Probability of boundary condition in quantum cosmology

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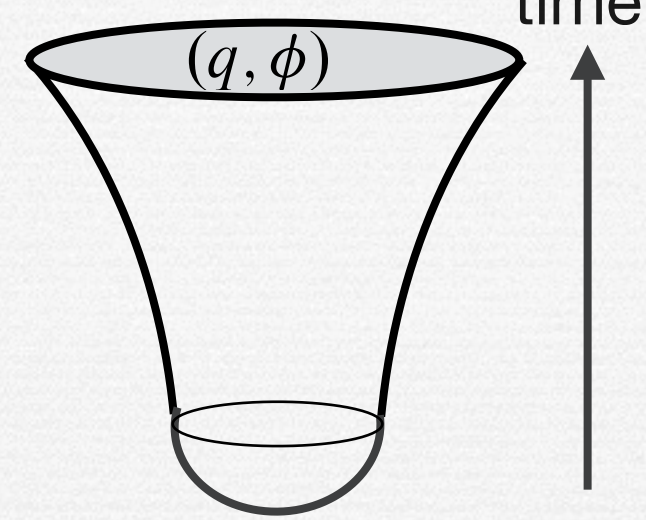
Introduction



Quantum cosmology (QC):

treat a universe as a single quantum system

- Canonical quantization of the universe

$$\hat{H}|\Psi\rangle = 0$$ Wheeler-DeWitt (WD) equation

Hamiltonian constraint

Quantum state of the universe is contained in the wave function of the universe $\Psi[q] = \langle q|\Psi\rangle$

We expect to obtain origin and history of our universe by analyzing the wave function of the universe

- There are several issues to be considered:

problem time: how can we derive dynamics of the universe?

probability: conserved charge is not positive definite

how can we define probability?

Prediction of the wave function : relies on WKB analysis

boundary condition: how do we determine BC of WD eq.?

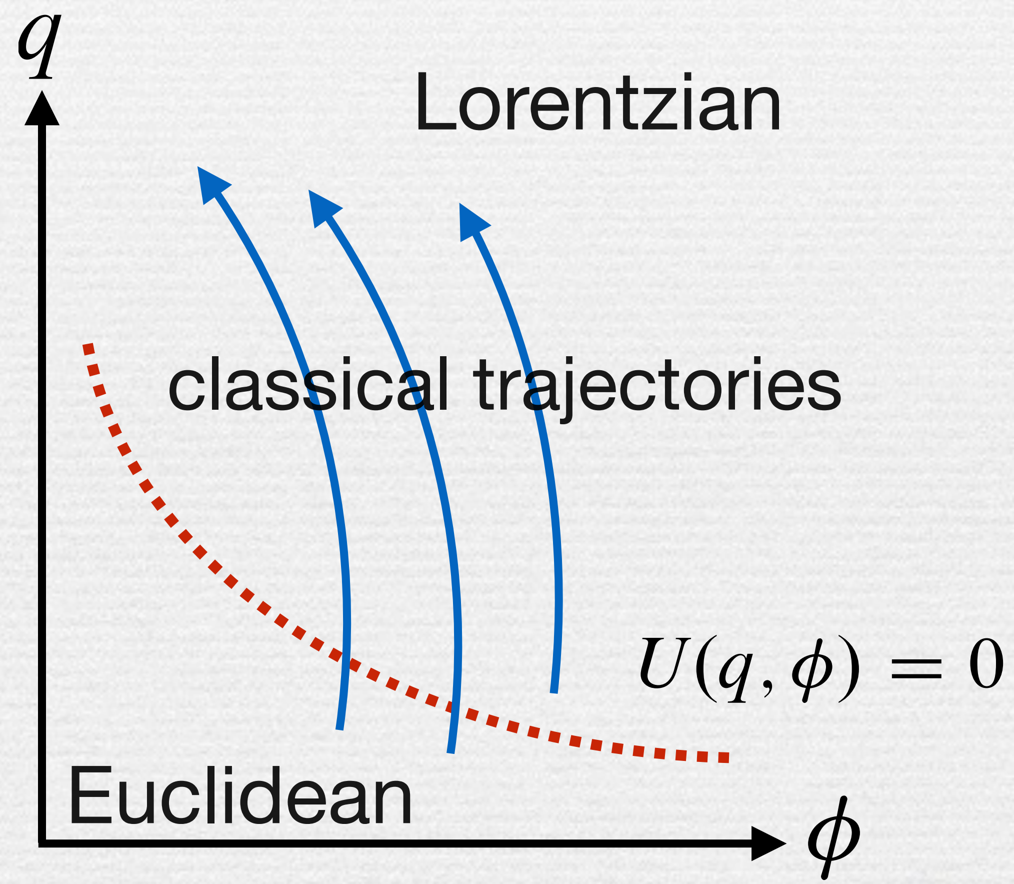
Structure of mini-superspace (q, ϕ)

scale factor matter field

WD equation (KG type eq.)

$$\left[-\frac{1}{2}G^{AB}\partial_A\partial_B + U(q,\phi)\right]\Psi(q,\phi) = 0$$

$$G^{AB} = \mathrm{diag}(-4, \frac{1}{q^2})$$
$$U(q,\phi) = -\frac{1}{2} + qV(\phi)$$



Lorentzian region: $U(q,\phi) > 0$

$q \gg 1 \qquad \Psi \sim e^{iS}$

Wave function has WKB form

"semi-classical" universe

Euclidean region: $U(q,\phi) < 0$

classically forbidden region

$q \ll 1 \qquad \Psi \sim e^{-S_E}$

"quantum" universe

How can wave functions predict classical trajectories (universe)?

Boundary condition

Hartle-Hawking (HH): sum over compact Euclidean geometries

$$\Psi(q) = \int [dNdq] \exp(-S[q,N])$$

path integral is dominated by regular Euclidean classical solutions

Vilenkin (V): wave function is purely outgoing at the infinity of superspace

tunneling type

(HH) prefers small values of cosmological constant

$$P(\phi) \sim \exp\left(\frac{1}{\Lambda(\phi)}\right)$$

(V) prefers large values of cosmological constant

$$P(\phi) \sim \exp\left(-\frac{1}{\Lambda(\phi)}\right)$$

Our present universe: large scale structure, isotropy of CMB

- we expect our universe has experienced **inflation** with $\mathcal{N} \geq 60$
- our universe has small value of cosmological constant

The purpose of QC is to explain these features of our universe

Purpose of this research

We want to say something about boundary conditions of WD eq.
by imposing observational constraints

(HH) or (V) or others ?

- model: closed FRW universe with a massive scalar field with a cosmological constant (toy cosmological model)

- constraint: sufficient number of e-foldings of inflation

$$\mathcal{N} \geq 60$$

We investigate which type of BCs of the universe is preferable

Contents

Mini-superspace model

Mini-superspace model

A closed FRW universe + massive scalar, cosmological constant

action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda) - \frac{1}{2}\int d^4x\sqrt{-g}\,\left[(\partial_\mu \Phi)^2 + m^2\Phi^2\right]$$

metric

$$ds^2 = \frac{3}{\Lambda}\left(-\frac{N^2}{q}d\lambda^2 + q\,d\Omega_3^2\right)$$

mini-superspace (q, ϕ)

Hamiltonian

$$H_T = \frac{KN}{2}\left[\frac{1}{K^2}\left(-4p_q^2 + \frac{p_\phi^2}{q^2}\right) - 1 + q(1+\mu^2\phi^2)\right] = NH$$

dimensionless parameters

$$\phi = \left(\frac{4\pi G}{3}\right)^{1/2}\Phi \qquad \mu = \left(\frac{3}{\Lambda}\right)^{1/2} m \qquad K = \frac{9\pi}{2G\Lambda}$$

Classical solutions

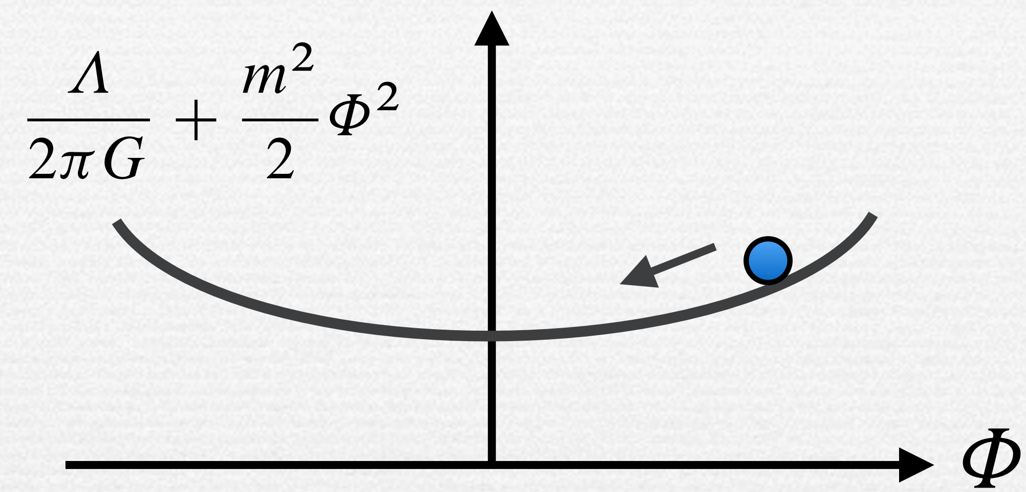
Hamiltonian constraint

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\Lambda}{3a^2} = \frac{\Lambda}{3} + \frac{4\pi G}{3}\left(\dot{\Phi}^2 + m^2\Phi^2\right)$$

scalar field eq.

$$\ddot{\Phi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\Phi} + m^2\Phi = 0$$

inflation potential



inflationary solution

slow roll condition

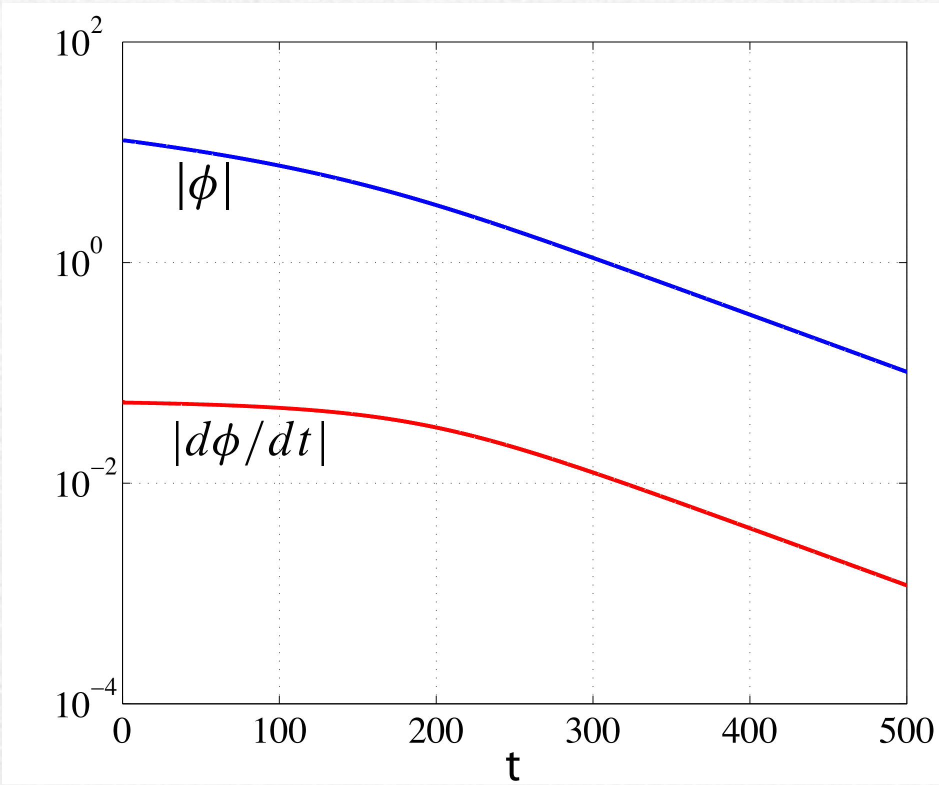
$$|\ddot{\Phi}| \lesssim \left(\frac{\dot{a}}{a}\right)|\dot{\Phi}|, \quad \dot{\Phi}^2 \lesssim m^2\Phi^2, \quad \frac{\Lambda}{3} \lesssim \frac{4\pi G}{3}m^2\Phi^2$$

