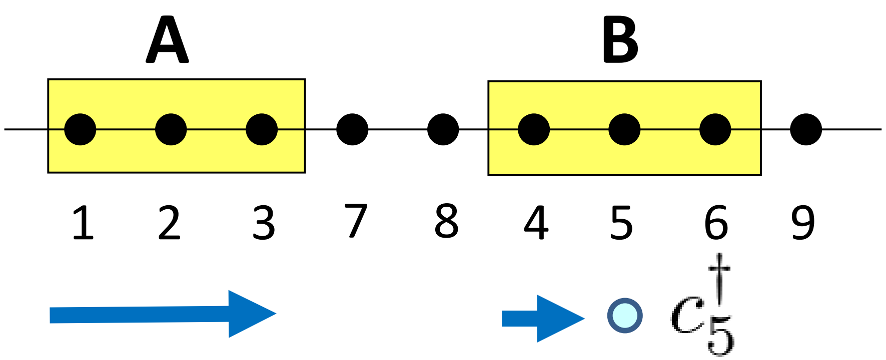
Density matrices for fermions

- Usual choice of the Fock basis



$$c_j^\dagger|\{n_i\}\rangle = \underbrace{(-1)^{\sum_{i<j} n_i}}_{\text{fermionic sign}} n_j|.., n_j+1, ..\rangle$$

- New basis

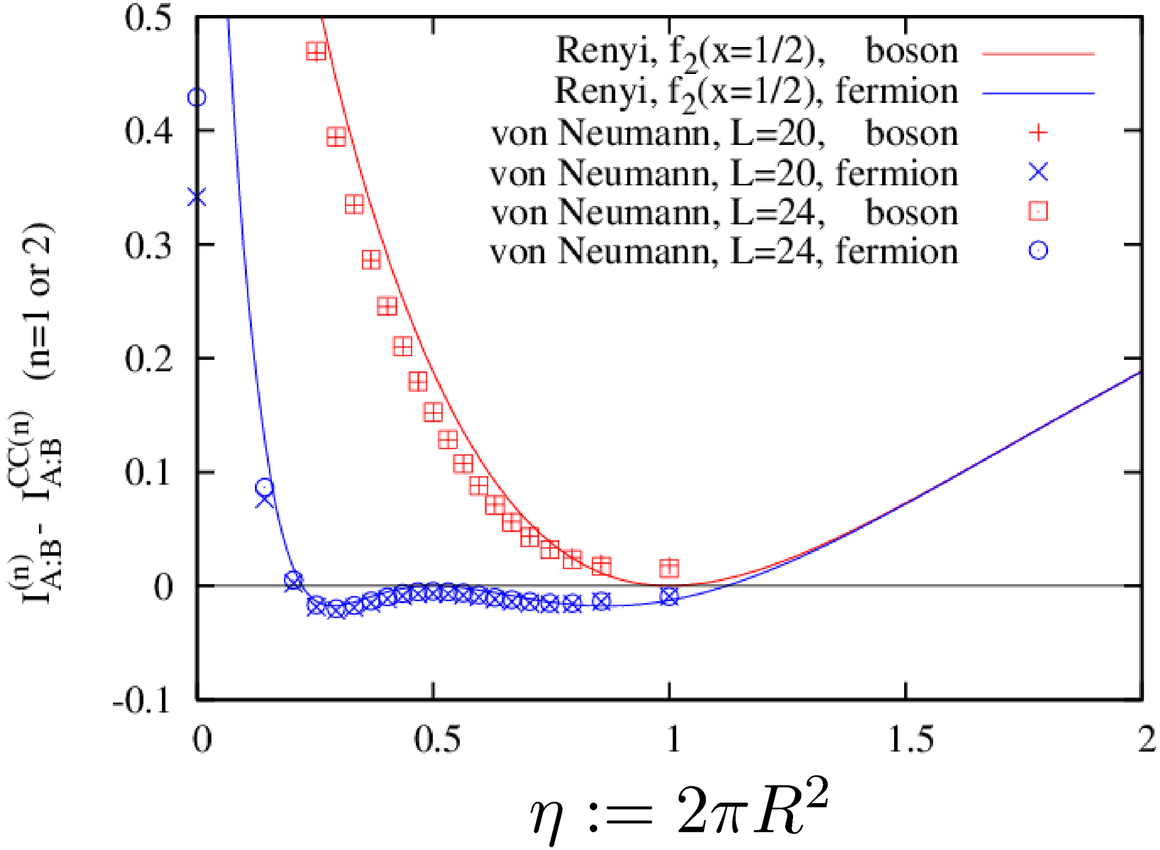


Fermionic operators act only on A U B.

