Finite Temperature Rényi Entropy and Modular Invariance

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[Modular Invariance and Entanglement Entropy",
Sagar Lokhande and Sunil Mukhi, arXiv: 1504.01921]
and work in progress.

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Outline

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Introduction: Entanglement and CFT

- We will consider von Neumann and Rényi entropies for real-space entanglement in a 2d CFT of central charge c.
- Partition the 1d space into an interval of length ℓ and the rest, called respectively A and B.
- Then $\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_B$. If ρ is any density matrix on $\mathcal{H}$, then let

$$\rho_A = \mathrm{tr}_B\, \rho$$

and the von Neumann entropy is:

$$S_A = -\mathrm{tr} \rho_A \log \rho_A$$

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  where n is an integer ≥ 2. This is easier to compute, using the replica trick – at least for free fields.
- By analytically continuing this to arbitrary real values of n, one can obtain the von Neumann entropy as a limit:

$$S_A = \lim_{n\to 1} S_A^{(n)}$$

- In CFT, if the total space is infinite and we work at zero temperature, it is well-known that:

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + c'$$

where:

$$\begin{aligned} \ell &= \text{size of the interval } A \\ c &= \text{central charge} \\ a &= \text{UV cutoff} \\ c' &= \text{non-universal constant} \end{aligned}$$

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- It is true that the von Neumann entropy is not a good entanglement measure at finite temperature, but it is still an interesting quantity (cf. Herzog's talk).

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- The reason is that the path integral treats temperature as a Euclidean time of length β, and it does not distinguish between space and (Euclidean) time.
- This suggests we study the case of finite interval and finite spatial size.

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- The precise goal is to compute $S_A(\ell, L, \beta)$ where ℓ is the size of the spatial interval, L is the size of the space (a circle) and β is the inverse temperature. In this case the CFT lives on a torus with a cut on it.

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- The precise goal is to compute $S_A(\ell, L, \beta)$ where ℓ is the size of the spatial interval, L is the size of the space (a circle) and β is the inverse temperature. In this case the CFT lives on a torus with a cut on it.
- From the preceding discussion, we know $\lim_{L\to\infty} S_A(\ell, L, \beta)$ as well as $\lim_{\beta\to\infty} S_A(\ell, L, \beta)$. In both cases the torus decompactifies to a cylinder and the answer is again universal.

- There is a general result [Cardy-Herzog] about the universal thermal correction to the Rényi/von Neumann entropies at lowest order in $q = e^{2\pi i \tau} = e^{-2\pi \frac{\beta}{L}}$ for an arbitrary CFT:

$$\delta S_A^{(n)} = f(\ell, \Delta, n) q^{\Delta} + \cdots$$

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- However the full computation of von Neumann entropy at finite size and finite temperature has been carried out only for free theories and not for any other 2d CFT.
- Even for free theories, there are some issues – as I will discuss in what follows.

- For free field theories, the Rényi entropy can be expressed in terms of a quantity called the "replica partition function":

$$\mathrm{tr}(\rho_A)^n = \frac{Z_n}{(Z_1)^n}$$

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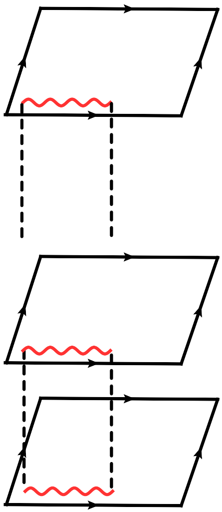

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- By a suitable diagonalisation of the problem, one reduces the problem to a set of fields ψ_k on a single copy of the space. The twist field acts on each one by a phase:

$$\sigma_k : \psi_k \to \omega^k \psi_k$$

where $\omega = e^{2\pi i/n}$ and $k = -\frac{n-1}{2}, -\frac{n-1}{2} + 1, \cdots, \frac{n-1}{2}$.

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- Then, the replica trick tells us that:

$$\operatorname{tr} \rho_A^n = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \sigma_k(\ell,\ell)\, \sigma_{-k}(0,0) \rangle$$

Outline

1. Introduction: Entanglement and CFT
2. Entanglement and modular invariance
3. Thermal entropy relation
4. Free boson CFT
5. Free fermion entanglement
6. Multiple fermions and lattice bosons
7. Conclusions

Entanglement and modular invariance

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- We already saw that $\beta \leftrightarrow L, \ell \to i\ell$ is a symmetry of von Neumann entropy at very large or small $\frac{\beta}{L}$. This is an example of modular invariance (in a limit).
- It is therefore interesting to ask how the von Neumann/Rényi entropies of a CFT at arbitrary finite size L and inverse temperature β transform under the modular group SL(2,Z).

