Mutual information of two disjoint intervals in Tomonaga-Luttinger liquids: bosons vs. fermions

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Outline

- Mutual information of two disjoint intervals in Tomonaga-Luttinger liquids: case of bosons

  S. F., V. Pasquier, and J. Shiraishi,
  Phys. Rev. Lett. 102, 170602 (2009)

- Case of interacting fermions

  Discussions with H. Katsura, A. Furusaki, S. Ryu, ...

Correspondence: Jordan-Wigner trans. or bosonization
Non-local!
Different Mutual info.

\[
I_{A:B} \equiv S_A + S_B - S_{A \cup B}
\]
Introduction

Wide variety of spatially 1D quantum critical (gapless) systems ↔ Conformal field theory (CFT)

• 1D conductor (e.g., Carbon nanotube)

• spin-1/2 XXZ chain → Interacting Dirac fermions (c=1)
  [Jordan-Wigner transformation]

• Ising chain in transverse field (critical point) → Majorana (c=1/2)

Q. Given a microscopic model, how can we address the information of underlying CFT?
Entanglement entropy (EE) in 1D

- Gapped (non-critical) system
  \[ S_A \to \text{const.} \quad (r \to \infty) \]
- Critical system
  \[ S_A \sim \frac{c}{3} \log r + s_1 \]

Has become a standard tool in DMRG analyses of 1D systems
(spin Bose metal with c=3: D. N. Sheng et al., PRB, 2009)

References:
- Vidal, Latorre, Rico, & Kitaev, PRL, 2003
Q. How can we obtain more detailed info of CFT?

**Ans.:** Use two intervals. Calculate the mutual information.

\[ I_{A:B} = S_A + S_B - S_{A \cup B} \]

- **Information of CFT beyond the central charge (or "operator content")**


- **Calculations in Tomonaga-Luttinger liquids (TLL) with c=1**
  

  **Numerical and half-analytical calculations**

  **Full analytical calculation**

  Mutual information is directly related to the TLL parameter K.  
  (Critical exponent in correlation functions: \( \eta = 1/2K \))
Description of interacting particles

- Interacting fermions in $D>1$: Fermi liquid
  
  single-particle picture
  renormalized dispersion relation; finite lifetime

- Interacting fermions/bosons/spins in $D=1$: Tomonaga-Luttinger liquid (TLL)

  Density fluctuations
  
  ➔ bosonic description

  $$\psi^\dagger(x) = \left[\rho(x)\right]^{1/2}e^{-i\sqrt{\pi}\theta(x)}$$

  density  phase
Phenomenological bosonization

\[ \psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i \sqrt{\pi} \theta(x)} \]

\( \psi^\dagger(x) \) represents the density, and \( \theta(x) \) represents the phase.

\[ \phi(x) = \sqrt{\pi} \int_x^\infty dx' [\rho(x') - \rho_0] \]

\[ [\partial_x \phi(x), \theta(x')] = i \delta(x - x') \]

\( \phi(x) \) is a phase field, and \( \theta(x') \) is a density field. The commutator shows the relation between the phase and density.

- **Hamiltonian**

\[ H = \int dx \frac{v}{2} \left[ K (\partial_x \theta)^2 + \frac{1}{K} (\partial_x \phi)^2 \right] \]

\( H \) is the Hamiltonian, \( v \) is the velocity, and \( K \) is the TLL parameter.

\[ U \rho(x)^2 = \frac{U}{\pi} (\partial_x \phi)^2 \]

The interaction term depends on the density field.

- **Field redefinition**

\[ \Theta = \sqrt{K} \theta \quad \Phi = \phi / \sqrt{K} \]

\[ [\partial_x \Phi(x), \Theta(x')] = i \delta(x - x') \]

\[ H = \int dx \frac{v}{2} \left[ (\partial_x \Theta)^2 + (\partial_x \Phi)^2 \right] \]

\( H \) is rewritten in terms of \( \Theta \) and \( \Phi \).