Use this basis in defining the density matrix on A U B.

Related and equivalent definition:
S.-A. Cheong and C. L. Henley, Phys. Rev. B 74, 165121 (2006).

Numerical result for interacting fermions



XXZ chain

t-V model

Analytical curves: explained later

- ✓ Different curves for bosons (spins) and fermions
- ✓ CC result corresponds to $\eta = 1/2$ (free Dirac fermion).
- ✓ Non-monotonic dependence, negative values for some eta

Field-theoretical understanding

- General CFT formula for n=2 Renyi case

$$F_2(x) = [x(1-x)]^{c/6} Z_{\text{torus}}(\tau, -\tau)/\mathcal{N}$$

$$\lim_{x\to 0} F_2(x) = 1 \qquad x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

It is sufficient to calculate the torus partition fn!

- Interacting fermion on a torus with <u>antiperiodic</u> BCs

Matsubara formalism & modular inv.

Bosonization formulae

$$\psi_R^\dagger(x) = \frac{1}{\sqrt{2\pi}} e^{-i\sqrt{4\pi}\phi_R(x)} \qquad \psi_L^\dagger(x) = \frac{1}{\sqrt{2\pi}} e^{+i\sqrt{4\pi}\phi_L(x)}$$

$$\psi_{R/L}(L) = -\psi_{R/L}(0)$$ (Antiperiodic BCs)

$$\Longleftrightarrow \quad \exp\left[i\sqrt{4\pi}(\phi_{R/L}(L) - \phi_{R/L}(0))\right] = 1$$

Minus sign disappears!
Subtle effect of point splitting.

$$\Longleftrightarrow \quad \phi_{R/L}(L) - \phi_{R/L}(0) = \frac{1}{\sqrt{4\pi}} N_{R/L} \qquad N_R, N_L \in \mathbb{Z}$$

Field-theoretical understanding (cont'd)

$$\Phi(L) - \Phi(0) = 2\pi R \Delta N \qquad \Delta N = N_R + N_L$$

$$\Theta(L) - \Theta(0) = 2\pi \tilde{R} M \qquad M = \frac{N_R - N_L}{2} \qquad N_R, N_L \in \mathbb{Z}$$

$$\boxed{\begin{aligned} \Phi(x) &= [\phi_R(x) + \phi_L(x)]/\sqrt{K} \\ \Theta(x) &= \sqrt{K}[\phi_R(x) - \phi_L(x)] \end{aligned}}$$

"Twisted structure"

Wong & Affleck, Nucl. Phys. 1994

Oshikawa et al., J. Sta. Mech, 2006

$$Z_{\text{torus}}(\tau, -\tau) = \frac{1}{|\eta_D(\tau)|^2} \sum_{N_R, N_L \in \mathbb{Z}} q^{\frac{\eta}{2}\Delta N^2 + \frac{1}{2\eta} M^2} \qquad q = e^{2i\pi\tau}$$