⇨ A universe expands with acceleration $\ddot{a} > 0$

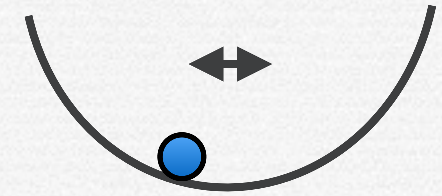
duration of inflation depends on initial values of ϕ ⇦

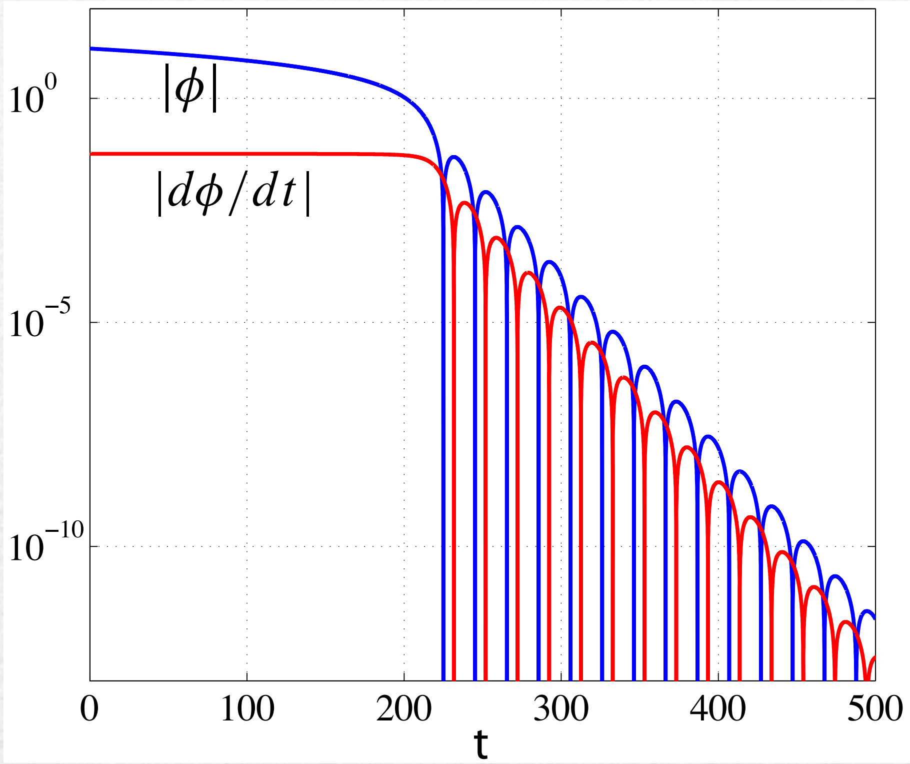
e-foldings $\mathcal{N} = \ln\left(\frac{a_{\mathrm{f}}}{a_{\mathrm{i}}}\right)$

predicted by the wave function of the universe small mass $\mu < 3$,



large mass $\mu > 3$





small m^2/Λ

slow roll ➡ over damp

large m^2/Λ

slow roll ➡ damped oscillation

The universe continues accelerated expansion forever due to the cosmological constant in this model

Wheeler-DeWitt equation

Hamiltonian constraint

$$H(q, p_q, \phi, p_\phi) = 0 \qquad p_a \to -i\frac{\partial}{\partial q}, \quad p_\phi \to -i\frac{\partial}{\partial \phi}$$

$$\left[\frac{1}{2K^2}\left(4\frac{\partial^2}{\partial q^2} - \frac{1}{q^2}\frac{\partial^2}{\partial \phi^2}\right) - \frac{1}{2} + qV(\phi)\right]\Psi(q, \phi) = 0$$

$$V(\phi) \equiv \frac{1}{2} + \frac{\mu^2}{2}\phi^2$$

As we cannot solve this equation analytically, we obtain the wave function numerically.

Two dimensional wave equation and can be solved with suitable BCs

de Sitter case $V(\phi) = \text{const.}$

$$\left(-8\frac{d^2}{dq^2} + \underbrace{1 - 2qV}_{\text{potential}}\right)\Psi(q) = 0$$

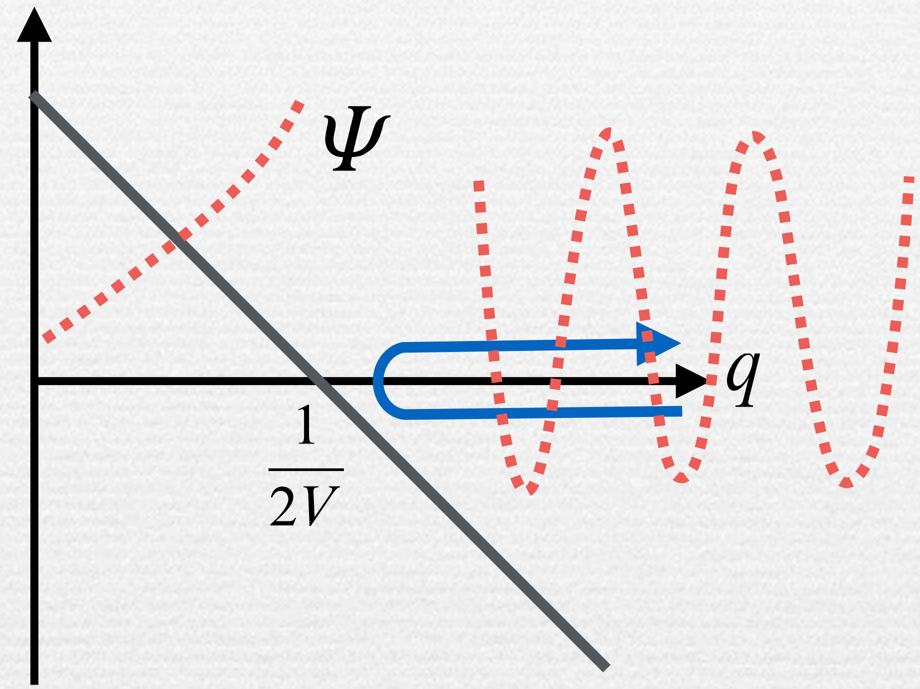
Schroedinger eq. with zero energy

(HH): superposition of expanding and contracting universes

$$\Psi \sim e^{+q/2}$$

$q \gg 1$

$$\Psi \sim e^{iS} + e^{-iS}$$

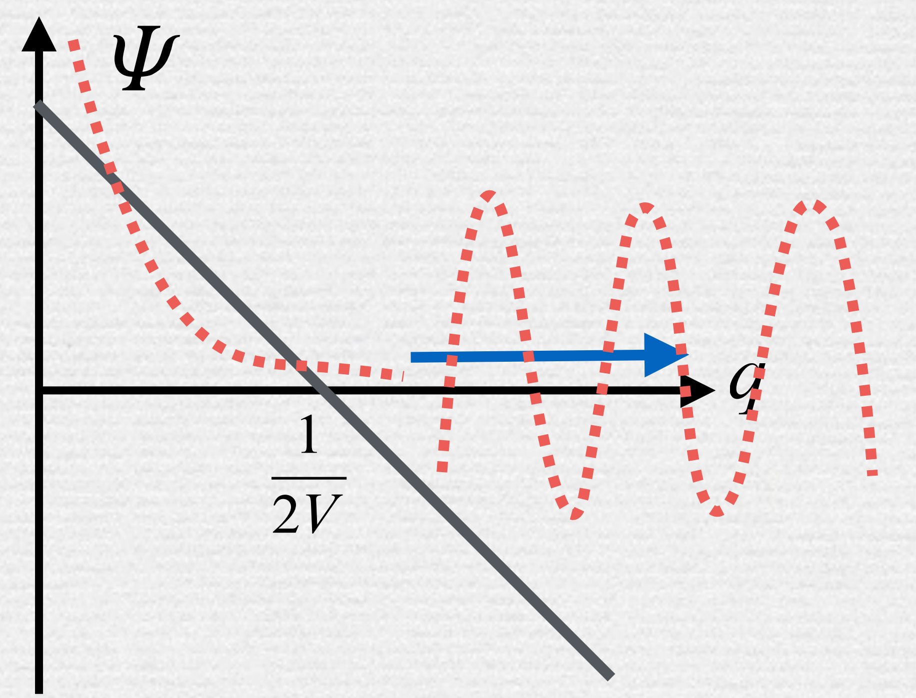


(V): purely outgoing wave tunneling type

$$\Psi \sim e^{-q/2}$$

$q \gg 1$

$$\Psi \sim e^{iS}$$



de Sitter case $V(\phi) = \text{const.}$

Halliwell, Louko 1989

- General solutions of WD eq. in terms of Airy function

$$G(q|q_0) = c_1 \mathrm{Ai}(z_0)\mathrm{Ai}(z) + c_2 \mathrm{Bi}(z_0)\mathrm{Bi}(z) + c_3(\mathrm{Ai}(z_0)\mathrm{Bi}(z) + \mathrm{Bi}(z_0)\mathrm{Ai}(z))$$

(wave functions as transition amplitude from $q_0 \to q$)

$$z = z(q) = \left(\frac{4V}{K}\right)^{-2/3} (1 - 2qV), \quad z_0 = z(0) = \left(\frac{4V}{K}\right)^{-2/3}$$

typical wave functions

$$\boxed{\begin{aligned} \Psi_{\mathrm{HH}} &= \Psi_2 + \Psi_3 \\ \Psi_{\mathrm{V}} &= \Psi_1 + i\Psi_3 \end{aligned}}$$

$$\sim \exp\left(+\frac{K}{6V}\right) \cos S_0$$

$$\sim \exp\left(-\frac{K}{6V}\right) \exp(-iS_0)$$

$$\Psi_1 \equiv (2V)^{-1/3} \mathrm{Ai}(z_0)\mathrm{Ai}(z)$$

$$\Psi_2 \equiv (2V)^{-1/3} \mathrm{Bi}(z_0)\mathrm{Ai}(z)$$

$$\Psi_3 \equiv (2V)^{-1/3} \mathrm{Ai}(z_0)\mathrm{Bi}(z)$$

$$S_0(q,\phi) = \frac{K}{6V(\phi)} (2V(\phi)q - 1)^{3/2} - \frac{\pi}{4}$$

(HH) and (V) can be represented using three functions (solutions)