No parameter in the Hamiltonian except for \( v \).
Boson compactification conditions

In fact, a characteristic parameter exists in the boson compactification conditions.

Case of TLL of bosons (PBC on a ring of length L)

• Phase winding \( \Theta(L) - \Theta(0) = \sqrt{\frac{K}{\pi}} \cdot 2\pi M = 2\pi \tilde{R} M \)

• Excess number of particles (relative to GS): \( \Phi(L) - \Phi(0) = \sqrt{\frac{\pi}{K}} \int_0^L dx' [\rho(x') - \rho_0] = \sqrt{\frac{\pi}{K}} \Delta N = 2\pi R \Delta N \)

\[
\Phi \equiv \Phi + 2\pi R \\
\Theta \equiv \Theta + 2\pi \tilde{R}
\]

Compactification radii:

\[
R = \frac{1}{\sqrt{4\pi K}}, \quad \tilde{R} = \sqrt{\frac{K}{\pi}} \quad (R \tilde{R} = 1/2\pi)
\]

Case of TLL of fermions (discussed later)

\[
\Delta N = N_R + N_L \quad M = \frac{N_R - N_L}{2} \quad N_R, N_L \in \mathbb{Z}
\]
Physical operators: vertex operators

\[ \Phi \equiv \Phi + 2\pi R \quad \Theta \equiv \Theta + 2\pi \tilde{R} \quad (R\tilde{R} = 1/2\pi) \]

- Vertex operators: \( e^{in\Phi/R} \), \( e^{im\Theta/\tilde{R}} \) etc.
  \( n,m=\text{integer} \)

- Correlation functions:

\[
\left\langle e^{in\Phi(x)/R} e^{-in\Phi(x')/R} \right\rangle = \frac{1}{|x-x'|n^2/(2\pi R^2)} = \frac{1}{|x-x'|n^2/\eta}
\]

\[
\left\langle e^{im\Theta(x)/\tilde{R}} e^{-im\Theta(x')/\tilde{R}} \right\rangle = \frac{1}{|x-x'|m^2/(2\pi \tilde{R}^2)} = \frac{1}{|x-x'|m^2/\eta}
\]

R or \( \eta := 2\pi R^2 \) controls power-law behavior of correlations.
Boson compactification radius $R$ in XXZ model

$\eta := 2\pi R^2$

XX model
free fermion
SU(2)

0 1/2 1 2

duality $\eta \to 1/\eta$

XXZ chain in a magnetic field

$H = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) - \hbar \sum_j S_j^z$

Power-law decay of correlations:

$\langle S_0^x S_r^x \rangle \sim \frac{(-1)^r}{r \eta}, \quad \langle S_0^z S_r^z \rangle \sim \frac{(-1)^r}{r^{1/\eta}}$

$M = \frac{1}{L} \sum_j S_j^z$

TLL phase

$\eta = \frac{1}{2}$

$\eta \to 0 \quad \eta = \frac{1}{2} \quad \eta = 1 \quad \eta = 2$
Mutual information

\[ I_{A:B} := S_A + S_B - S_{A \cup B} \geq 0 \]

Non-zero value signals the presence of a correlation:

\[ \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle \neq 0 \text{ for some } O_A, \ O_B \]

Idea: Use this as a "region-region" correlation function!

Expected behavior in critical systems

- \( r = \text{const.}, \ d \to \infty : \ I_{A:B} \to 0 \)
  due to power-law decaying correlation

- \( \frac{r}{d} = \text{const.}, \ d \to \infty : \ I_{A:B} \to \text{nonzero} \)
  What determines this value?
Calabrese-Cardy (initial) prediction

\[ S_A = \frac{c}{3} \log(x_2 - x_1) + s_1 \]

non-universal constant

\[ S_{A \cup B} = \frac{c}{3} \log \left( \frac{x_{21} x_{32} x_{43} x_{41}}{x_{31} x_{42}} \right) + 2s_1 \]

\[ I_{CC}^{A:B} = \frac{c}{3} \log \left( \frac{x_{31} x_{42}}{x_{32} x_{41}} \right) \]

finite chain of length \( L \): \( x_{ij} \rightarrow \frac{L}{\pi} \sin \frac{\pi x_{ij}}{L} \)

``chord distance''

Caution: This prediction is not always valid.