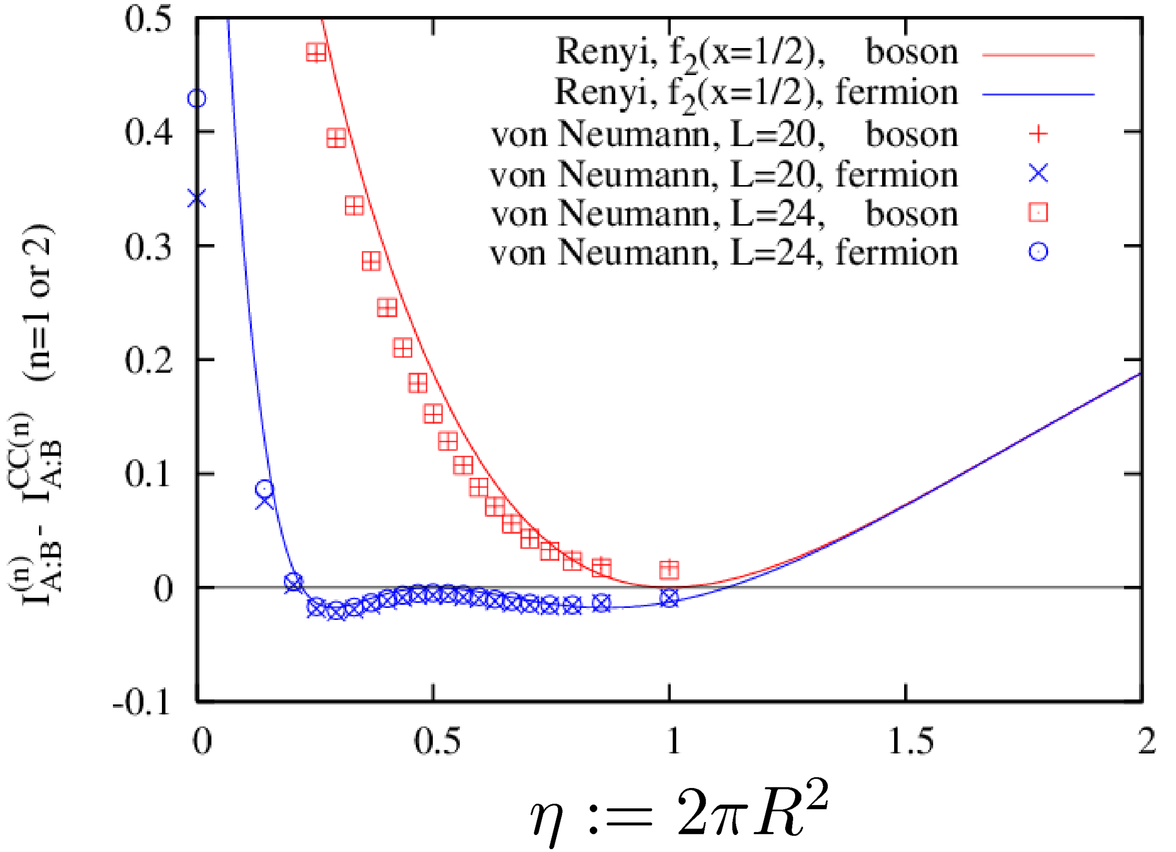
$\eta_D(\tau)$: Dedekind's η fn.

$$= \frac{\theta_3(4\eta\tau)\theta_3(\tau/\eta) + \theta_2(4\eta\tau)\theta_2(\tau/\eta)}{[\theta_3(\tau)]^2}$$

Related argument for free fermion but for general integer n>1:

M. Headrick, A. Lawrence,and M. M. Roberts, J. Stat. Mech. (2013) P02022

Analytical (n=2) vs. numerical (n=1)

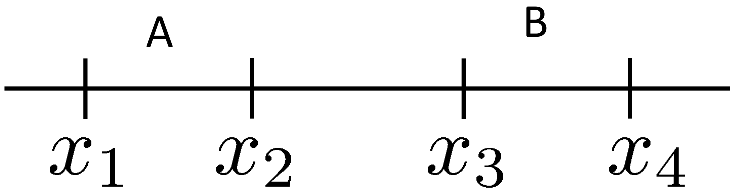


XXZ chain

t-V model

- ✓ Different curves for bosons (spins) and fermions
- ✓ CC result corresponds to $\eta = 1/2$ (free Dirac fermion).
- ✓ Non-monotonic dependence, negative values for some eta