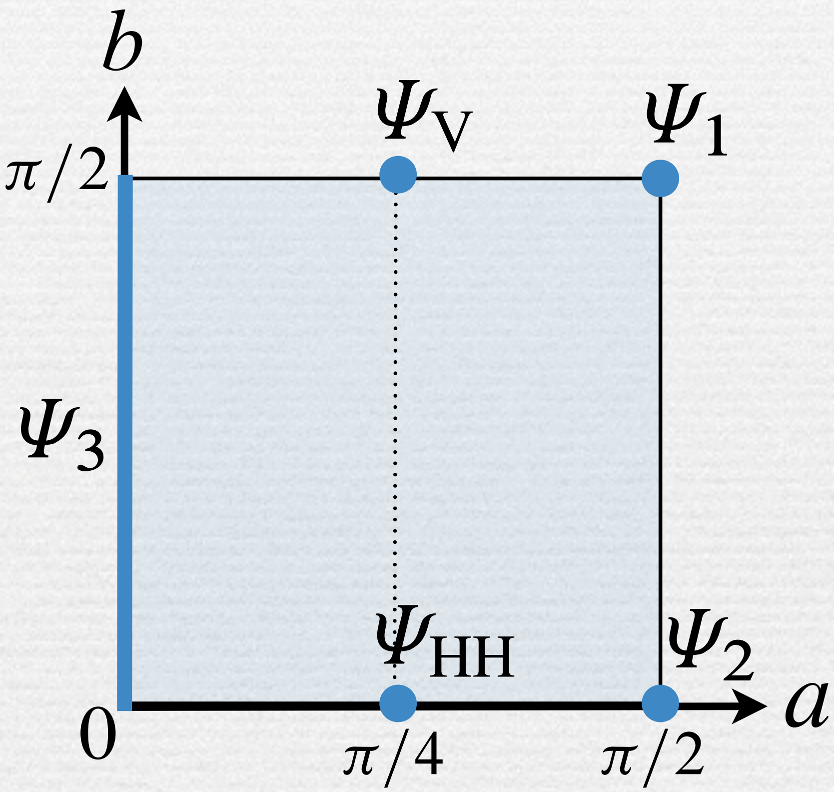
We parametrize solutions including (HH) and (V) using two real parameters

$$\Psi_C = \tan a(\cos b\, \Psi_2 - i \sin b\, \Psi_1) + \Psi_3, \quad 0 \leq a, b \leq \pi/2$$

$$\Psi_1 \equiv (2V)^{-1/3}\mathrm{Ai}(z_0)\mathrm{Ai}(z)$$

$$\Psi_2 \equiv (2V)^{-1/3}\mathrm{Bi}(z_0)\mathrm{Ai}(z)$$

$$\Psi_3 \equiv (2V)^{-1/3}\mathrm{Ai}(z_0)\mathrm{Bi}(z)$$



space of BCs

wave function	parameter (a, b)	asymptotic form for $q \gg 1$
Ψ_{HH}	$(\frac{\pi}{4}, 0)$	$\sim \exp\left(+\frac{K}{6V}\right) \cos S_0$
Ψ_{V}	$(\frac{\pi}{4}, \frac{\pi}{2})$	$\sim \exp\left(-\frac{K}{6V}\right) \exp(-i S_0)$
Ψ_1	$(\frac{\pi}{2}, \frac{\pi}{2})$	$\sim \exp\left(-\frac{K}{6V}\right) \cos S_0$
Ψ_2	$(\frac{\pi}{2}, 0)$	$\sim \exp\left(+\frac{K}{6V}\right) \cos S_0$
Ψ_3	(0, any values)	$\sim -\exp\left(-\frac{K}{6V}\right) \sin S_0$

We specify a BC of WD eq. for non-constant potential case using this parametrization

WKB analysis and probability

Hartle, Hawking and Hertog 2008

$$\left[-\frac{1}{2}G^{AB}\partial_A\partial_B + U(q,\phi)\right]\Psi(q,\phi) = 0$$

WKB anzatz

$$\Psi(q^A) = C(q^A)e^{-\frac{1}{\hbar}I(q^A)}$$

phase function is complex in general

$$I = I_R - iS$$

$$O(\hbar^0): \quad -\frac{1}{2K^2}(\nabla I)^2 + U(q^A) = 0,$$

$$O(\hbar^1): \quad 2\nabla I \cdot \nabla C + C\nabla^2 I = 0,$$

conservation of current

If the condition holds

$$\frac{|\nabla I_R|^2}{|\nabla S|^2} \ll 1$$

"classicality" condition

we obtain the Hamilton-Jacobi equation

$$\frac{1}{2K^2}(\nabla S)^2 + U = 0. \qquad p_A = \frac{\partial S}{\partial q_A}$$

WKB wave function

$$\Psi(q^A) = \sum_{i=\text{saddle}} C^{(i)}(q^A)e^{-I_R^{(i)}(q^A)}e^{iS^{(i)}(q^A)}$$

Conserved current of WD eq.

$$\mathcal{J}_A = \frac{i}{2}(\Psi^* \nabla_A \Psi - \Psi \nabla_A \Psi^*), \quad \nabla \cdot \mathcal{J}_A = 0.$$

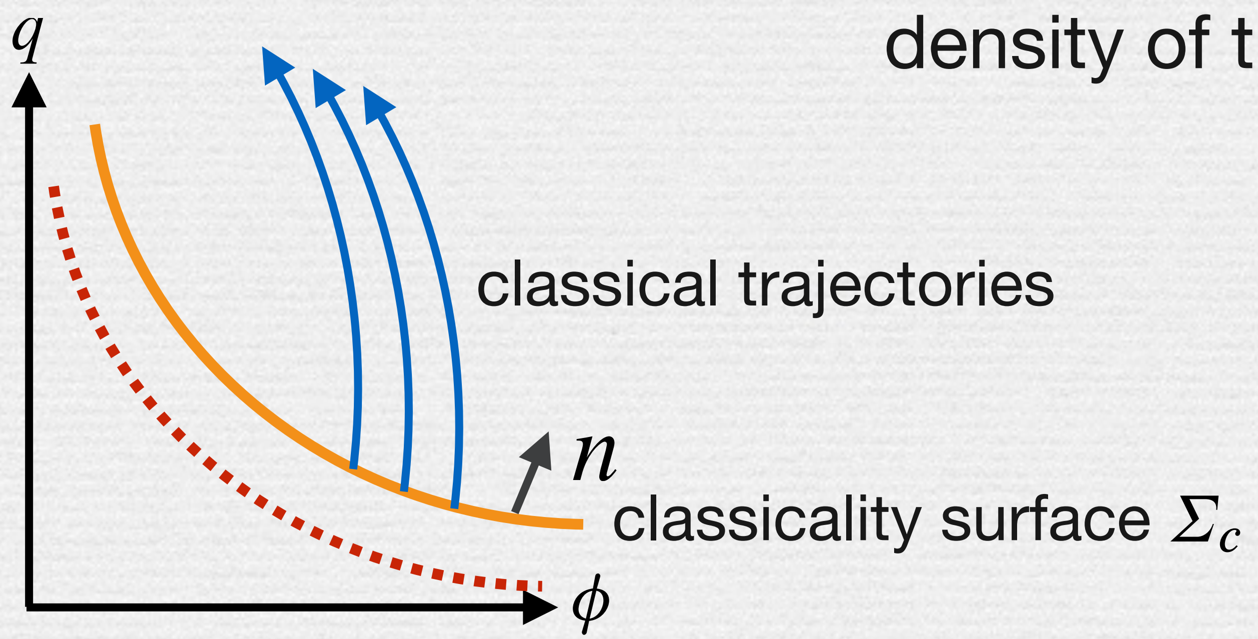
For WKB wave function,

$$J_A^{(i)} \equiv -|C^{(i)}|^2 \exp(-2I_R^{(i)}) \nabla_A S^{(i)} \qquad \nabla \cdot J^{(i)} = 0.$$

We consider a hypersurface Σ_c on which the classicality condition is satisfied. We can define probability measure for expanding universes on Σ_c by

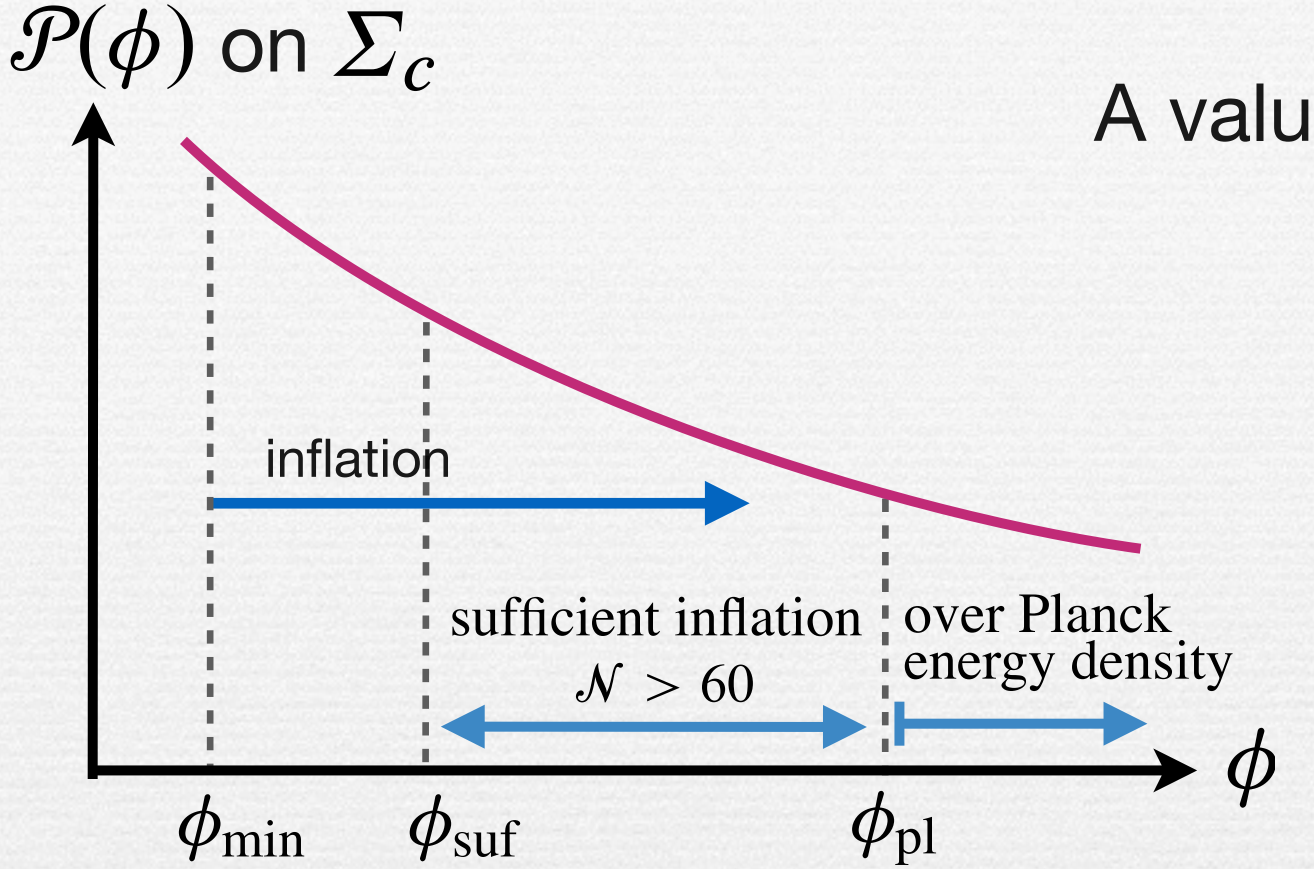
$$\mathcal{P}(\phi) \equiv J^+ \cdot n = \underline{-|C|^2 \exp(-2I_R)}\nabla_n S \qquad p_q = -\partial_q S < 0$$

density of trajectories crossing Σ_c



$\Rightarrow$ probability for ϕ

$\mathcal{P}(\phi)$



A value ϕ on Σ_c determines $\mathcal{N}$

Conditional probability for sufficient e-foldings of inflation

$$P_{\rm suf} \equiv P(\mathcal{N} \geq 60) = \frac{\int_{\phi_{\rm suf}}^{\phi_{\rm pl}} d\phi\, \mathcal{P}(\phi)}{\int_{\phi_{\rm min}}^{\phi_{\rm pl}} d\phi\, \mathcal{P}(\phi)}.$$