Invarient under global scale transformations.

Non-universal constants are canceled!
Numerical analysis

XXZ model: \[ H = \sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) \]

\[-1 < \Delta \leq 1 : c=1 \text{ TLL phase}\]

Deviation from CC result when we go away from SU(2) point

Exact diag., L=24, 28

\[
\begin{align*}
\mathcal{I}_{A:B} &= S_A + S_B - S_{A:B} \\
\mathcal{I}_{A:B}^{\text{CC}} &= -(1/3) \log \left[ \cos^2(\pi r/L) \right]
\end{align*}
\]

\[
\Delta = -0.8 (\eta=0.205), -0.6 (\eta=0.295), 0 (\eta=0.5), 1 (\eta=1)
\]

black line: CC result
Dependence on the exponent $\eta$

\[ H = \sum_{j=1}^{L} \left( S_{j}^{x}S_{j+1}^{x} + S_{j}^{y}S_{j+1}^{y} + \Delta S_{j}^{z}S_{j+1}^{z} \right) - h \sum_{j=1}^{L} S_{j}^{z} \]

\[ M = \frac{1}{L} \sum_{j} S_{j}^{z} \]

$\eta = \frac{1}{2}$

$c=1$ critical phase

Direct relation between $I_{A:B}$ and $\eta$

Fixed divisions

$r_{A}:r_{C}:r_{B}:r_{D} = 1:1:1:1, 1:2:1:2$

\[ \eta := 2\pi R^{2} \]

$1/\eta$
CFT calculation of entanglement entropy

We now try to understand the deviation from CC result.

We first follow the CFT calculation in the single-interval case.


Then we consider how to extend it to the two-interval case.

Starting point: Replica trick

$$S_A = \lim_{n \to 1} \frac{-1}{n - 1} \log(\text{Tr} \rho_A^n)$$

Renyi entropy \(S_A^{(n)}\)

We compute \(\text{Tr} \rho_A^n\) for integer \(n>1\).

Then we take an analytic continuation \(n \to 1\).
Path integral representation

\[ \rho = \frac{1}{Z} e^{-\beta H} \]

finite-temperature total density matrix

\[ \langle \{\phi_1(x)\}|\rho|\{\phi_2(x)\} \rangle = \frac{1}{Z} \int D\phi \exp \left[ -\int dx dt E \mathcal{L} [\phi] \right] \]

\[ \langle \phi_1^A | \rho_A | \phi_2^A \rangle = \frac{1}{Z} \]

\[ \text{Tr} \rho_A^n = \frac{Z \mathcal{R}_n(A)}{Z^n} \]

\( \mathcal{R}_n(A) \)

n-sheeted Riemann surface

Branch-point twist field

$$\phi(z|k) \text{ on } R_n(A)$$

$$\Phi(z) = t(\phi(z|1), \ldots, \phi(z|n))$$

$$\mathcal{T}$$

$$\tilde{\mathcal{T}}$$

$$\Phi(z) \rightarrow \sigma \Phi(z)$$

$$\Phi(z) \rightarrow \sigma^{-1} \Phi(z)$$

Cyclic permutation

$$\sigma = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & \\ & & & 1 & 0 \end{pmatrix}$$

$$\mathcal{L}_{tot}[\Phi(x, t_E)] = \sum_k \mathcal{L}[\phi(x, t_E|k)]$$

$$\text{Tr} \rho^n_A = \frac{Z_{R_n(A)}}{Z^n}$$

$$= \frac{\int \mathcal{D} \Phi \mathcal{T}(x_1, 0) \tilde{\mathcal{T}}(x_2, 0) e^{-S_{tot}[\Phi]}}{\int \mathcal{D} \Phi e^{-S_{tot}[\Phi]}} = \langle \mathcal{T}(x_1, 0) \tilde{\mathcal{T}}(x_2, 0) \rangle_\mathbb{C}$$