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- These computations were carried out at fixed torus boundary conditions for the fermions. Therefore they were not modular invariant.
- Subsequently [Datta-David] attempted to compute the Rényi entropy for a free compact scalar field at radius R. This was later corrected by [Chen-Wu]. These authors did not comment on the modular properties of their results.

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- Our motivation was to understand whether von Neumann entropy at finite temperature and size is modular invariant, and whether it obeys Bose-Fermi duality. Accordingly, we investigated it for the modular-invariant free fermion theory and compared the result with that for free bosons.
- We were partially successful, but some puzzles remain.

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Thermal entropy relation

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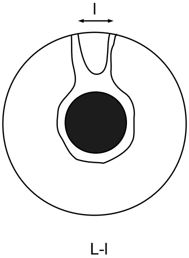
- At finite temperature and spatial size, we are dealing with a Euclidean 3d bulk that is asymptotic to Euclidean AdS_3. The boundary is a (conformally) flat Euclidean 2d torus.
- Now suppose we are at high temperature. Then there is a black hole in the bulk.

- For a large entangling region the "drooping geodesic" can sense the black hole. Hence the geodesics with boundary ℓ and $L-\ell$ are not the same.



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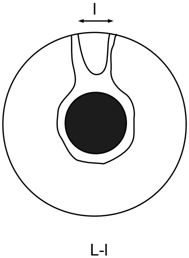

- This leads to the thermal entropy relation [Azeyanagi-Nishioka-Takayanagi]: As $\ell \to 0$ the difference is the geodesic wrapping the black hole horizon, which gives the thermal entropy of the CFT state. Hence we get the constraint:

$$\lim_{\ell \to 0} \Big(S_A(L-\ell) - S_A(\ell) \Big) = S_{\text{thermal}}(\beta)$$

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- This also suggests that the high- and low-temperature limits of the boundary CFT are related by the same S-transformation.
- Then, the thermal entropy relation should hold at any temperature.

- Although originally arising from holography, the thermal entropy relation can be derived directly within CFT. In fact a stronger relation holds ([Cardy-Herzog], [Chen-Wu]):

$$\lim_{\ell \to 0} Z_n(\ell, L, \beta) = \left(\frac{\ell}{L}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} (Z_1(L, \beta))^n$$

$$\lim_{\ell \to L} Z_n(\ell, L, \beta) = \left(\frac{L-\ell}{L}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} Z_1(L, n\beta)$$

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- For a small interval the replicas are effectively decoupled, so one finds n copies of the usual partition function. On the other hand for a large interval, the replicas are effectively "joined" into a single torus of n times the height.
- These relations are not merely intuitive guesses but have been proved by formal manipulations in CFT.

- They immediately imply the thermal entropy relation:

$$\begin{aligned}\lim_{\ell\to 0}\big(S_A(L-\ell)-S_A(\ell)\big) &= \lim_{\ell\to 0}\lim_{n\to 1}\frac{1}{1-n}\log\frac{Z_n(L-\ell,L,\beta)}{Z_n(\ell,L,\beta)}\\ &= \lim_{n\to 1}\frac{1}{1-n}\log\frac{Z_1(L,n\beta)}{(Z_1(L,\beta))^n}\\ &= \log Z_1\left(\frac{\beta}{L}\right)-\frac{\beta}{L}\frac{Z_1'(\frac{\beta}{L})}{Z_1(\frac{\beta}{L})}\\ &= Z_{\mathrm{thermal}}\end{aligned}$$

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\end{aligned}$$

- An implicit assumption is that the order of limits $\ell\to 0$ and $n\to 1$ can be interchanged.

Outline

Free boson CFT

- For the free boson replica partition function, one considers a complex boson ($c = 2$) compactified on a square torus of size R. The twist fields $\mathcal{T}_k$ satisfy:

$$\mathcal{T}_k(z, \bar{z})\phi(w) \sim (z - w)^{\frac{k}{n}}$$

and one has:

$$Z_n(\ell, L; \beta) = \prod_{k=0}^{n-1} \langle\langle \mathcal{T}_k(z, \bar{z})\mathcal{T}_{-k}(0, 0)\rangle\rangle_\tau$$

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  where $z = \frac{\ell}{L}$ and $\tau = i\frac{\beta}{L}$.
- At the end one can take a square root to get the $c = 1$ theory.
- This problem was studied by [Datta-David] and [Chen-Wu] using techniques developed many years ago for orbifold compactifications.