⇦ prob. of inflation with $\mathcal{N} > 60$

⇦ prob. of inflation

$\phi_{\rm min}$: end of inflation driven by scalar field

Probability of boundary conditions

Probability of boundary conditions

For a wave function with a specific BC, it is possible to obtain probability of sufficient inflation

$$P_{\mathrm{suf}} = P(S|B_i)$$

On the other hand,
probability of BC under the condition of sufficient inflation is

$$\boxed{P(B_i|S) = \frac{P(B_i)P(S|B_i)}{\sum_k P(B_k)P(S|B_k)}}$$

Bayes theorem

$P(S|B_i)$ probability of sufficient inflation under a specific BC

$P(B_k)$ prior probability: assume uniformly distributed

We represented $\{B_i\}$ using two parameters a, b

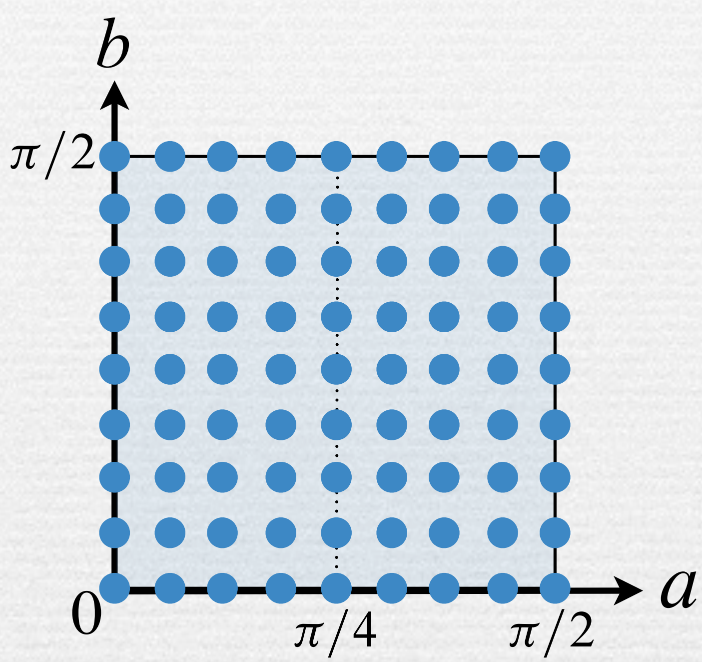
Probability of boundary condition under restriction of sufficient inflation

$$\boxed{P(a,b|S) = \frac{P(S|a,b)}{\int da'db' P(S|a',b')}}$$

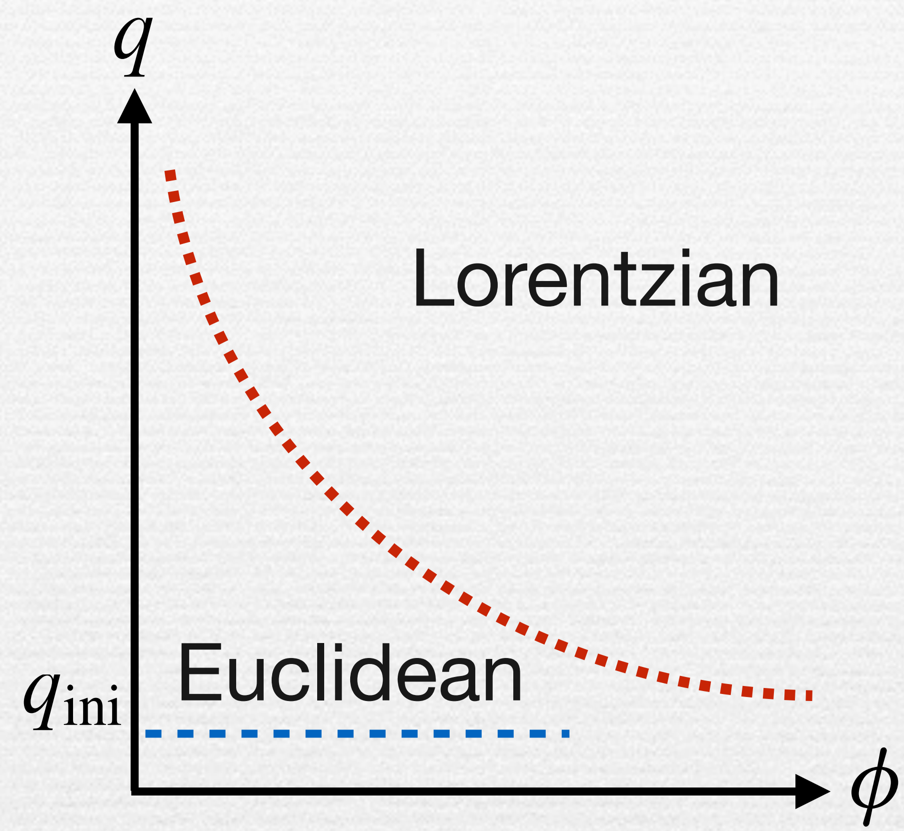
Numerical analysis of the wave function

H.Suenobu & YN
arXiv:1603.08172

We solved the WD equation numerically
and obtained wave functions for 9x9 BCs



space of BCs



2000x200 grids in mini-superspace

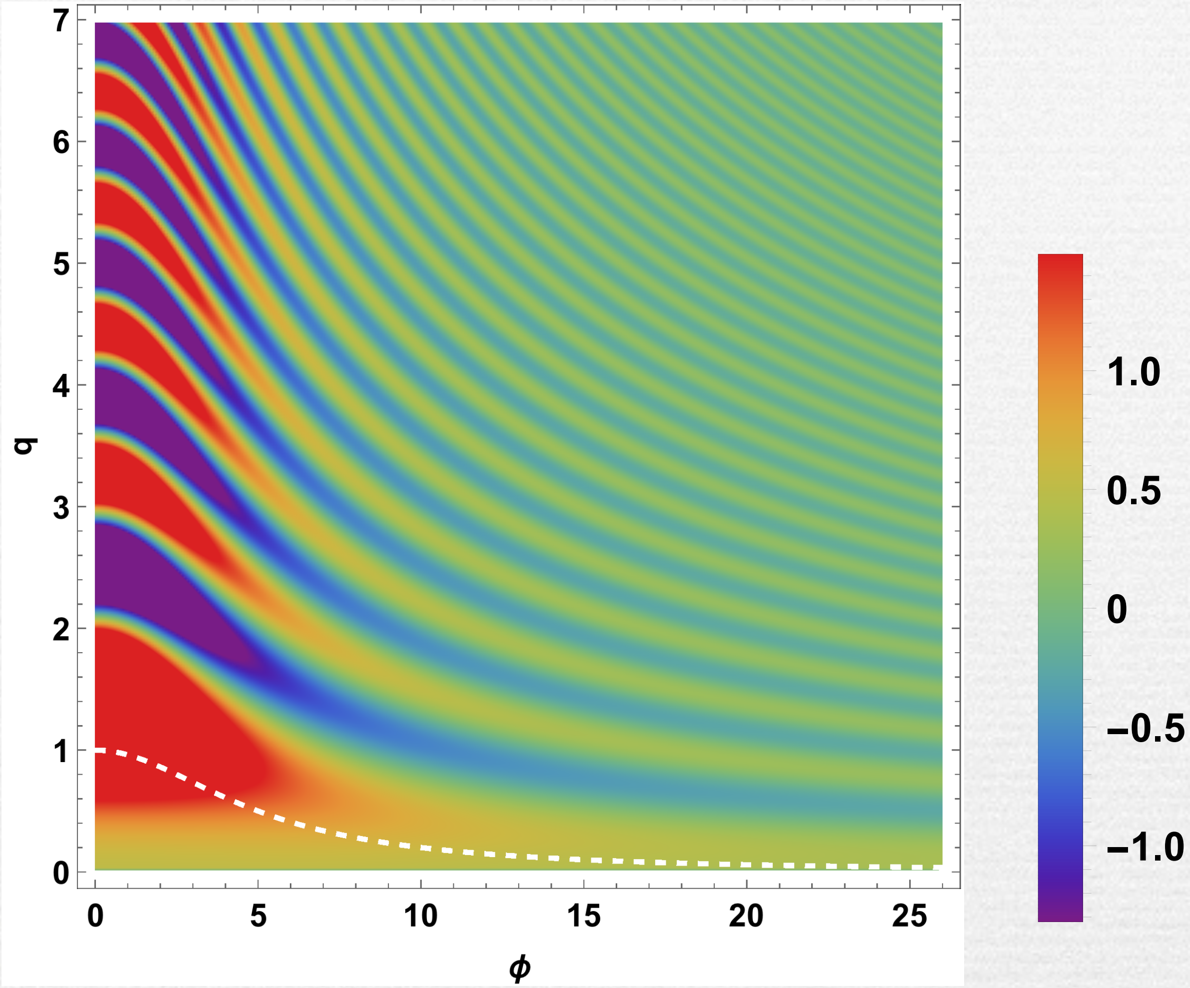
5-step Adams-Bashforth method

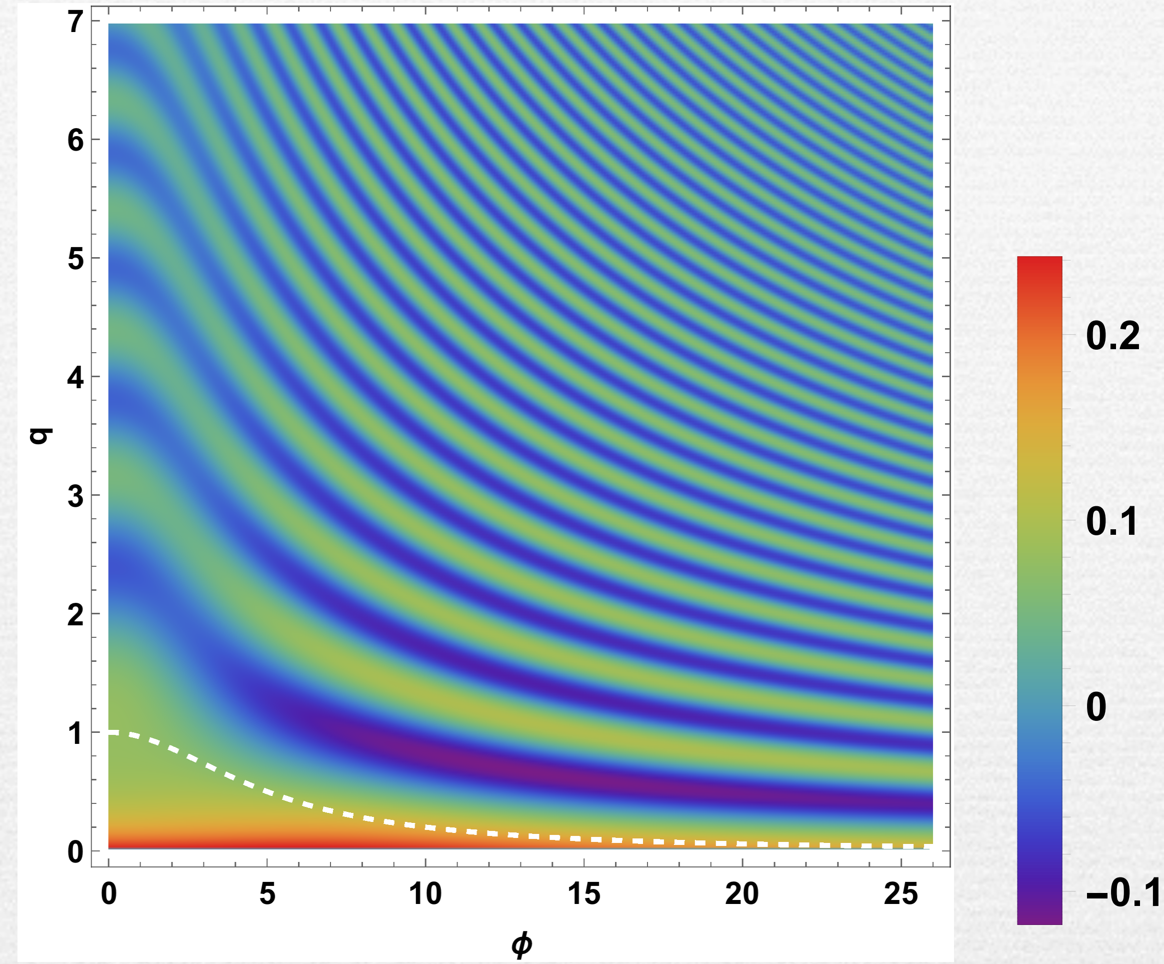
Boundary condition for WD equation: exact solution for constant potential