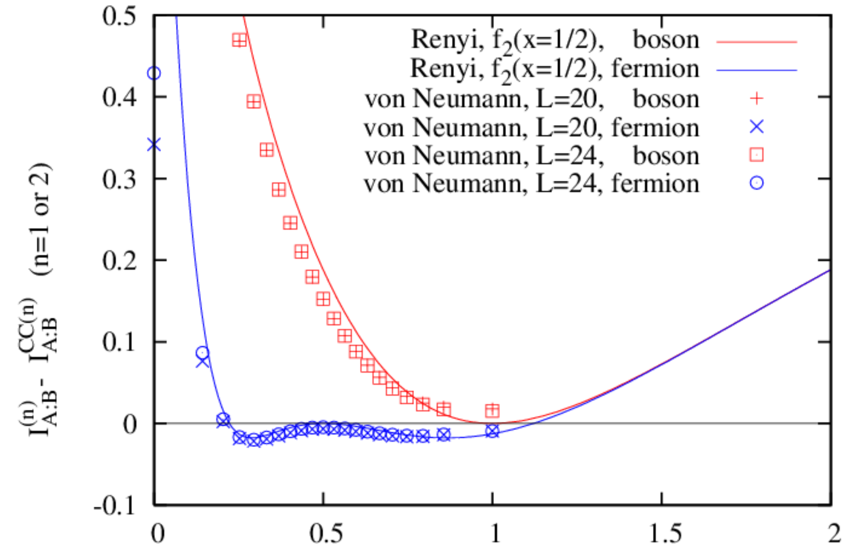
Summary: Mutual information in TLLs



$$I_{A:B} := S_A + S_B - S_{A \cup B}$$

$$= I^{\mathrm{CC}}_{A:B} + \lim_{n \to 1} \frac{1}{n-1} \log F_n(x, x, \eta)$$

$$x = \frac{x_{21} x_{43}}{x_{31} x_{42}}$$



$$\eta := 2\pi R^2$$

- Two-interval mutual information contains detailed information of CFT beyond the central charge.
- Universal relation between mutual info and boson radius R
- Difference between bosons and fermions in spite of the equivalence of the models
- Full analytical solution for interacting fermions: under investigation

Conformal dimensions of twist fields - I

Examine the correlation fn. with the stress tensor:

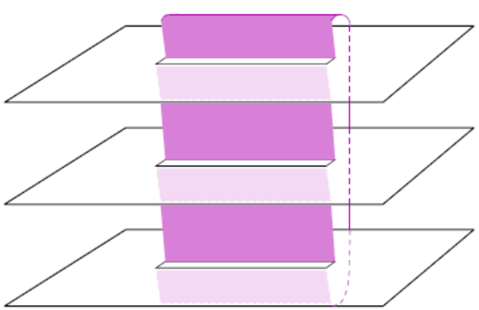
$$\frac{\langle T_{\text{tot}}(z)\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}}}{\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}}} = n\langle T(z)\rangle_{\mathcal{R}_n(A)} \qquad \boxed{T_{\text{tot}}(z) := \sum_k T(z|k)}$$

$$\langle T(z)\rangle = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w,z\} = \left(\frac{dw}{dz}\right)^2 \underbrace{\langle T(w)\rangle}_{=0} + \frac{c}{24}\left(1-\frac{1}{n^2}\right)\frac{(x_1-x_2)^2}{(z-x_1)^2(z-x_2)^2}$$

$$\{w,z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

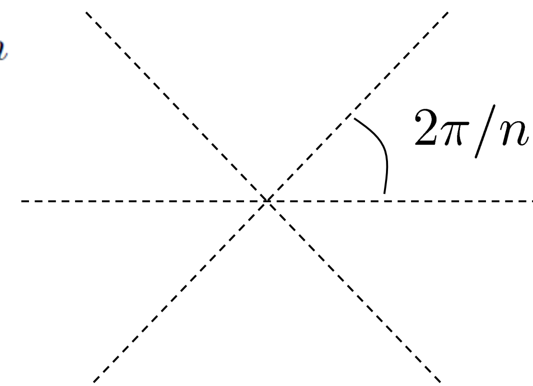
Mapping to a simple surface:

$z \in \mathcal{R}_n(A)$



$$w = \left(\frac{z-x_1}{z-x_2}\right)^{1/n}$$

$w \in \mathbb{C}$



Conformal dimensions of twist fields - II

$$\frac{\langle T_{\text{tot}}(z)\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}}}{\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}}} = \frac{c}{24}\left(n - \frac{1}{n}\right)\frac{(x_1-x_2)^2}{(z-x_1)^2(z-x_2)^2}$$

Compare with the conformal Ward identity:

$$\langle T_{\text{tot}}(z)\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}} = \sum_j \left(\frac{\Delta_n}{(z-x_j)^2} + \frac{1}{z-x_j}\frac{\partial}{\partial x_j} + \text{reg.}\right)\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle_{\mathbb{C}}$$

Conformal dimensions of $\mathcal{T}$ & $\tilde{\mathcal{T}}$ $\quad \Delta_n = \bar{\Delta}_n = \frac{c}{24}\left(n - \frac{1}{n}\right)$