Entanglement entropy as two-point correlation functions

\[ \text{Tr } \rho^n_A = \frac{Z_{\mathcal{R}^n}}{Z^n} \propto \langle \mathcal{T}(x_1) \bar{\mathcal{T}}(x_2) \rangle = x_{21}^{-2\Delta_n} x_{21}^{-2\bar{\Delta}_n} \]

\[ x_i = \bar{x}_i \]
\[ x_{ij} = x_i - x_j \]

conformal dimensions:
\[ \Delta_n = \bar{\Delta}_n = \frac{c}{24} \left( n - \frac{1}{n} \right) \]

\[ S_A^{(n)} = \frac{-1}{n-1} \log(\text{Tr } \rho^n_A) = \frac{1+n}{6n} c \log x_{21} + s_n \]

\[ n \to 1 \quad S_A = \frac{c}{3} \log x_{21} + s_1 \]
Why is the single-interval case so simple?

\[ \zeta = \frac{z - x_1}{z - x_2} \]

\[ w = \zeta^{1/n} \]

\[ z \in \mathcal{R}_n(A) \]

\[ w \in \mathbb{C} \]

\[ w = \left( \frac{z - x_1}{z - x_2} \right)^{1/n} \]
Consider \( w = \frac{x_{21} x_4 - z}{x_{42} z - x_1} : (x_1, x_2, x_3, x_4) \rightarrow (\infty, 1, x, 0) \)

\[
\text{Tr } \rho_{AUB}^n \propto \langle \mathcal{T}(x_1) \tilde{\mathcal{T}}(x_2) \mathcal{T}(x_3) \tilde{\mathcal{T}}(x_4) \rangle
\]

\[
\propto \left( \frac{x_{31} x_{42}}{x_{21} x_{32} x_{43} x_{41}} \right)^{2\Delta_n} \left( \frac{\bar{x}_{31} \bar{x}_{42}}{\bar{x}_{21} \bar{x}_{32} \bar{x}_{43} \bar{x}_{41}} \right)^{2\bar{\Delta}_n} F_n(x, \bar{x}; \eta)
\]

\[
S_{AUB} = \frac{c}{3} \log \left( \frac{x_{21} x_{32} x_{43} x_{41}}{x_{31} x_{42}} \right) + 2S_1 - \lim_{n \rightarrow 1} \frac{1}{n-1} \log F_n(x, x; \eta)
\]

Calabrese-Cardy result

new part
Cross-ratio-dependent part

\[ I_{A:B} = I_{A:B}^{CC} + \lim_{n \to 1} \frac{1}{n-1} \log F_n (x, x; \eta) \]

Calabrese-Cardy result

new part

cross ratio
\[ x = \frac{x_{21} x_{43}}{x_{31} x_{42}} \]

How to calculate \( F_n \)?

\[ x = \frac{\sin \frac{\pi r_A}{L} \sin \frac{\pi r_B}{L}}{\sin \frac{\pi (r_A+r_C)}{L} \sin \frac{\pi (r_C+r_B)}{L}} \]
Non-trivial topologies

$n=2$
genus=1

$n=4$
genus=3

In general, genus=$n-1$
n=2 case

Renyi mutual information: \( I_{A:B}^{(n)} = I_{A:B}^{CC(n)} + \frac{1}{n-1} \ln F_n(x) \)

- General result:

\[ F_2(x) = [x(1-x)]^{c/6} Z_{\text{torus}}(\tau, -\tau) / N \]
\[ \lim_{x \to 0} F_2(x) = 1 \]

- Case of compactified boson

\[ F_2(x) = \frac{\theta_3(\eta \tau) \theta_3(\tau/\eta)}{[\theta_3(\tau)]^2} \]

Coincides with the old results for Z2 twist field correlations

Al. B. Zamolodchikov,
Sov. Phys. JETP, 1986;