- The result of [Chen-Wu] is of the form:

$$Z_n(R) = Z_n^{(1)}\, Z_n^{(2)}\, Z_n^{(3)}(R)\, Z_n^{(3)}\!\left(\frac{2}{R}\right)$$

where:

$$Z^{(1)} = \frac{1}{|\eta(\tau)|^{2n}} \prod_{k=0}^{n-1} \frac{1}{|W_1^1(k,n;z|\tau)|}$$

$$Z^{(2)} = \left|\frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)}\right|^{\frac{1}{6}\left(n-\frac{1}{n}\right)}$$

$$Z^{(3)}(R) = \sum_{m_j\in\mathbb{Z}} \exp\left(-\frac{\pi R^2}{2n}\sum_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right| \times \sum_{j,j'=0}^{n-1}\left[\cos 2\pi(j-j')\frac{k}{n}\right] m_j m_{j'}\right)$$

- Here $W_1^1(k,n;z|\tau)$ and $W_2^2(k,n;z|\tau)$ are integrals of the cut differentials over the different periods of the torus:

$$W_1^1 = \int_0^1 dz'\ \theta_1(z'|\tau)^{-\left(1-\frac{k}{n}\right)}\theta_1\left(z'-z|\tau\right)^{-\frac{k}{n}}\,\theta_1\left(z'-\frac{k}{n}z|\tau\right)$$

$$W_2^2 = \int_0^{\bar\tau} d\bar z'\ \bar\theta_1(\bar z'|\tau)^{-\frac{k}{n}}\bar\theta_1\left(\bar z'-\bar z|\tau\right)^{-\left(1-\frac{k}{n}\right)}\,\bar\theta_1\left(\bar z'-\left(1-\frac{k}{n}\right)\bar z|\tau\right)$$

- Here $W_1^1(k,n;z|\tau)$ and $W_2^2(k,n;z|\tau)$ are integrals of the cut differentials over the different periods of the torus:

$$W_1^1 = \int_0^1 dz'\ \theta_1(z'|\tau)^{-\left(1-\frac{k}{n}\right)}\theta_1\left(z'-z|\tau\right)^{-\frac{k}{n}}\,\theta_1\left(z'-\frac{k}{n}z|\tau\right)$$

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- We investigated the modular transformation of this expression. To this end, we note the following results:

$$\eta\Big(-\tfrac{1}{\tau}\Big) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$$

$$W_1^1\Big(k,n;\tfrac{z}{\tau}|-\tfrac{1}{\tau}\Big) = \frac{1}{\tau}e^{-\frac{i\pi z^2}{\tau}\frac{k}{n}\left(1-\frac{k}{n}\right)}W_2^2(k,n;z|\tau)$$

$$\frac{\theta_1'\left(0|-\frac{1}{\tau}\right)}{\theta_1\left(\frac{z}{\tau}|-\frac{1}{\tau}\right)} = i\tau e^{-\frac{i\pi z^2}{\tau}}\frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)}$$

- Next, performing a multi-variable Poisson resummation, we find that:

$$Z^{(3)}\left(R;\frac{z}{\tau}\bigg|-\frac{1}{\tau}\right)=\frac{2^{\frac{n}{2}}}{R^n}\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|^{\frac{1}{2}}\right)Z^{(3)}\left(\frac{2}{R};z\bigg|\tau\right)$$

$$Z^{(3)}\left(\frac{2}{R};\frac{z}{\tau}\bigg|-\frac{1}{\tau}\right)=\frac{R^n}{2^{\frac{n}{2}}}\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|^{\frac{1}{2}}\right)Z^{(3)}\left(R;z\bigg|\tau\right)$$

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- Thus the product transforms as:

$$Z^{(3)}(R)\,Z^{(3)}\left(\frac{2}{R}\right)\to\left(\prod_{k=0}^{n-1}\left|\frac{W_2^2(k,n)}{W_1^1(k,n)}\right|\right)Z^{(3)}(R)\,Z^{(3)}\left(\frac{2}{R}\right)$$

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- Putting everything together, we find that:

$$Z_n\left(R;\frac{z}{\tau}\Big|-\frac{1}{\tau}\right)=|\tau|^{\frac{1}{6}\left(n-\frac{1}{n}\right)}Z_n(R;z|\tau)$$

Thus, it is modular covariant (rather than invariant).