<u>Single-interval entanglement entropy</u>

$$\text{Tr}\ \rho_A^n = \frac{Z_{\mathcal{R}_n}}{Z^n} \propto \langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle = x_{21}^{-2\Delta_n}\bar{x}_{21}^{-2\bar{\Delta}_n} \qquad \begin{gathered} x_i = \bar{x}_i \\ x_{ij} = x_i - x_j \end{gathered}$$

$$R_A^{(n)} = \frac{-1}{n-1}\log(\text{Tr}\ \rho_A^n) = \frac{1+n}{6n}c\log x_{21} + s_n$$

$$n \longrightarrow 1 \qquad S_A = \frac{c}{3}\log x_{21} + s_1$$

n=2 case

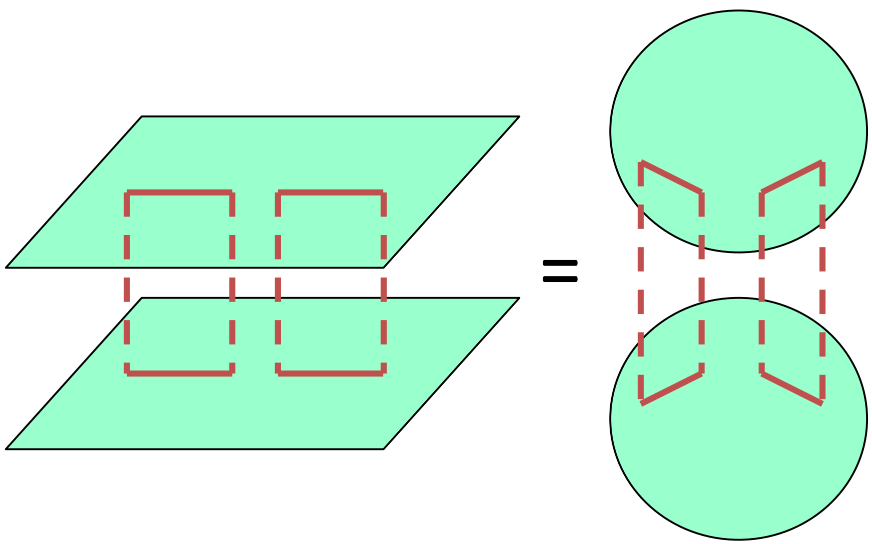
We follow the idea of Dixon,Friedan,Martinec,Shenker, Nucl.Phys.B, 1987

$$\frac{\langle T_{\rm tot}(z)\mathcal{T}(\infty)\mathcal{T}(1)\mathcal{T}(x)\mathcal{T}(0)\rangle_{\mathbb{C}}}{\langle \mathcal{T}(\infty)\mathcal{T}(1)\mathcal{T}(x)\mathcal{T}(0)\rangle_{\mathbb{C}}} = 2\langle T(z)\rangle_{\mathcal{R}_2(A\cup B)}$$

$$\langle T(w)\rangle_{\rm torus} = \left(\frac{dz}{dw}\right)^2 \langle T(z)\rangle_{\mathcal{R}_2(A\cup B)} + \frac{c}{12}\{z,w\}$$

Compare with the conformal Ward identity:

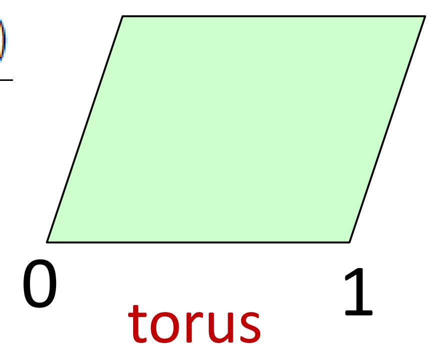
$$\langle T_{\rm tot}(z)\mathcal{T}(\infty)\mathcal{T}(1)\mathcal{T}(x)\mathcal{T}(0)\rangle_{\mathbb{C}} = \left[\frac{\Delta_2}{z^2} + \frac{\Delta_2}{(z-1)^2} + \frac{\Delta_2}{(z-x)^2} + \frac{1}{z-x}\frac{\partial}{\partial x} + \dots\right]\langle \mathcal{T}(\infty)\mathcal{T}(1)\mathcal{T}(x)\mathcal{T}(0)\rangle_{\mathbb{C}}$$



Weierstrass's elliptic fn. τ :modulus

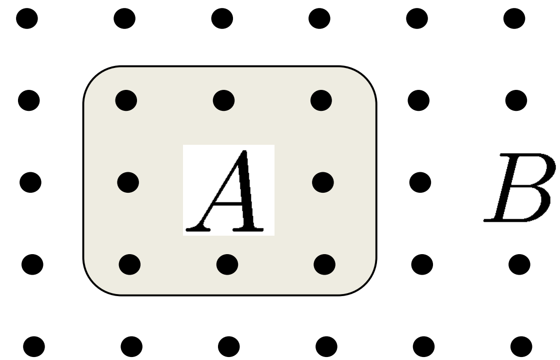
$$z = \frac{\wp(w|\tau) - e_2(\tau)}{e_1(\tau) - e_2(\tau)}$$

$$x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$



What is entanglement?