Dixon, Friedan, Martinec, Shenker,

Agrees relatively well with numerics, but numerics show oscillations
Full analytical results for $n>1$


Calculation of the partition fn. for a surface of genus $n-1$

\[ F_n(x) = \frac{\Theta(0|\eta \Gamma) \Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2} \]

$n=4$
genus=3

\[ \Theta: \text{Riemann-Siegel theta function} \]

\[ \Gamma: n \times n \text{ matrix produced from } \beta_y \]

\[ \beta_y = \frac{F_y(1-x)}{F_y(x)} \]

\[ F_y(x) \equiv {}_2F_1(y, 1-y; 1; x) \quad \text{hypergeometric fn.} \]

How to perform the analytical continuation $n \rightarrow 1$?
Numerical extrapolation $n \rightarrow 1$


Excellent agreement with exact diag. results!
What can we do with this result?

- New method for determining R using the GS wave fn.
- General critical systems in 1D

\[ I_{A:B} - I_{A:B}^{CC} = \lim_{n \to 1} \frac{1}{n - 1} \log F_n(x, x) =: f(x) \]

- Mutual information \iff Compactification radius R
- Direct relation

Calculations in different CFTs

- Ising criticality (c=1/2)  Alba, Tagliacozzo, Calabrese, PRB, 2010
- AdS / CFT  Headrick, PRD, 2010
Question from many researchers

S. Ryu, H. Casini, M. Huerta, V. E. Korepin, H. Katsura, ...

- Field-theoretical calculation for the free fermion case


  Simple expressions of twist operators (in terms of vertex ops.) are available.

  \[
  I_{A:B} = I_{A:B}^{CC}
  \]

  Some researchers have also checked this numerically with correlation matrix method.

  Why doesn’t the result in the XY case agree with this?

- Ans.: Jordan-Wigner transformation relating the two models are non-local!

  \[ S_j^+ = c_j^\dagger \exp(i\pi \sum_{i<j} n_i) \]

  Equivalence of the two models does not apply to the two-interval EE.

  Then, what is the behavior of mutual information in interacting fermions?
Density matrices for fermions

- Usual choice of the Fock basis
  
  Fermionic operators act only on \( A \cup B \).
  Use this basis in defining the density matrix on \( A \cup B \).

- New basis

Related and equivalent definition:
Numerical result for interacting fermions

Different curves for bosons (spins) and fermions

CC result corresponds to $\eta = 1/2$ (free Dirac fermion).

Non-monotonic dependence, negative values for some eta.

XXZ chain

t-V model

Analytical curves: explained later
Field-theoretical understanding

- General CFT formula for $n=2$ Renyi case

$$F_2(x) = [x(1 - x)]^{c/6} Z_{\text{torus}}(\tau, -\tau)/\mathcal{N}$$

$$\lim_{x \to 0} F_2(x) = 1 \quad x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

- It is sufficient to calculate the torus partition fn!

- Interacting fermion on a torus with antiperiodic BCs

  Matsubara formalism & modular inv.

Bosonization formulae

$$\psi_R^\dagger(x) = \frac{1}{\sqrt{2\pi}} e^{-i\sqrt{4\pi} \phi_R(x)} \quad \psi_L^\dagger(x) = \frac{1}{\sqrt{2\pi}} e^{+i\sqrt{4\pi} \phi_L(x)}$$

$$\psi_{R/L}(L) = -\psi_{R/L}(0) \quad \text{(Antiperiodic BCs)}$$

$$\exp \left[ i\sqrt{4\pi}(\phi_{R/L}(L) - \phi_{R/L}(0)) \right] = 1$$

Minus sign disappears!