parameter a, b

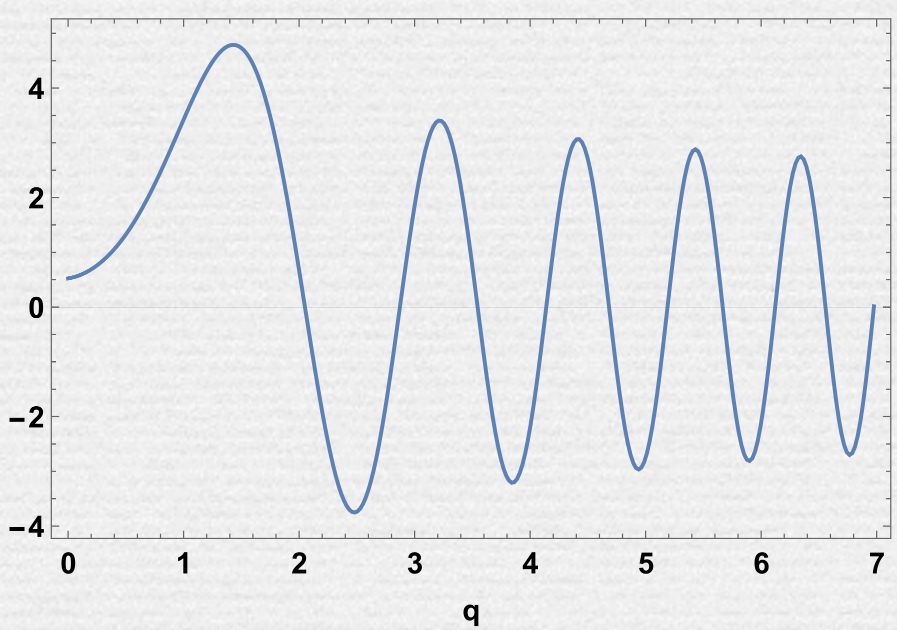
$$\Psi(q_{\rm ini}, \phi) = \Psi_C(q_{\rm ini}, \phi), \quad \partial_q \Psi(q_{\rm ini}, \phi) = \partial_q \Psi_C(q_{\rm ini}, \phi)$$

$\Psi(\phi, q)$ (HH) $\mu = 0.2$ $\mathrm{Im}[\Psi(\phi, q)]$ (V)

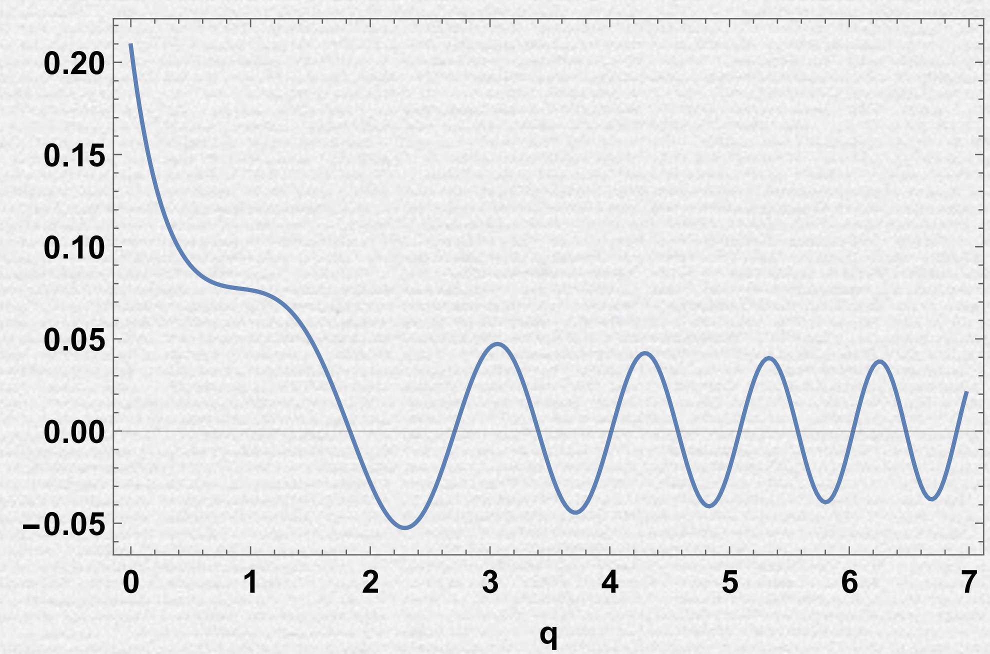




slice at $\phi = 10$

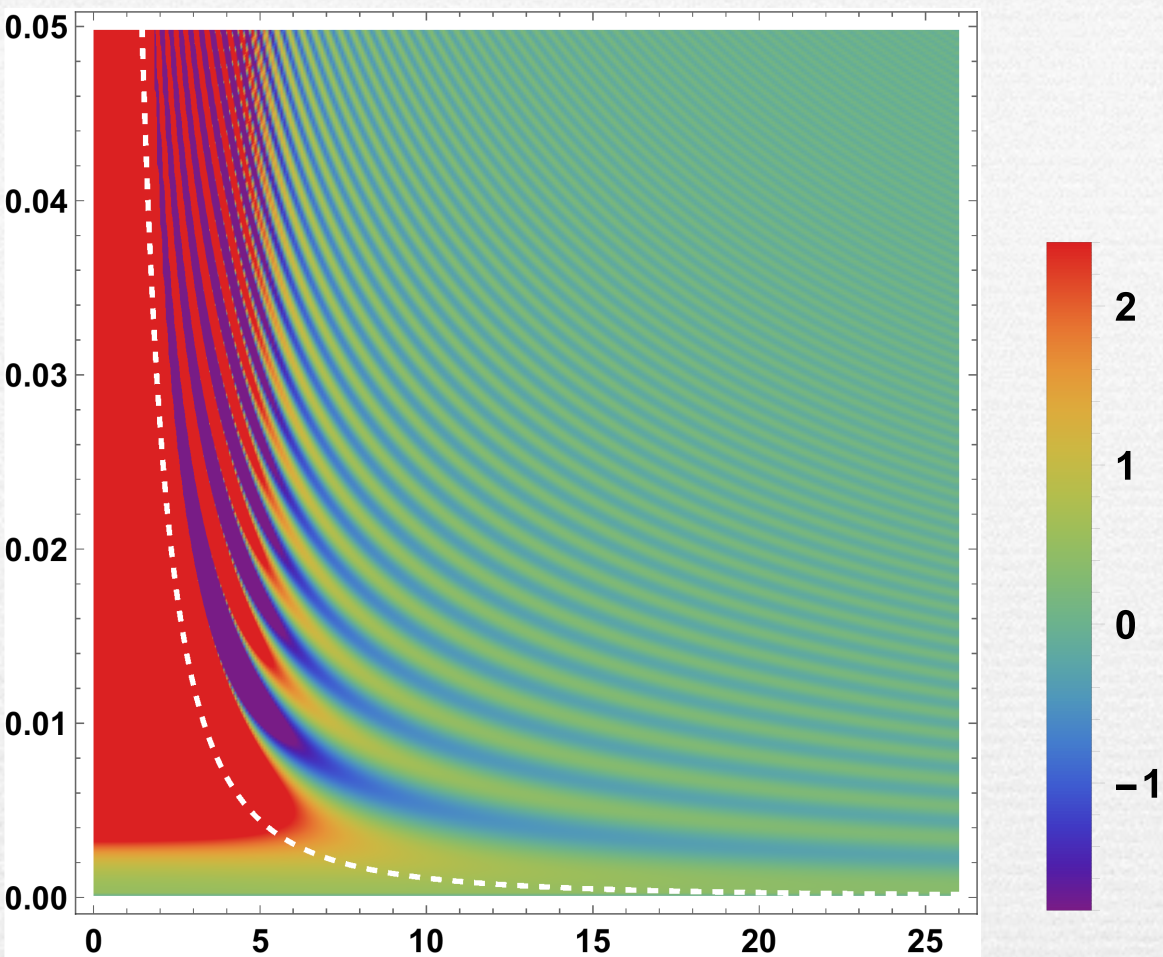


slice at $\phi = 10$

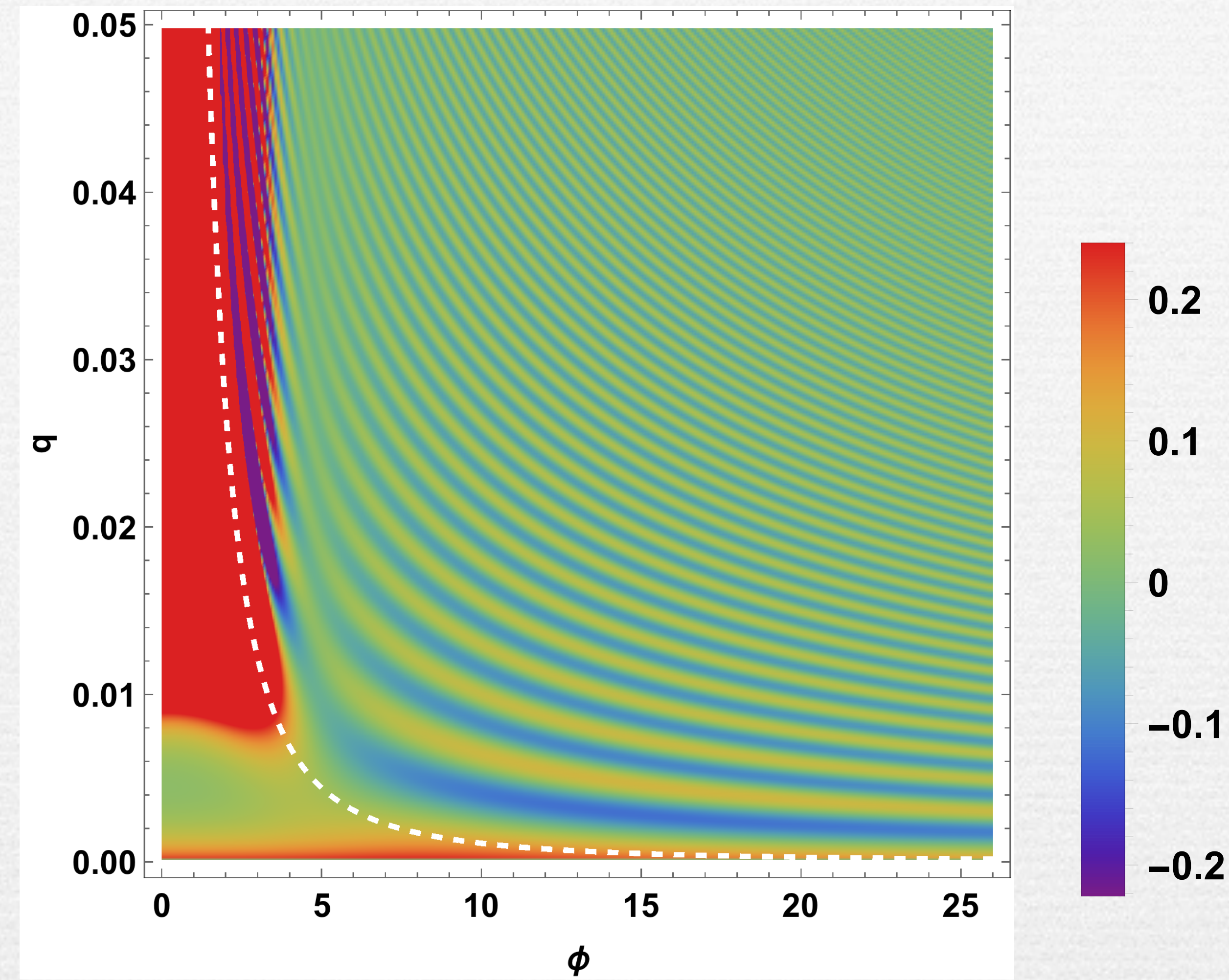


$\Psi(\phi, q)$ (HH)