Structure of a quantum state which cannot be represented as a product form



- $|\Psi\rangle = |\Psi_A\rangle|\Psi_B\rangle$

No entanglement between A and B

$\Leftrightarrow$ Reduced density matrix on A is a pure state.

$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = |\Psi_A\rangle\langle\Psi_A|$$

- $|\Psi\rangle \neq |\Psi_A\rangle|\Psi_B\rangle$

There is entanglement between A and B

$\Leftrightarrow$ Reduced density matrix on A is a mixed state.

How to quantify entanglement ?

⇨ Measure how mixed the reduced density matrix is.

Entanglement entropy (von Neumann entropy)

$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A \ \ (= S_B)$$

$$= -\sum_i p_i \log p_i \quad \{p_i\}\text{:eigenvalues of } \rho_A$$

- product state ⟶ pure state $\quad p_i = 1,\ 0$

$$|\Psi\rangle = |00\rangle \qquad \rho_A = |0\rangle\langle 0| \qquad \boxed{S_A = 0}$$

- entangled state ⟶ mixed state $\quad p_i = \frac{1}{2},\ \frac{1}{2}$

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \rho_A = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\boxed{S_A = \log 2}$$

How to use it in many-body systems?

Look at the scaling of $S_A = -\mathrm{Tr}\ \rho_A \log \rho_A$

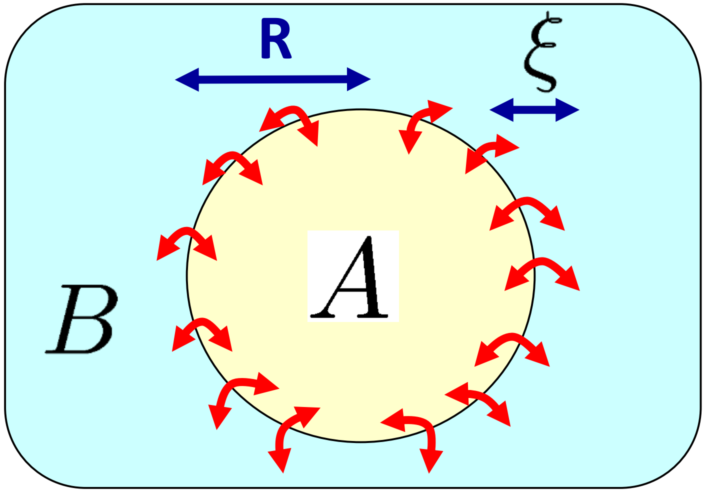
Information on the universality class

- Short-range correlations only

$$S_A \approx \alpha R^{d-1}$$

boundary law (or area law)

Srednicki, PRL, 1993
Wolf,Verstraete,Hastings,Cirac,PRL,2008

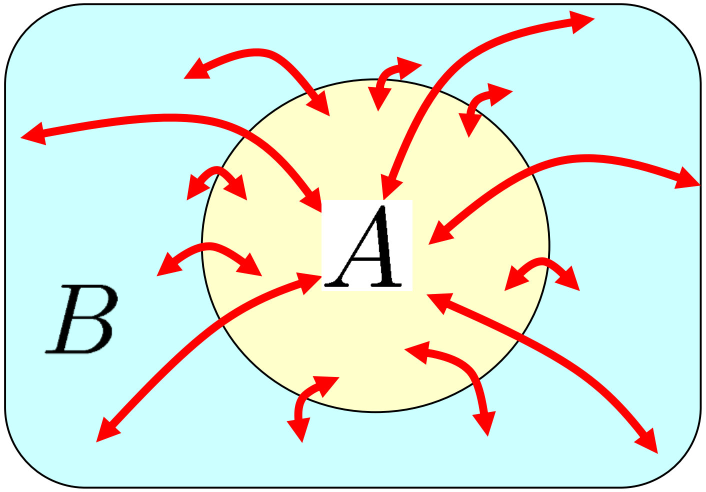


- Power-law decaying correlations

Deviation from boundary law

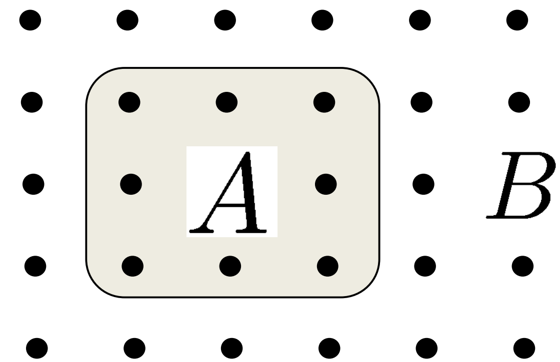
e.g., free fermion: $S_A \approx \alpha R^{d-1} \log R$