Subtle effect of point splitting.

$$\phi_{R/L}(L) - \phi_{R/L}(0) = \frac{1}{\sqrt{4\pi}} N_{R/L} \quad N_R, N_L \in \mathbb{Z}$$
Field-theoretical understanding (cont’d)

\[ \Phi(L) - \Phi(0) = 2\pi R \Delta N \]
\[ \Theta(L) - \Theta(0) = 2\pi \tilde{R} M \]

\[ \Phi(x) = \frac{[\phi_R(x) + \phi_L(x)]}{\sqrt{K}} \]
\[ \Theta(x) = \sqrt{K} [\phi_R(x) - \phi_L(x)] \]

\[ \Delta N = N_R + N_L \]
\[ M = \frac{N_R - N_L}{2} \quad N_R, N_L \in \mathbb{Z} \]

“Twisted structure”

Wong & Affleck, Nucl. Phys. 1994
Oshikawa et al., J. Sta. Mech, 2006

\[ Z_{torus}(\tau, -\tau) = \frac{1}{|\eta_D(\tau)|^2} \sum_{N_R, N_L \in \mathbb{Z}} q^{\frac{\eta}{2} \Delta N^2 + \frac{1}{\eta} M^2} q = e^{2i\pi \tau} \eta_D(\tau) : \text{Dedekind’s } \eta \text{ fn.} \]

\[ = \frac{\theta_3(4\eta \tau)\theta_3(\tau/\eta) + \theta_2(4\eta \tau)\theta_2(\tau/\eta)}{[\theta_3(\tau)]^2} \]

Related argument for free fermion but for general integer n>1:

Analytical (n=2) vs. numerical (n=1)

- Different curves for bosons (spins) and fermions
- CC result corresponds to $\eta = 1/2$ (free Dirac fermion).
- Non-monotonic dependence, negative values for some eta
Summary: Mutual information in TLLs

Two-interval mutual information contains detailed information of CFT beyond the central charge.

Universal relation between mutual info and boson radius $R$

Difference between bosons and fermions in spite of the equivalence of the models

Full analytical solution for interacting fermions: under investigation
Conformal dimensions of twist fields - I

Examine the correlation fn. with the stress tensor:

\[
\frac{\langle T_{\text{tot}}(z)\, T(\, x_1 \, )\, \tilde{T}(\, x_2 \, ) \rangle_C}{\langle T(\, x_1 \, )\, \tilde{T}(\, x_2 \, ) \rangle_C} = n \langle T(z) \rangle_{\mathcal{R}_n(A)}
\]

\[
T_{\text{tot}}(z) := \sum_k T(z \mid k)
\]

\[
\langle T(z) \rangle = \left( \frac{dw}{dz} \right)^2 T(w) + \frac{c}{12} \{w, z\} = \left( \frac{dw}{dz} \right)^2 \langle T(w) \rangle + \frac{c}{24} \left( 1 - \frac{1}{n^2} \right) \frac{(x_1 - x_2)^2}{(z - x_1)^2(z - x_2)^2}
\]

\[
\{w, z\} = \frac{w'''}{w'} - \frac{3}{2} \left( \frac{w''}{w'} \right)^2
\]

Mapping to a simple surface:

\[
\mathcal{R}_n(A)
\]

\[
w = \left( \frac{z - x_1}{z - x_2} \right)^{1/n}
\]

\[
w \in \mathbb{C}
\]

\[
2\pi/n
\]
Conformal dimensions of twist fields - II

\[
\frac{\langle T_{\text{tot}}(z) T(x_1) \tilde{T}(x_2) \rangle_C}{\langle T(x_1) \tilde{T}(x_2) \rangle_C} = \frac{c}{24} \left( n - \frac{1}{n} \right) \frac{(x_1 - x_2)^2}{(z - x_1)^2(z - x_2)^2}
\]

Compare with the conformal Ward identity:

\[
\langle T_{\text{tot}}(z) T(x_1) \tilde{T}(x_2) \rangle_C = \sum_j \left( \frac{\Delta_n}{(z - x_j)^2} + \frac{1}{z - x_j} \frac{\partial}{\partial x_j} + \text{reg.} \right) \langle T(x_1) \tilde{T}(x_2) \rangle_C
\]

Conformal dimensions of \( \mathcal{T} \) & \( \tilde{\mathcal{T}} \): \( \Delta_n = \bar{\Delta}_n = \frac{c}{24} \left( n - \frac{1}{n} \right) \)