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- This precisely agrees with what we found.

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- Alternatively we can live with the additive term in the Renyi and von Neumann entropies, given that they anyway have finite, non-universal additive terms.

Outline

Free fermion entanglement

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- This will lead to a puzzle.

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- At $R = 1$ the physical vertex operators for a boson are:

$$\mathcal{O}_{e,m} = e^{i\left(e+\frac{m}{2}\right)\phi(z)} e^{i\left(e-\frac{m}{2}\right)\bar{\phi}(\bar{z})}$$

with $(\Delta_{e,m}, \bar{\Delta}_{e,m}) = \left(\frac{1}{2}\left(e + \frac{m}{2}\right)^2, \frac{1}{2}\left(e - \frac{m}{2}\right)^2\right)$.

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- The fermion is $D(z) \ \sim \ e^{i\phi(z)}$.

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- These operators have $(\Delta, \bar{\Delta}) = (\frac{k^2}{2n^2}, \frac{k^2}{2n^2})$. They are nonlocal operators with the desired OPE:

$$\mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z})\, D(w) \sim (z-w)^{\frac{k}{n}}$$

- A standard computation now gives:

$$\langle\langle \mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z})\mathcal{O}_{0,-\frac{2k}{n}}(0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2k^2}{n^2}} \times \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

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- Note that this involves a sum over spin structures, or $(\pm,\pm)$ boundary conditions, of free fermions on the torus.
- [Azeyanagi et al] restricted to a specific spin structure, to get:

$$\langle\langle \mathcal{O}_{0,\frac{2k}{n}}(z,\bar{z})\mathcal{O}_{0,-\frac{2k}{n}}(0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2k^2}{n^2}} \times \frac{|\theta_3(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

(recall that θ_3 corresponds to $(--)$ boundary conditions).

- Taking the product over replicas they got:

$$Z_n(\ell, L, \beta) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}(n-\frac{1}{n})} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{|\theta_3(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

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- Under modular transformations θ_3 goes into other θ-functions, so this result is not modular invariant. Therefore it cannot be equal to the modular-invariant answer for free bosons that we exhibited earlier.
- One can expand it and show that it satisfies the thermal entropy relation of [Cardy-Herzog], with $\Delta = \frac{1}{2}$. But we know that in the modular-invariant theory, the primary of lowest dimension is the spin field of dimension $\Delta = \frac{1}{8}$.

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- One way would be to sum over spin structures before we carry out replication, leading to the "uncorrelated replica partition function":

$$Z_n^{\mathrm{u}}(\ell, L, \beta) = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}(n-\frac{1}{n})} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{2}\frac{\sum_{\nu=1}^{4}|\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

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- Another way is to take the product over replicas before summing over spin structures, leading to the "correlated replica partition function":

$$Z_n^{\rm c}(\ell, L, \beta) = \frac{1}{2}\left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{1}{6}(n-\frac{1}{n})} \sum_{\nu=1}^{4}\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{|\theta_\nu(\frac{k\ell}{nL}|\tau)|^2}{|\eta(\tau)|^2}$$

- Notice that the two types of replica partition functions coincide at $n = 1$:

$$Z_1^{\mathrm{u}} = Z_1^{\mathrm{c}} = Z_1 = \frac{1}{2} \frac{\sum_{\nu=1}^{4} |\theta_\nu(0|\tau)|^2}{|\eta(\tau)|^2}$$

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- Also both types of replica partition functions are modular-covariant with the same prefactor:

$$Z_n^{\mathrm{u,c}}(i\ell, \beta, L) = \left(\frac{\beta}{L}\right)^{\frac{1}{6}\left(n - \frac{1}{n}\right)} Z_n^{\mathrm{u,c}}(\ell, L, \beta)$$

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- Taking this limit on our candidate answers, we get:

$$Z_n^{\mathrm{u}}(\ell, L, \beta) \sim \left(\frac{\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \left(\frac{1}{2}\frac{\sum_{\nu=1}^{4} |\theta_\nu(0|\tau)|^2}{|\eta(\tau)|^2}\right)^n$$

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- Only in the first case do we obtain the expected answer $\sim (Z_1)^n$. Thus on this basis it seems that Z_n^{u} is the correct Rényi entropy.