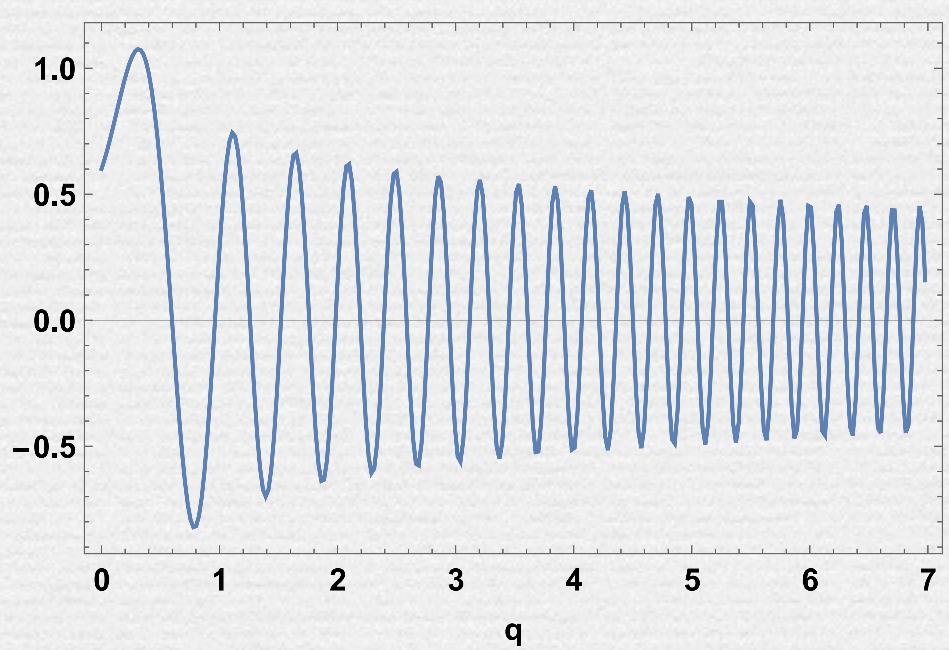
$\mu = 3$



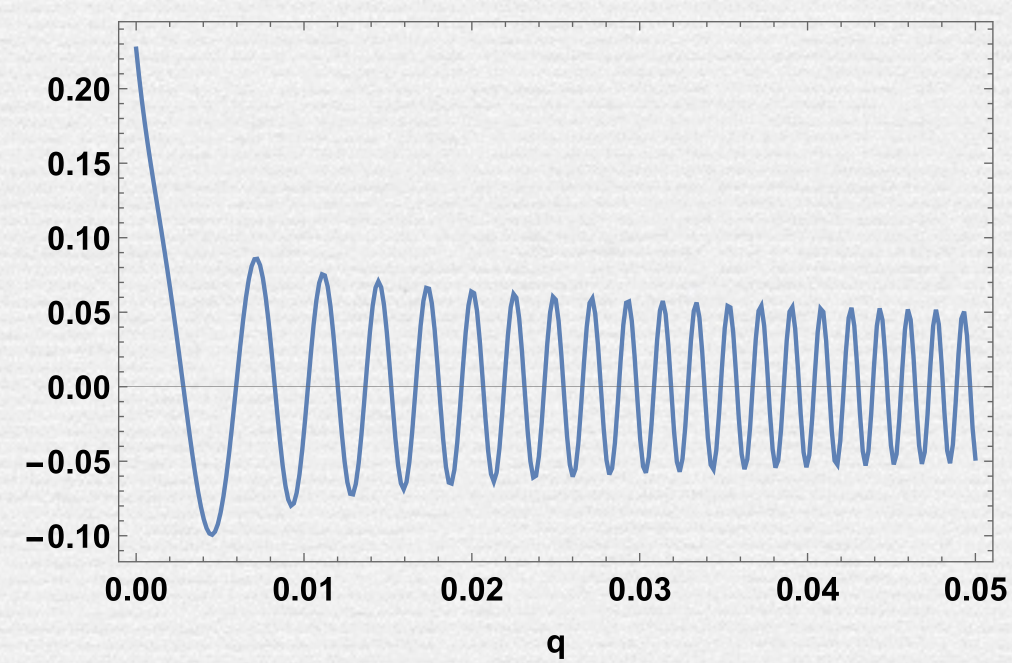
$\mathrm{Im}[\Psi(\phi, q)]$ (V)



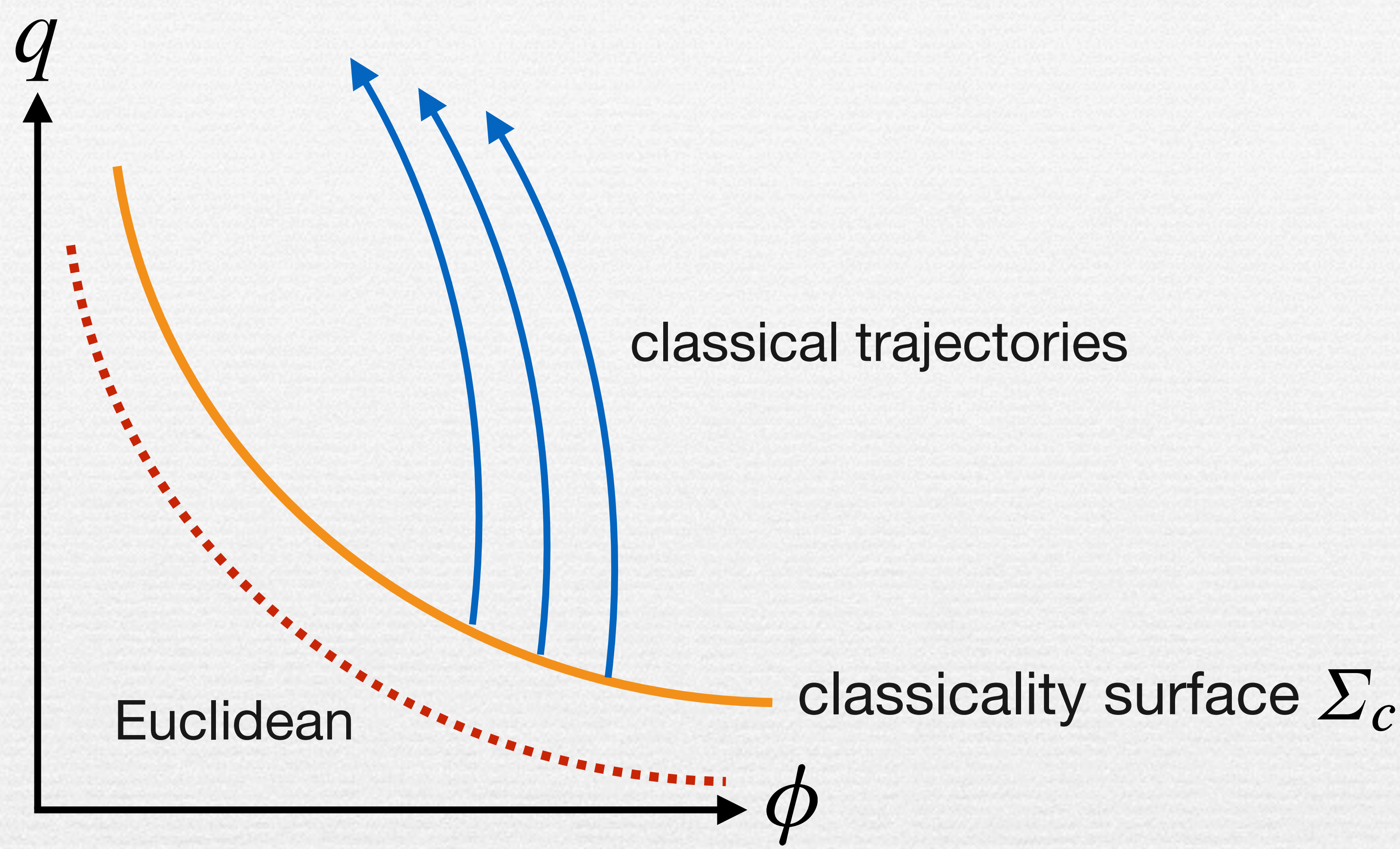
slice at $\phi = 10$



slice at $\phi = 10$



Extraction of probability from wave functions



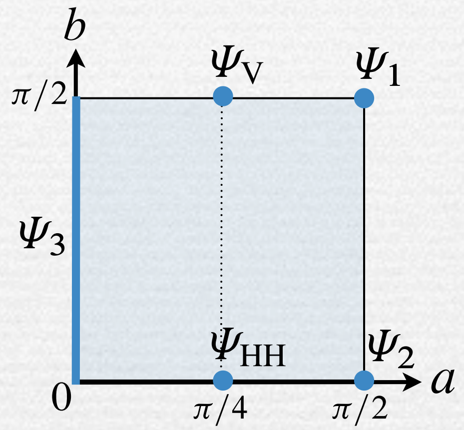
(0) Obtain wave function with (a,b)

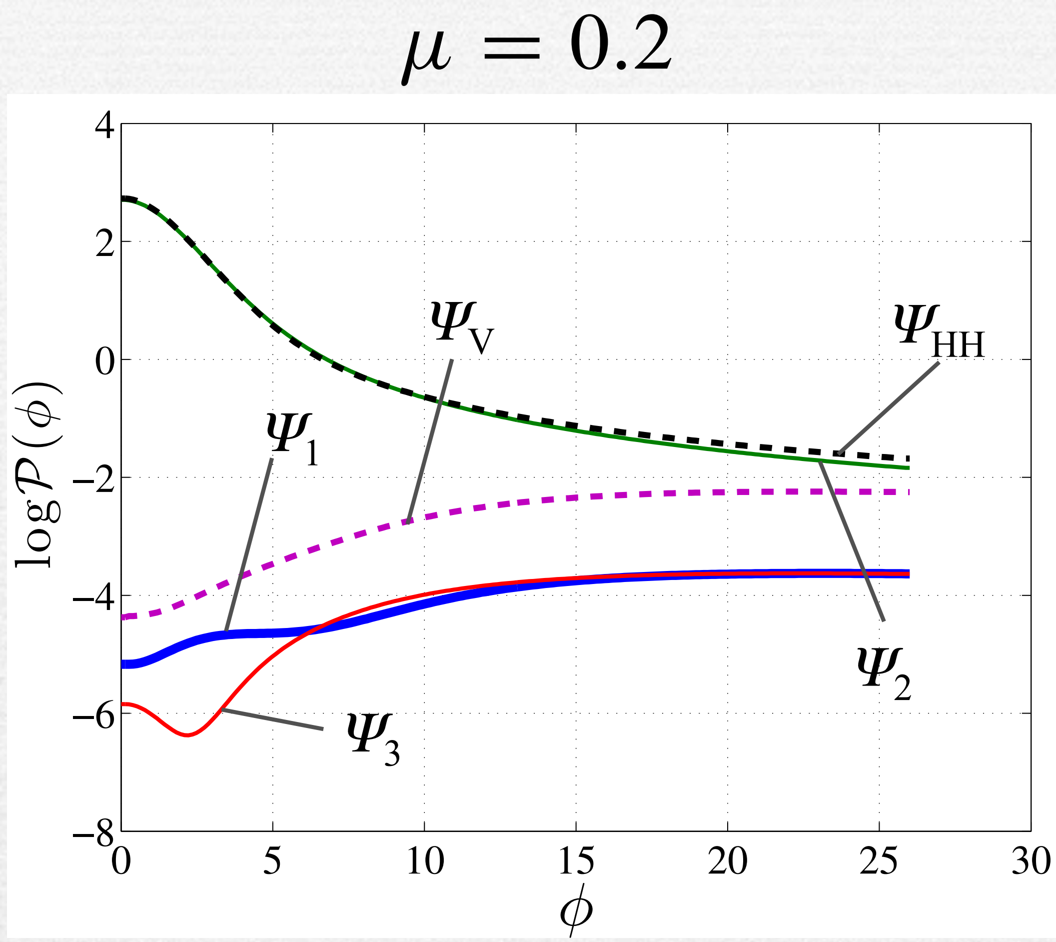
(1) Specify a classicality surface $S_0(q,\phi) = \frac{K}{6V(\phi)}\left(2V(\phi)q - 1\right)^{3/2} - \frac{\pi}{4}$