Single-interval entanglement entropy

\[
\text{Tr} \rho^n_A = \frac{Z_R^n}{Z_n} \propto \langle T(x_1) \tilde{T}(x_2) \rangle = x_{21}^{-2\Delta_n} \bar{x}_{21}^{-2\bar{\Delta}_n} \\
x_i = \bar{x}_i\\nx_{ij} = x_i - x_j
\]

\[
R_A^{(n)} = -\frac{1}{n-1} \log(\text{Tr} \rho^n_A) = \frac{1 + n}{6n} c \log x_{21} + s_n
\]

\[
n \to 1 \quad S_A = \frac{c}{3} \log x_{21} + s_1
\]
n=2 case

\[
\frac{\langle T_{\text{tot}}(z)T(\infty)T(1)T(x)T(0)\rangle_C}{\langle T(\infty)T(1)T(x)T(0)\rangle_C} = 2\langle T(z)\rangle_{\mathcal{R}_2(A\cup B)}
\]

\[
\langle T(w)\rangle_{\text{torus}} = \left(\frac{dz}{dw}\right)^2 \langle T(z)\rangle_{\mathcal{R}_2(A\cup B)} + \frac{c}{12} \{z, w\}
\]

Compare with the conformal Ward identity:

\[
\langle T_{\text{tot}}(z)T(\infty)T(1)T(x)T(0)\rangle_C = \left[\frac{\Delta_2}{z^2} + \frac{\Delta_2}{(z-1)^2} + \frac{\Delta_2}{(z-x)^2} + \frac{1}{z-x} \frac{\partial}{\partial x} + \ldots\right] \langle T(\infty)T(1)T(x)T(0)\rangle_C.
\]

We follow the idea of Dixon, Friedan, Martinec, Shenker, Nucl. Phys. B, 1987

Weierstrass’s elliptic fn.

\[
z = \frac{\wp(w|\tau) - e_2(\tau)}{e_1(\tau) - e_2(\tau)}
\]

\[
x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}
\]

0 torus

1
What is entanglement?

Structure of a quantum state which cannot be represented as a product form

\[ |\Psi\rangle = |\Psi_A\rangle |\Psi_B\rangle \]

No entanglement between A and B

Reduced density matrix on A is a pure state.

\[ \rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| = |\Psi_A\rangle \langle \Psi_A| \]

There is entanglement between A and B

Reduced density matrix on A is a mixed state.
How to quantify entanglement?

Measure how mixed the reduced density matrix is.

**Entanglement entropy (von Neumann entropy)**

\[ S_A = - \text{Tr} \rho_A \log \rho_A \quad (= S_B) \]
\[ = - \sum_i p_i \log p_i \quad \{p_i\}: \text{eigenvalues of } \rho_A \]

- **product state**
  \[ |\Psi\rangle = |00\rangle \]
  \[ \rho_A = |0\rangle \langle 0| \]
  \[ p_i = 1, 0 \]
  \[ S_A = 0 \]

- **entangled state**
  \[ |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \]
  \[ \rho_A = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) \]
  \[ p_i = \frac{1}{2}, \frac{1}{2} \]
  \[ S_A = \log 2 \]
How to use it in many-body systems?

Look at the scaling of \( S_A = -\text{Tr} \; \rho_A \log \rho_A \)

Information on the universality class

- Short-range correlations only
  \( S_A \approx \alpha R^{d-1} \) \textit{boundary law (or area law)}
  Srednicki, PRL, 1993
  Wolf, Verstraete, Hastings, Cirac, PRL, 2008

- Power-law decaying correlations
  Deviation from boundary law
  e.g., free fermion: \( S_A \approx \alpha R^{d-1} \log R \)
  Wolf, PRL, 2006; Gioev & Klich, PRL, 2006
Schmidt decomposition