Wolf, PRL, 2006; Gioev & Klich, PRL,2006



Schmidt decomposition

Express as a sum of product state:



$$|\Psi\rangle = \sum_i \lambda_i |\psi_i^A\rangle |\psi_i^B\rangle$$

$$\langle \psi_i^A | \psi_j^A \rangle = \langle \psi_i^B | \psi_j^B \rangle = \delta_{ij}$$

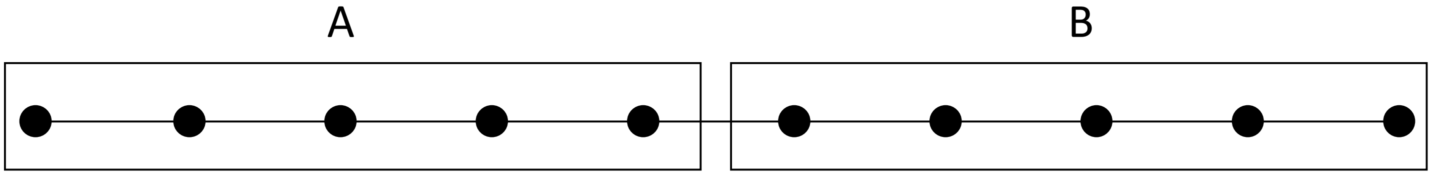
$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \sum_i |\psi_i^A\rangle \lambda_i^2 \langle \psi_i^A| \qquad \rho_B = \sum_i |\psi_i^B\rangle \lambda_i^2 \langle \psi_i^B|$$

$$S_A = S_B = -\sum_i \lambda_i^2 \log(\lambda_i^2)$$

e.g. $$\lambda_i^2 = \begin{cases} 1/m & (i = 1, 2, \ldots, m) \\ 0 \ (\text{otherwise}) \end{cases} \quad \Rightarrow \quad S_A = \log m$$

(number of relevant states in decomposition) $\sim e^{S_A}$

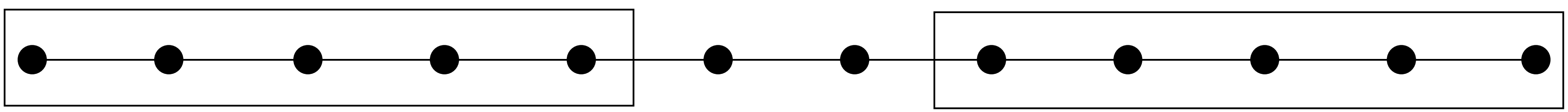
DMRG (Density Matrix Renormalization Group)



$$|\Psi\rangle = \sum_i \lambda_i |\psi_i^A\rangle|\psi_i^B\rangle$$

m states

Hilbert space dim:
$s^2 m^2$



How many states to keep? $\longrightarrow$ $m \gg e^{S_A}$

Vidal, Latorre, Rico, Kitaev, PRL, 2003

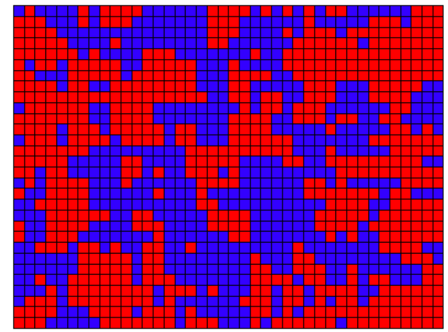
- Gapped system: $e^{S_A} \to \text{const.} \ (r \to \infty)$
- Gapless system: $S_A \simeq \frac{c}{6} \log\left[\frac{2L}{\pi} \sin \frac{\pi r}{L}\right] + \text{const.}$ (open chain)