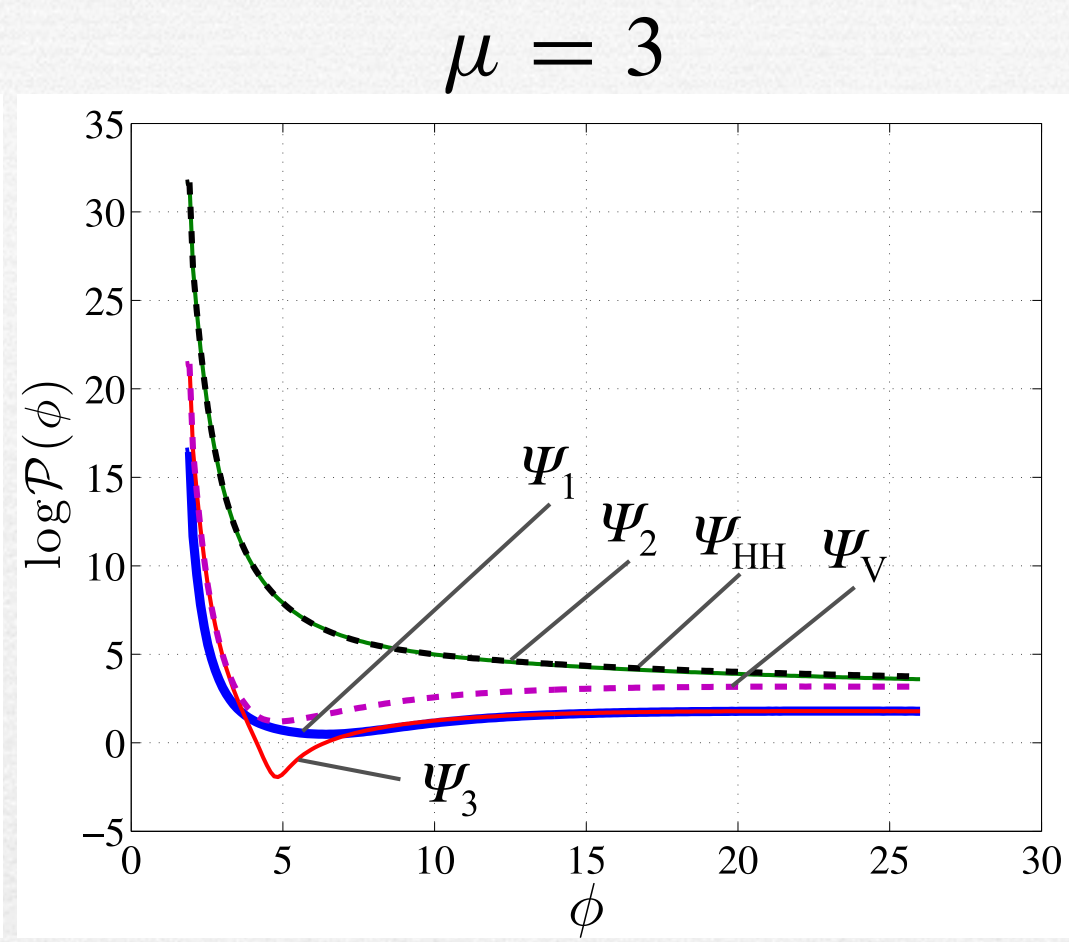
(2) On the classicality surface, we obtain probability measure of inflaton field from the wave function. ⇨ initial value of classical equation

(3) Then, integrate classical eq. motion to obtain e-foldings for scalar field driven inflation and obtain probability for e-foldings.

Probability of inflaton field for various BCs (unnormalized) on the classicality surface







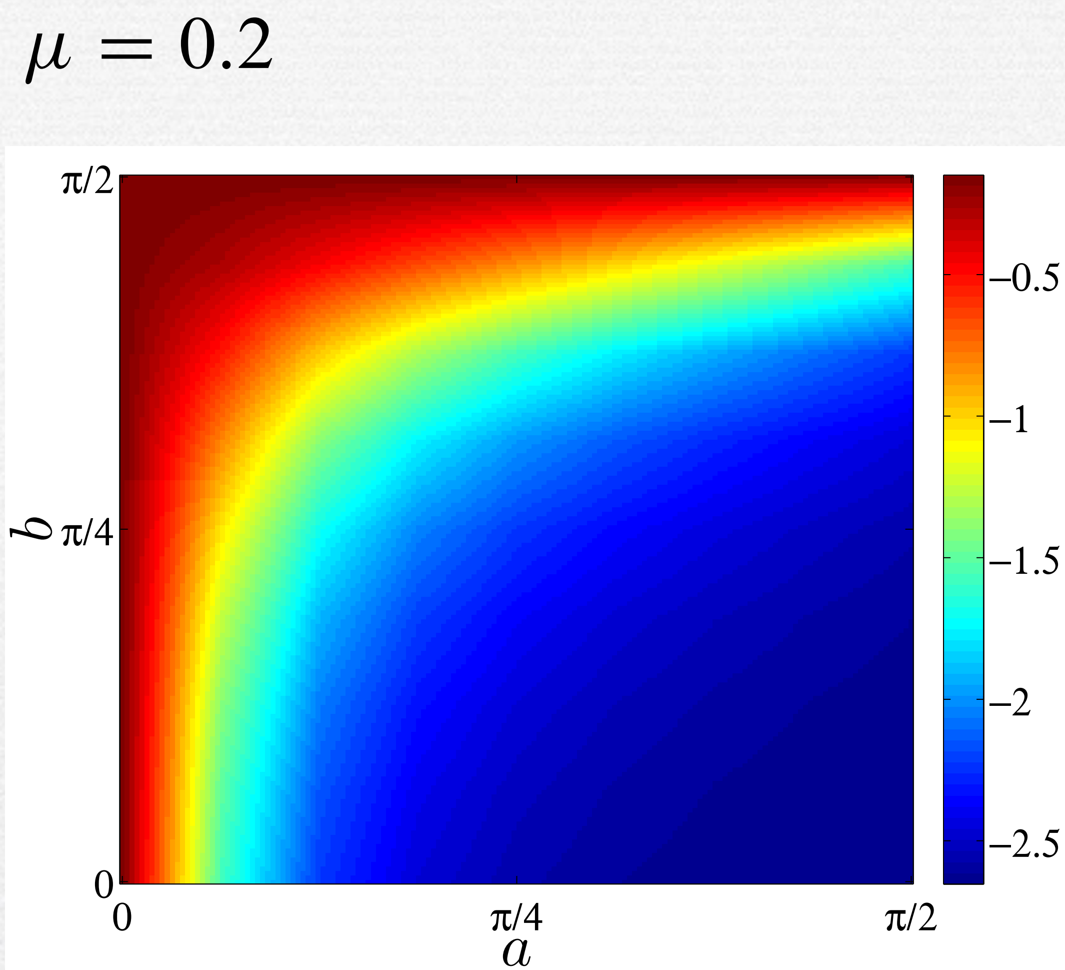
- The wave function with (HH) prefers small values of potential
- The behavior of the wave function with (V) depends on the value of

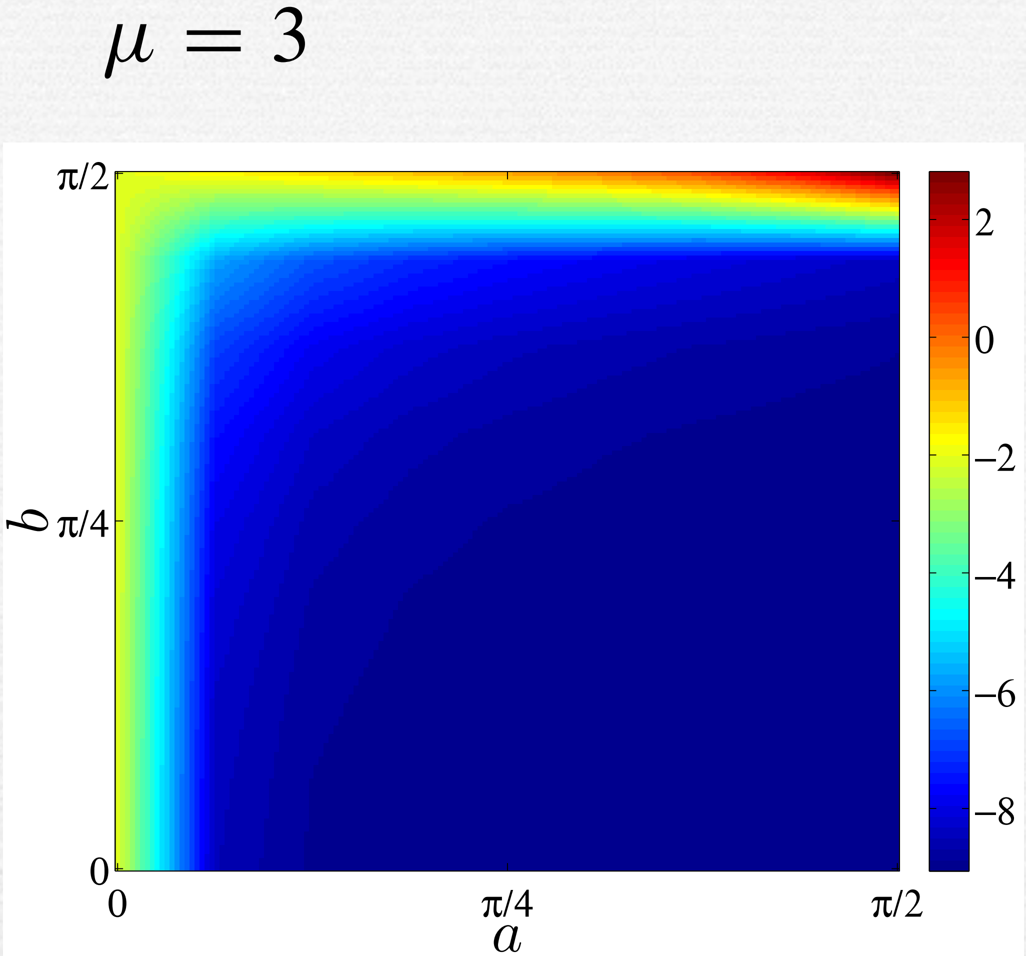
$$\mu \propto m/\Lambda^{1/2}$$

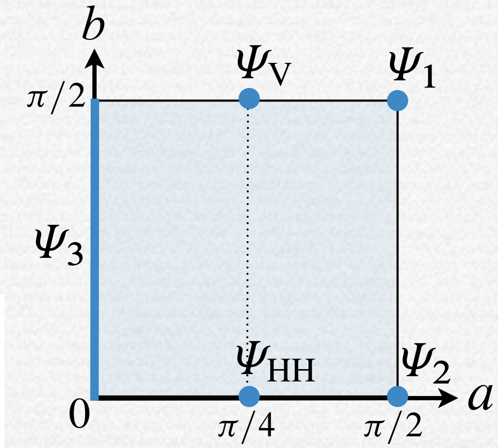
small μ prefers large values of potential

large μ prefers small values of potential

Probability of boundary conditions $P(a, b)$







- Ψ_{V} is superior to Ψ_{HH} for large e-foldings of inflation
- Location and spread of peak depends on mass parameter μ
- For fixed value of mass, the probability distribution becomes more steep for smaller value of the cosmological constant.