Express as a sum of product state:

$$|\Psi\rangle = \sum_i \lambda_i |\psi_i^A\rangle |\psi_i^B\rangle$$

$$\langle \psi_i^A | \psi_j^A \rangle = \langle \psi_i^B | \psi_j^B \rangle = \delta_{ij}$$

$$\rho_A = \text{Tr}_B |\Psi\rangle\langle \Psi| = \sum_i |\psi_i^A\rangle \lambda_i^2 \langle \psi_i^A|$$

$$\rho_B = \sum_i |\psi_i^B\rangle \lambda_i^2 \langle \psi_i^B|$$

$$S_A = S_B = -\sum_i \lambda_i^2 \log(\lambda_i^2)$$

e.g.

$$\lambda_i^2 = \begin{cases} 
1/m & (i = 1, 2, \ldots, m) \\
0 & \text{(otherwise)}
\end{cases}$$

$$S_A = \log m$$

(number of relevant states in decomposition) \(\sim e^{S_A}\)
DMRG (Density Matrix Renormalization Group)

\[ |\Psi\rangle = \sum_{i} \lambda_i |\psi_i^A\rangle |\psi_i^B\rangle \]

Hilbert space dim: \[s^2m^2\]

How many states to keep?

- Gapped system: \[e^{S_A} \rightarrow \text{const.} \ (r \rightarrow \infty)\]
- Gapless system: \[S_A \approx \frac{c}{6} \log \left[ \frac{2L}{\pi} \sin \frac{\pi r}{L} \right] + \text{const.}\] \(r = L/2\) \[e^{S_A} \approx (\text{const.}) \times L^{c/6}\]

e.g., XX chain (c=1), L=2000 \[e^{S_A} \approx 4.7\]

Vidal, Latorre, Rico, Kitaev, PRL, 2003
Critical phenomena and conformal field theory (CFT)

d-dimensional quantum system = (d+1)-dimensional classical system

- Critical phenomena
  - Scale invariance
  - Local scale invariance (conformal invariance)

- For d=1, the conformal symmetry forms an infinite-dimensional algebra!
  - Powerful constraints on the possible CFTs. Systematic framework.

- Examples: \( \{ \text{central charge } c \sim \text{number of gapless modes} \} \)
  - Ising model, gapless Majorana: \( c=1/2 \)
  - XY model (KT phase), gapless Dirac, Tomonaga-Luttinger liquids: \( c=1 \) (fixed) but with continuously varying exponent \( \langle S_0 S_r \rangle \sim r^{-\eta} \)
What is conformal field theory (CFT)?

- Field theory which is invariant under local scale transformation (conformal transformation)

- In 1+1 D, any conformal transformation can be written as a holomorphic function
  \[ z = v \tau - ix = -i(x - vt) \]
  \[ \bar{z} = v \tau + ix = i(x + vt) \]
  \[ w = f(z) \quad \bar{w} = \bar{f}(\bar{z}) \]

- Hamiltonian: \[ H = H_R(z) + H_L(\bar{z}) \]

  CFTs are massless relativistic theories.

- In 1+1 D, the conformal invariance strictly constrains possible field theories.
(von Neumann) mutual information

\[ I_{A:B} \equiv S_A + S_B - S_{A \cup B} \geq 0 \]

Adami & Cerf, PRA, 1997
Vedral, Plenio, Rippin, & Knight, PRL, 1997

- Reduction of info due to correlations between A & B
- Measure of correlations
- Equality condition: \( \rho_{A \cup B} = \rho_A \otimes \rho_B \)

\[ \text{i.e., } \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle = 0 \text{ for any } O_A, O_B \]

Ex.: \( \rho_{A \cup B} = \frac{1}{2}(| \uparrow \uparrow \rangle \langle \uparrow \uparrow | + | \downarrow \downarrow \rangle \langle \downarrow \downarrow |) \)

\[ I_{A:B} = \log 2 + \log 2 - \log 2 = \log 2 \]

\[ \text{Ferromagnetic correlation} \]

\[ \text{or} \quad \text{or} \quad \text{or} \quad \text{or} \]