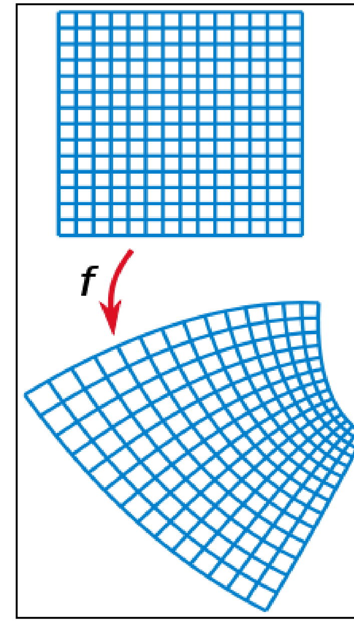
$\xrightarrow[r = L/2]{}$ $e^{S_A} \simeq (\text{const.}) \times L^{c/6}$

e.g., XX chain (c=1), L=2000 $\longrightarrow$ $e^{S_A} \simeq 4.7$

Critical phenomena and conformal field theory (CFT)

d-dimensional quantum system = (d+1)-dimensional classical system





- Critical phenomena
  - Scale invariance
  - Local scale invariance (conformal invariance)
- For d=1, the conformal symmetry forms an infinite-dimensional algebra!
  - Powerful constraints on the possible CFTs. Systematic framework.
- Examples: (central charge c $\simeq$ number of gapless modes)

• Ising model, gapless Majorana: c=1/2

• XY model (KT phase), gapless Dirac, Tomonaga-Luttinger liquids: c=1 (fixed) but with continuously varying exponent $\langle S_0 S_r \rangle \sim r^{-\eta}$

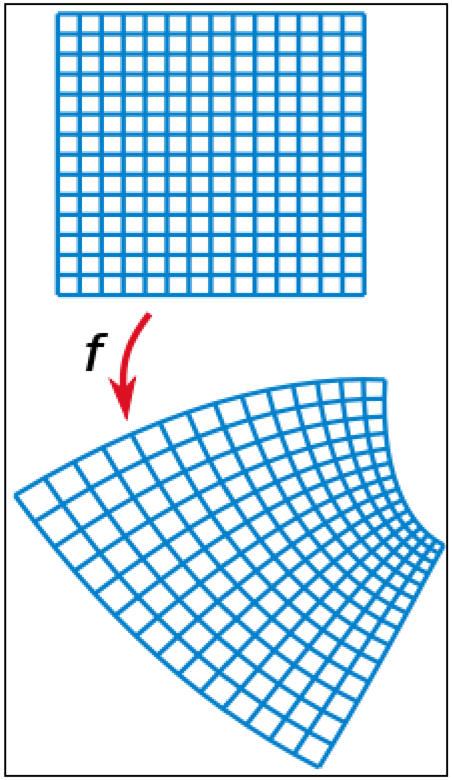
What is conformal field theory (CFT) ?

- Field theory which is invariant under local scale transformation (conformal transformation)

- In 1+1 D, any conformal transformation can be written as a holomorphic function

$$z = v\tau - ix = -i(x - vt)$$
$$\bar{z} = v\tau + ix = i(x + vt)$$

$$w = f(z) \qquad \bar{w} = \bar{f}(\bar{z})$$



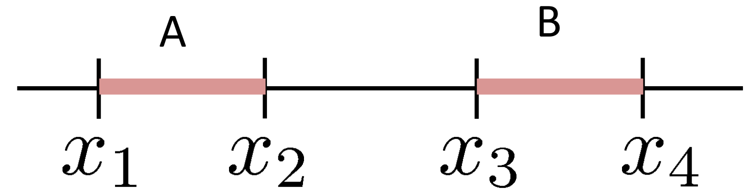
- Hamiltonian: $H = H_R(z) + H_L(\bar{z})$

CFTs are massless relativistic theories.

- In 1+1 D, the conformal invariance strictly constrains possible field theories.

(von Neumann) mutual information

$$I_{A:B} \equiv S_A + S_B - S_{A\cup B} \geq 0$$



Adami & Cerf, PRA, 1997

Vedral,Plenio,Rippin,& Knight, PRL, 1997

- Reduction of info due to correlations between A & B

  ⇒ Measure of correlations

- Equality condition : $\rho_{A\cup B} = \rho_A \otimes \rho_B$

  i.e., $\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle = 0$ for any $O_A,\ O_B$

<u>Ex. :</u> $\rho_{A\cup B} = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$

$$I_{A:B} = \log 2 + \log 2 - \log 2 = \log 2$$

$\uparrow$ or $\downarrow$ $\quad$ $\uparrow$ or $\downarrow$ $\quad$ $\uparrow\uparrow$ or $\downarrow\downarrow$

Ferromagnetic correlation