Does this something to do with the cosmological constant problem?

Fisher information analysis

Summary

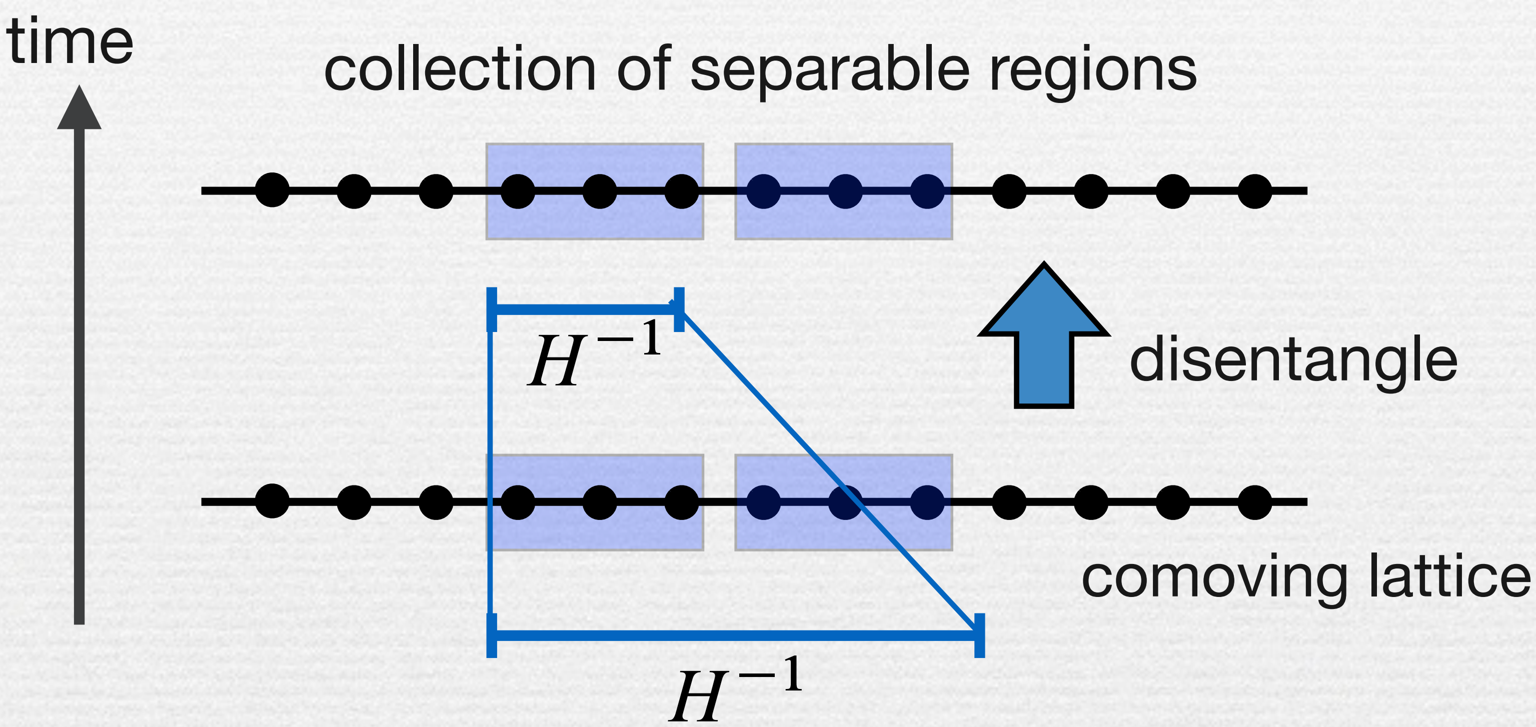
Summary

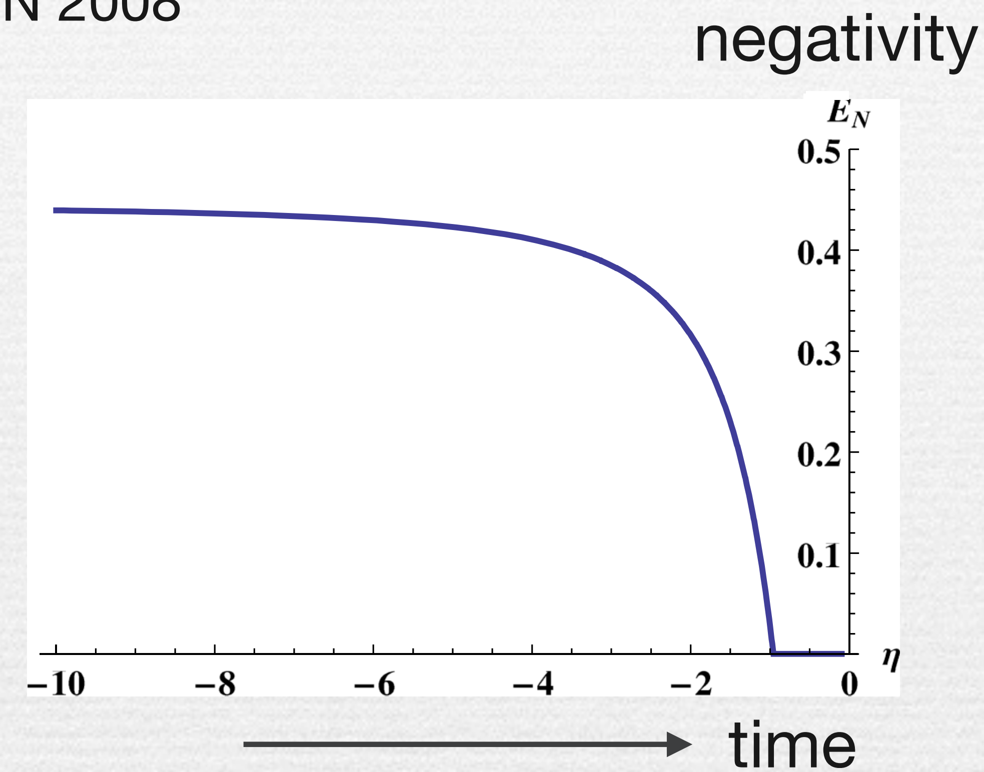
- One purpose of quantum cosmology is to predict inflationary universe.
- Introducing parametrization in BC space of the wave function, we evaluated probability of BC under the condition of sufficient e-foldings of inflation. The probability depends of the value of parameters in the model.
- Fisher information (parameter estimation?)
- Beyond mini-superspace model of quantum cosmology?

Quantum informational approach to quantum cosmology

How can we predict inflationary universe?
(emergence of de Sitter spacetime)

- Quantum field in de Sitter (harmonic chain) YN 2008

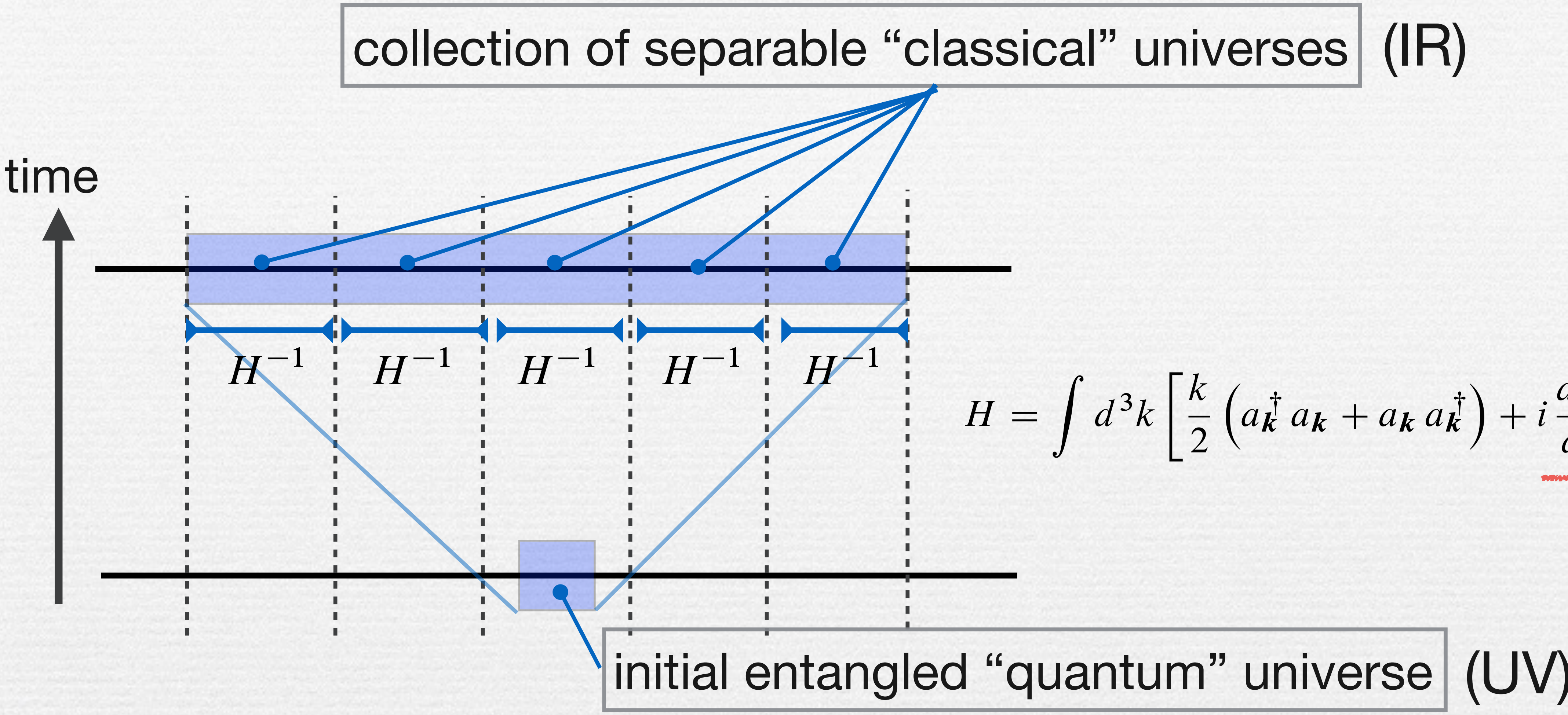




- Two regions become separable after H^{-1}

⇨ Independent separable “classical” universes with classical correlations appear.

originated from entangled vacuum fluctuations
and become seed of large scale structures in our universe



$$H = \int d^3k \left[\frac{k}{2} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger \right) + \underbrace{i \frac{a'}{a} \left(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger - a_{\mathbf{k}} a_{-\mathbf{k}} \right)}_{\text{disentangler}} \right]$$

- Emergence of classical universes needs specific type of expansion law

  ⇒ separability requires emergence of de Sitter expansion

> We expect to obtain a new perspective for origin of inflationary universe in this direction of research using QI.