- Now we consider the same quantities in the limit $\ell \to L$. This time we find:

$$Z_n^{\mathrm{u}}(\ell, L, \beta) \sim \left(\frac{L-\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{1}{2} \frac{\sum_{\nu=1}^{4} |\theta_\nu(\frac{k}{n}|\tau)|^2}{|\eta(\tau)|^2}$$

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- This time, neither of the answers looks like the desired $Z_1(L, n\beta)$. However there is a beautiful θ-identity that allows us to evaluate the correlated case:

$$\prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left|\theta_\nu\left(\frac{k}{n} - z \middle| \tau\right)\right| = \left(\prod_{p=1}^{\infty} \left|\frac{(1-q^{2p})^n}{1-q^{2pn}}\right|\right) |\theta_\nu(nz|n\tau)|$$

- It follows easily that:

$$Z_n^{\mathrm{c}}(\ell \to L, L, \beta) = \frac{1}{2}\left(\frac{L-\ell}{L}\right)^{-\frac{1}{6}\left(n-\frac{1}{n}\right)} \sum_{\nu=1}^{4} \frac{|\theta_\nu(0|n\tau)|^2}{|\eta(n\tau)|^2}$$

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- This time it is the "correlated" replica partition function, where the sum over spin structures is taken after the product over replicas, that satisfies the desired relation.
- It is easy to check that, due to cross terms, the uncorrelated one does not satisfy any similar relation.

- To summarise: as $\ell \to 0$ the sum over spin structures must be performed before the product over replicas. As $\ell \to L$ it must be performed after the product over replicas.

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- To summarise: as $\ell \to 0$ the sum over spin structures must be performed before the product over replicas. As $\ell \to L$ it must be performed after the product over replicas.
- There should of course be a unique Rényi entropy for this theory at all ℓ, but it is not (yet) clear what is the prescription for it.
- In contrast, older works where spin structures were not summed were able to write the complete answer in terms of a single θ-function valid for all ℓ.

- Ideally one would like to compare with the free boson result at $c = 1$ to verify Bose-Fermi duality.

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- However, as $\ell \to 0$ and $\ell \to L$ the bosonic expression has been evaluated by [Chen-Wu] and found to agree with the predictions $(Z_1(\tau))^n$ and $Z_1(n\tau)$ respectively.
- Since at $R = 1$, the function Z_1 is equal to the free Dirac fermion partition function, this means our results and theirs are in full agreement in the regions $\ell \to 0$ and $\ell \to L$ where comparison is possible.

Outline

Multiple fermions and lattice bosons

- The theory of d free Dirac fermions with correlated spin structures is dual to a specific compactification of d free bosons on a target-space torus:

$$T^c = R^d / \Gamma_d$$

where Γ_d is the root lattice of $\mathrm{Spin}(2d)$.

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- This can be achieved by starting with a rectangular torus and choosing a suitable constant metric and B-field.
- In this case the d different bosons are not orthogonal to each other, while the fermions have correlated spin structures, so on both sides of the Bose-Fermi duality we are dealing with CFT's that are not direct sums of simpler ones.

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- Elements of the weight lattice can be parametrised as:

$$w^i = \frac{1}{\sqrt{2}} g^{ij} v_j, \qquad \bar{w}^i = \frac{1}{\sqrt{2}} g^{ij} \bar{v}_j$$

where $v_i, \bar{v}_i$ are integers and g^{ij} is the inverse of g_{ij} which is the half the Cartan matrix of $\mathrm{Spin}(2d)$.

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- We have $\frac{1}{\sqrt{2}}(v_i - \bar{v}_i) = \sqrt{2} n_i$ where n_i are integers.

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- If $\vec{\alpha}_i$ are the d simple roots of $\mathrm{Spin(2d)}$ and $\vec{\lambda}^i$ are the fundamental weights then one finds:

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  where $w^{(p)i} = \sqrt{2}(\vec{\lambda}^i)_p$.
- We can now look for the twist field, which induces a monodromy:

$$\sigma_k : D_p(z) \to e^{\frac{2\pi ik}{n}} D_p(z)$$

  corresponding to a shift:

$$w^{(p)i}\phi_i(z) \to w^{(p)i}\phi_i(z) + \frac{2\pi k}{n}$$

- This will be induced by a shift $\phi_i \to \phi_i + 2\pi\zeta_i^{(k)}$ where $\zeta_i^{(k)}$ is a constant vector satisfying:

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  Thus the twist field only acts on the last scalar ϕ_d.
- It takes the form:

$$\sigma_k = \mathcal{O}_{\zeta^{(k)i}, -\zeta^{(k)i}} = e^{i\zeta^{(k)i}\phi_i(z)}e^{-i\zeta^{(k)i}\bar{\phi}_i(\bar{z})}$$

  and has the desired conformal dimension $\sum_k \Delta_k = \frac{d}{24}\left(n - \frac{1}{n}\right)$.

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- Recall that the ordinary partition function for these theories is:

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- The un-normalised two-point function of twist fields is:

$$\langle\langle \sigma_k(z,\bar{z})\sigma_{-k}(0)\rangle\rangle = \left|\frac{\theta_1'(0|\tau)}{\theta_1(\frac{\ell}{L}|\tau)}\right|^{\frac{2dk^2}{n^2}} \frac{1}{|\eta(\tau)|^{2d}}\times$$

$$\sum_{\substack{w,\bar{w}\in\Lambda_W \\ w-\bar{w}\in\Lambda_R}} q^{w^2}\bar{q}^{\bar{w}^2} e^{2\pi i\frac{\ell}{L}g_{ij}(w^i+\bar{w}^i)\zeta^{(k)j}}$$

- Now we have:

$$
\begin{aligned}
g_{ij}(w^i + \bar{w}^i)\zeta^{(k)j} &= \frac{k}{n}\sum_{p=1}^{d}(n_p + m_p), \quad w, \bar{w} \in \Lambda_R \cup \Lambda_V \\
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- Taking the product over k after/before the sum over spin structures gives us the uncorrelated/correlated Z_n.
- As before, we choose the former as $\ell \to 0$ and the latter as $\ell \to L$, and the thermal entropy relation follows.

Outline

- We have argued that Rényi and von Neumann entropies for CFT's should be modular invariant (more precisely, covariant with a known prefactor).

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- The free boson result for arbitrary radius R is known and satisfies this requirement, but is very complicated and implicit.
- The free fermion result offers a puzzle: the order of the sum over spin structures and product over replicas needs to be reversed when going from $\ell \to 0$ to $\ell \to L$. Thus we do not know the answer for intermediate values of ℓ.

- For the future, some directions are suggested:

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  - Can one write the replica partition function for fermions at intermediate values of ℓ as a linear combination of correlated/uncorrelated quantities? Alternatively, should the twist fields depend on the spin structure?

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  - For free bosons, there is a result but it is very implicit. Can its form be simplified?
  - Can such computations be extended to other CFT's? Can modular invariance be used as a constraint for this purpose?
  - Can one compute entanglement negativity (a better measure for mixed states) for CFT at finite size and temperature, and is it modular-invariant?

Thank you

ありがとうございます

Spin structures and modular invariance

- Boundary conditions on a torus of sides L, β:

$$\psi(z+L) = \pm\psi(z)$$
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- With these boundary conditions, denote the path integral by $Z_{\pm\pm}(L,\beta)$ and the Hamiltonian by $H_{\pm}(L)$. Then:

$$Z_{--} = \mathrm{tr} e^{-\beta H_-} \qquad Z_{+-} = \mathrm{tr} e^{-\beta H_+}$$
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- Let $\tau = i\frac{\beta}{L}$. Then only Z_{++} is invariant under modular transformations:

$$\tau \to \tau + 1, \quad \tau \to -\frac{1}{\tau}$$

while the other three are permuted. However, $Z_{++} = 0$ (and it is not a thermal ensemble anyway).

- As shown long ago by Seiberg and Witten, the following combination is modular-invariant:

$$Z(L,\beta) = \tfrac{1}{2}(Z_{--} + Z_{-+} + Z_{+-} + Z_{++})$$
$$= \mathrm{tr}\left(\frac{1+(-1)^F}{2}\right) e^{-\beta H_-} + \mathrm{tr}\left(\frac{1+(-1)^F}{2}\right) e^{-\beta H_+}$$

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- The modular-invariant partition function of the free Dirac fermion is therefore:

$$Z_{\mathrm{Dirac}} = \frac{1}{2}\sum_{\nu=2,3,4}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|^2$$

- Next consider a free boson $\phi(z, \bar{z})$ that takes a compact set of values:

  $$\phi(z, \bar{z}) \sim \phi(z, \bar{z}) + 2\pi R$$

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where $q = e^{i\pi\tau}$.

- The statement of Bose-Fermi duality at $c = 1$ is then:

$$Z_{\text{Dirac}} = Z_{\text{boson}}(R = 1)$$

Notice that this holds only with the spin-structure-summed partition function on the LHS.

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- The two theories have very different spectra and correlation functions. In particular the latter theory is not the direct sum of two CFT's.